

Riordan Arrays and Probability Distributions

Masaaki Sibuya, Keio University

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1 Preliminaries

Generating function ω GF / coefficient extraction

$$\begin{aligned} \text{weight : } & \omega = (\omega_0, \omega_1, \omega_2, \dots), \omega_0 = 1, \omega_n \neq 0. \\ a_\omega(t) := \sum_{n \geq 0} a_n \frac{t^n}{\omega_n} & \iff a_n = \left[\frac{t^n}{\omega_n} \right] a_\omega(t), \forall a = (a_0, a_1, a_2, \dots), \\ \forall a = (a_0, a_1, a_2, \dots), \forall b = (b_0, b_1, b_2, \dots), & \\ \text{convolution } c := a \star b = b \star a, c = (c_0, c_1, c_2, \dots), & \iff \frac{c_n}{\omega_n} = \sum_{n \geq k \geq 0} \frac{a_k}{\omega_k} \frac{b_{n-k}}{\omega_{n-k}}, \\ \text{and } c_\omega(t) = a_\omega(t)b_\omega(t) \text{ or } & \left[\frac{t^n}{\omega_n} \right] a_\omega(t)b_\omega(t) = \sum_{n \geq k \geq 0} \left[\frac{u^k}{\omega_k} \right] a_\omega(u) \left[\frac{t^{n-k}}{\omega_{n-k}} \right] b_\omega(t). \end{aligned}$$

Self-convolution A sequence of the k -th order self-convolution

$$\begin{aligned} a^{(2)}(t) &= a_\omega(t) \star a_\omega(t), \quad a^{(k)}(t) = a^{(k-1)}(t) \star a_\omega(t), \quad k \geq 2, \\ a^{(1)}(t) &= a_\omega(t) \quad a^{(0)}(t) = 1 =: \epsilon, \quad \epsilon a_\omega(t) = a_\omega(t) \epsilon = a_\omega(t), \quad \text{identity} \\ (a^{(0)}(t), a^{(1)}(t), a^{(2)}(t), \dots) &= (\epsilon, a_\omega(t), a_\omega^2(t), \dots). \end{aligned}$$

Reciprocal (convolution inverse)

$$\begin{aligned} a_\omega(t)b_\omega(t) = \epsilon, \quad a_0 \neq 0 \implies b_n &= \frac{-1}{a_0} \left(\frac{a_1}{\omega_1} \frac{b_{n-1}}{\omega_{n-1}} + \cdots + \frac{a_n}{\omega_n} \frac{b_0}{\omega_0} \right), \quad n > 0; \quad b_0 = \frac{1}{a_0}. \\ b_\omega(t)a_\omega(t) = \epsilon, \quad b_\omega(t) &=: a_\omega^{-1}(t) = \frac{1}{a_\omega(t)}. \end{aligned}$$

Composition

$$\begin{aligned} \forall a = (a_0, a_1, a_2, \dots), \forall b = (b_0, b_1, b_2, \dots), & \\ \text{composition } c := a \circ b, c = (c_0, c_1, c_2, \dots), & \iff \\ \frac{c_n}{\omega_n} := \sum_{k \geq 0} \frac{a_k b_k^n}{\omega_k}, \quad b_k^n := \left[\frac{t^n}{\omega_n} \right] b_\omega^k(t). & \iff c_\omega(t) = a_\omega(b_\omega(t)). \end{aligned}$$

Compositional inverse

$$\begin{aligned} a &= (a_0, a_1, a_2, \dots), \text{ such as } a_0 = 0, a_1 \neq 0, \\ a_\omega(b_\omega(t)) &= b_\omega(a_\omega(t)) = t : b_\omega(t) =: a_\omega^\leftarrow(t). \end{aligned}$$

Compositional identity in generating function is independent of ω .

Formal power series

\mathbf{F} : a field of characteristic 0,

$\mathcal{F}[t]$: the ring of power series in the indeterminate t with coefficients in \mathbf{F} ,

$[t^n/\omega_n] : \mathcal{F} \rightarrow \mathbf{F}$: linear functionals, “coefficient extraction”.

$$\begin{aligned} \left[\frac{t^n}{\omega_n} \right] \frac{t^k}{\omega_k} &= \delta_{n,k}, \\ \left[\frac{t^n}{\omega_n} \right] (\alpha a_\omega(t) + \beta b_\omega(t)) &= \alpha \left[\frac{t^n}{\omega_n} \right] a_\omega(t) + \beta \left[\frac{t^n}{\omega_n} \right] b_\omega(t). \end{aligned}$$

convolution, compositon, inversion; shift, differentiation. All results are ω -dependnt.

“ordinary (standard) GF”: $\omega_n \equiv 1$, $a(t) := \sum_{n \geq 0} a_n t^n$.

Comtet(1974), Riordan(1958)

2 Riordan arrays

Definition and basic properties [Shapiro, et al. 1991.]

$$\Lambda_0 = \{(n, k) \in \mathbb{N}_0^2; n \geq k \geq 0\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

$$g_\omega(t), f_\omega(t) \in \mathcal{F}[t], g_0 \neq 0; f_0 = 0, f_1 \neq 0;$$

$$\begin{aligned} R &= (R_{n,k})_{(n,k) \in \mathbb{N}_0^2}, \quad R_{n,k} = 0, \text{ if } (n, k) \notin \Lambda_0. \\ \sum_{n \geq k} \frac{R_{n,k} t^n}{\omega_n} &:= g_\omega(t) \frac{(f_\omega(t))^k}{\omega_k}, \quad k = 0, 1, 2, \dots \quad \omega\text{GF of the } k\text{-th column}. \end{aligned}$$

$R = R_\omega =: (g_\omega(t), f_\omega(t))$: “ ω -Riordan array” (ω RA).

$$\begin{aligned} R_{n,k} &= \left[\frac{t^n}{\omega_n} \right] g_\omega(t) \frac{(f_\omega(t))^k}{\omega_k}, \\ b = R * a : \quad b_n &= \sum_{n \geq k \geq 0} R_{n,k} a_{n-k}, \quad \iff \quad b_\omega(t) = g_\omega(t) a_\omega(f_\omega(t)), \quad \forall a_\omega(t) \in \mathcal{F}[t], \end{aligned} \tag{1}$$

A linear transform of a to b .

[Shapiro, Getu, Woan and Woodson (1991), Wang and Wang (2008)]

Riordan group

$$\begin{aligned} R_1 &:= (g_\omega(t), f_\omega(t)), \quad R_2 := (G_\omega(t), F_\omega(t)), \\ R_1 * R_2 &= (g_\omega(t) G_\omega(f_\omega(t)), F_\omega(f_\omega(t))). \end{aligned} \tag{2}$$

$$(g_\omega(t), f_\omega(t)) * (1, t) = (1, t) * (g_\omega(t), f_\omega(t)) = (g_\omega(t), f_\omega(t)) \tag{3}$$

$$R_1^{-1} = (g_\omega(t), f_\omega(t))^{-1} = (1/g_\omega(f_\omega^\leftarrow(t)), f_\omega^\leftarrow(t)) \tag{4}$$

$$(g_\omega(t), f_\omega(t)) = (g_\omega(t), t) * (1, f_\omega(t)) \tag{5}$$

Proof of (2) ω GF of the k -th column of R_2 :

$$G_\omega(t) \frac{(F_\omega(t))^k}{\omega_k}, \quad k = 0, 1, 2, \dots$$

Apply (1) to each k -th column, that is, regard this as a_ω , to get

$$g_\omega(t) G_\omega(f_\omega(t)) \frac{(F_\omega(f_\omega(t)))^k}{\omega_k},$$

and this is the ω GF of the k -th column of $R_1 * R_2$. \square

Riordan group structure (2) – (5) is independent of ω .

Riordan subgroup

1. Associated subgroup (Bell polynomials) $(1, f_\omega(t)) * (1, F_\omega(t)) = (1, F_\omega(f_\omega(t))),$
2. Appell subgroup $(g_\omega(t), t) * (G_\omega(t), t) = (g_\omega(t)G_\omega(t), t),$
3. Bell subgroup

$$(g_\omega(t), tg_\omega(t)) * (G_\omega(t), tG_\omega(t)) = (g_\omega(t)G_\omega(tg_\omega(t)), tg_\omega(t)G_\omega(tg_\omega(t))),$$

$$(f_\omega(t)/t, f_\omega(t)) * (F_\omega(t)/t, F_\omega(t)) = (F_\omega(f_\omega(t))/t, F_\omega(f_\omega(t))),$$

4. Checkerboard subgroup $(g_\omega(t), f_\omega(t))$: g_ω is odd, and f_ω is even.
5. Hitting time subgroup

$$(tf'_\omega(t)/f_\omega(t), f_\omega(t)) * (tF'_\omega(t)/F_\omega(t), F_\omega(t)) = (tf'_\omega(t)F'_\omega(f_\omega(t)), F_\omega(f_\omega(t))).$$

3 Ordinary and exponential Riordan arrays

Generating functions

$\omega,$	ω GF,	ω RA,
$\omega_n \equiv 1,$	$\text{oGF, "ordinary," } a(t) := \sum_{n \geq 0} a_n t^n,$	oRA,
$\omega_n = n!,$	$\text{eGF, "exponential," } \check{a}(t) := \sum_{n \geq 0} a_n \frac{t^n}{n!},$	eRA.

$(\rho_{n,k}; (n, k) \in \Lambda_0)$: a double triangular sequence,

$$\rho(t, u) = \sum_{n \geq k \geq 0} \rho_{n,k} t^n u^k, \quad \text{oGF, "ordinary",}$$

$$\check{\rho}(t, u) = \sum_{n \geq k \geq 0} \rho_{n,k} \frac{t^n u^k}{n!}, \quad \text{tGF, "triangular",}$$

Ordinary and exponential Riordan arrays, Definition

$g = (g_0, g_1, \dots), g_0 \neq 0,$	$\text{oGF } g(t), \quad \text{eGF } \check{g}(t).$
$f = (f_1, f_2, \dots), f_0 = 0, f_1 \neq 0,$	$\text{oGF } f(t), \quad \text{eGF } \check{f}(t).$

$\text{oGF: } R(t, u) = \frac{g(t)}{1 - uf(t)},$	$\text{tGF: } \check{R}(t, u) = \check{g}(t) \exp(u\check{f}(t)),$
$[u^k]R(t, u) = g(t)(f(t))^k, \quad k \in \mathbb{N}_0.$	$[u^k]\check{R}(t, u) = \check{g}(t) \frac{(\check{f}(t))^k}{k!}, \quad k \in \mathbb{N}_0.$

The elements of the array are determined by as follows, and the array is denoted by the pair of GF's.

$\text{oRA} \quad R_{n,k} = [t^n u^k]R(t, u) = [t^n]g(t)(f(t))^k,$	$\text{eRA} \quad \check{R}_{n,k} = \left[\frac{t^n u^k}{n!} \right] \check{R}(t, u) = \left[\frac{t^n}{n!} \right] \check{g}(t) \frac{(\check{f}(t))^k}{k!},$
$(R_{n,k})_{(n,k) \in \Lambda_0} =: (g(t), f(t)).$	$(\check{R}_{n,k})_{(n,k) \in \Lambda_0} =: (\check{g}(t), \check{f}(t)).$

The generating function of the n -th row ($n > 0$), Sheffer sequence, is

$$[t^n]R(t, u) = [t^n] \frac{g(t)}{1 - uf(t)} =: s_n(u). \quad [\frac{t^n}{n!}] \check{R}(t, u) = [\frac{t^n}{n!}] \check{g}(t) \exp(uf(t)) =: \check{s}_n(u).$$

Remarks

1. The Sheffer sequence is a polynomial on u of degree n .
2. $R(t, 1)$ (or $\check{R}(t, 1)$) is the oGF (or eGF) of the row-sums $s_n(u)$ (or $\check{s}_n(u)$).
3. Scale parametrization.

$$[t^n u^k] R(act, bu/c) = a^n b^k c^{n-k} R(t, u). \quad [t^n u^k] \check{R}(act, bu/c) = a^n b^k c^{n-k} \check{R}(t, u).$$

Relationship between ordinary and exponential Riordan arrays

Table 1: Riordan arrays, ordinary and exponential.

oRA					
$n \setminus m$	0	1	2	3	4
0	g_0	0	0	0	0
1	g_1	$f_1 g_0$	0	0	0
2	g_2	$f_1 g_1 + f_2 g_0$	$f_1^2 g_0$	0	0
3	g_3	$f_1 g_2 + f_2 g_1 + f_3 g_0$	$f_1^2 g_1 + 2f_1 f_2 g_0$	$f_1^3 g_0$	0
4	g_4	$f_1 g_3 + f_2 g_2 + f_3 g_1 + f_4 g_0$	$f_1^2 g_2 + 2f_1 f_2 g_1 + 2f_1 f_3 g_0 + f_2^2 g_0$	$f_1^3 g_1 + 3f_1^2 f_2 g_0$	$f_1^4 g_0$

eRA

$n \setminus m$	0	1	2	3	4
0	g_0	0	0	0	0
1	g_1	$f_1 g_0$	0	0	0
2	g_2	$2f_1 g_1 + f_2 g_0$	$f_1^2 g_0$	0	0
3	g_3	$3f_1 g_2 + 3f_2 g_1 + f_3 g_0$	$3f_1^2 g_1 + 3f_1 f_2 g_0$	$f_1^3 g_0$	0
4	g_4	$4f_1 g_3 + 6f_2 g_2 + 4f_3 g_1 + f_4 g_0$	$6f_1^2 g_2 + 12f_1 f_2 g_1 + 4f_1 f_3 g_0 + 3f_2^2 g_0$	$4f_1^3 g_1 + 6f_1^2 f_2 g_0$	$f_1^4 g_0$

Put $g_0 = 1$, $g_1 = g_2 = \dots = 0$, to obtain partial Bell Polynomials.

Relation 1 For a pair of sequences (g, f) , by definition, $(g(t), f(t)) = (R_{n,k})$, and $(\check{g}(t), \check{f}(t)) = (\check{R}_{n,k})$, See Table 1

Relation 2 For a suitable array $(\rho_{n,k}; (n, k) \in \Lambda_0)$

$$\rho_{n,k} = R_{n,k} = (g(t), f(t))_{n,k}, \quad \text{and} \quad \rho_{n,k} = \check{R}_{n,k} = (\check{g}(t), \check{f}(t))_{n,k},$$

Example 1

$$\begin{aligned} \rho_{n,k} &= \binom{n}{k}, \quad (g(t), f(t)) = \left(\frac{1}{1-t}, \frac{t}{1-t} \right), \quad \text{and} \quad (\check{g}(t), \check{f}(t)) = (e^t, t), \\ \rho_{n,k} &= [t^n u^k] \frac{1}{1-t(1+u)} = \left[\frac{t^n u^k}{n!} \right] \exp(t(1+u)) \end{aligned}$$

Example 2

$$\begin{aligned}\rho_{n,k} &= \mathbb{I}[(n, k) \in \Lambda_0], \quad (g(t), f(t)) = \left(\frac{1}{1-t}, t\right), \quad \text{and} \quad (\check{g}(t), \check{f}(t)) = (e^t, e^t - 1) \\ \rho_{n,k} &= [t^n u^k] \frac{1}{(1-t)(1-tu)} = \left[\frac{t^n u^k}{n!}\right] \exp(t + u(e^t - 1))\end{aligned}$$

Relation 3 For a pair of formal power series, $(\psi(t), \phi(t))$,

$$\begin{aligned}(\psi(t), \phi(t)) &= (g(t), f(t)) = (R_{n,k}), \quad \text{or} \quad (\psi(t), \phi(t)) = (\check{g}(t), \check{f}(t)) = (\check{R}_{n,k}), \\ R_{n,k} &= \frac{k!}{n!} \check{R}_{n,k}, \quad \text{or} \quad \check{R}_{n,k} = \frac{n!}{k!} R_{n,k}.\end{aligned}$$

Example 3

$$(\psi(t), \phi(t)) = \left(\frac{e^t - 1}{t}, \frac{t}{1-t}\right), \quad R_{n,k} = \frac{1}{(n-k+1)!}, \quad \text{and} \quad \check{R}_{n,k} = \frac{1}{n+1} \binom{n+1}{k},$$

See Table 5 for other examples.

Table 2: Summary of the relationship between oRA and eRA

	$(\psi(t), \phi(t))$	oGF/tGF	$\rho_{n,k}$
oRA	$(g(t), f(t))$	$R(t, u)$	$(R_{n,k})$
eRA	$(\check{g}(t), \check{f}(t))$	$\check{R}(t, u)$	$(\check{R}_{n,k})$

Notes on Table 2	φ	$[t^n]$	$[t^n/n!]$	$n!R_{n,k} = k!\check{R}_{n,k}.$
	$\varphi(t)$	φ_n	$n!\varphi_n$	
	$\check{\varphi}(t)$	$\varphi_n/n!$	φ_n	

4 Bell polynomials and compound distributions

ordinary and exponential Bell polynomials Partial ordinary and exponential Bell polynomials are special cases of ordinary and exponential Riordan arrays, respectively.

$$\begin{aligned}(R_{n,k})_{(n,k) \in \Lambda_0} &= (1, f(t)), & (\check{R}_{n,k})_{(n,k) \in \Lambda_0} &= (1, \check{f}(t)), \\ \text{or } R(t, u) &= \frac{1}{1-uf(t)}, & \text{or } \check{R}(t, u) &= \exp(u\check{f}(t)), \\ R_{n,k} &=: B_{n,k}(f), \quad (n, k) \in \Lambda_0, & \check{R}_{n,k} &=: \check{B}_{n,k}(f), \quad (n, k) \in \Lambda_0, \\ B_{n,k}(f) &= \sum_{s \in \mathcal{P}_{n,k}} k! \prod_{j=1}^n \frac{1}{s_j!} f_j^{s_j}. & \check{B}_{n,k}(f) &= \sum_{s \in \mathcal{P}_{n,k}} n! \prod_{j=1}^n \frac{1}{s_j!} \left(\frac{f_j}{j!}\right)^{s_j}. \\ (R_{n,k}) * (g_k) &= [t^n]g(f(t)), & (\check{R}_{n,k}) * (g_k) &= \left[\frac{t^n}{n!}\right] \check{g}(\check{f}(t)), \\ [t^n]R(t, u) &= \sum_{n \geq k \geq 1} u^k (f(t))^k =: Y_n(u), \quad n \in \mathbb{N}_0. & \left[\frac{t^n}{n!}\right] \check{R}(t, u) &= \sum_{n \geq k \geq 1} u^k (\check{f}(t))^k =: \check{Y}_n(u), \quad n \in \mathbb{N}_0. \\ Y_n(u+v) &= \sum_{n \geq k \geq 0} Y_k(u)Y_{n-k}(v), \quad Y_0(u) = 1. & \check{Y}_n(u+v) &= \sum_{n \geq k \geq 0} \binom{n}{k} \check{Y}_k(u)\check{Y}_{n-k}(v), \quad \check{Y}_0(u) = 1.\end{aligned}$$

- s denotes a partition of a number n into k terms.

$$\mathcal{P}_{n,k} = \{s; s = (s_1, s_2, \dots, s_n), \sum_{n \geq j \geq 1} s_j = k, \sum_{n \geq j \geq 1} js_j = n\}.$$

- $Y_n(u)$ is the oGF of the n -th row, and $\check{Y}_n(u)$ is the eGF of the n -th row, and they are the “total ordinary or exponential Bell polynomials”, and are Sheffer sequences.
- $Y_n(u)$ are “convolution polynomials”, and $\check{Y}_n(u)$ are “binomial type polynomials”.
- See Comtet (1974) for the two types of Bell polynomials.

return to oRA: $(g(t), f(t))$ **and eRA:** $(\check{g}(t), \check{f}(t))$

$$R_{n,k} = \sum_{n \geq k \geq 0} g_{n-k} B_{n,k}(f), \quad \check{R}_{n,k} = \sum_{n \geq k \geq 0} \binom{n}{k} g_{n-k} \check{B}_{n,k}(f).$$

5 Natural exponential family on lower triangle

NEF (power series) on Λ_0

$$\begin{aligned} R(t, u) &= \frac{g(t)}{1 - uf(t)}, \quad \omega_{n,k} = [t^n u^k] R(t, u) \geq 0, \quad \check{R}(t, u) = \check{g}(t) \exp(u\check{f}(t)), \quad \check{\omega}_{n,k} = \left[\frac{t^n u^k}{n!} \right] \check{R}(t, u) \geq 0, \\ P\{(X, Y) = (n, k)\} &= p_{n,k} = \frac{\omega_{n,k} \theta^n \eta^k}{\sum_{(n,k) \in \Lambda_0} \omega_{n,k} \theta^n \eta^k}, \quad P\{(X, Y) = (n, k)\} = \check{p}_{n,k} = \frac{\check{\omega}_{n,k} \frac{\theta^n \eta^k}{n!}}{\sum_{(n,k) \in \Lambda_0} \check{\omega}_{n,k} \frac{\theta^n \eta^k}{n!}}, \\ \text{pgf: } R(t\theta, u\eta) / R(\theta, \eta), &\quad \text{pgf: } \check{R}(t\theta, u\eta) / \check{R}(\theta, \eta). \\ P\{X = n\} &= \theta^n \sum_{n \geq k \geq 0} \omega_{n,k} \eta^k / R(\theta, \eta), \quad n \in \mathbb{N}_0, \quad P\{X = n\} = \frac{\theta^n}{n!} \sum_{n \geq k \geq 0} \check{\omega}_{n,k} \eta^k / \check{R}(\theta, \eta), \quad n \in \mathbb{N}_0, \\ \text{pgf: } R(t\theta, \eta) / R(\theta, \eta), &\quad \text{pgf: } \check{R}(t\theta, \eta) / \check{R}(\theta, \eta), \\ P\{Y = k\} &= \eta^k \sum_{n \geq k} \omega_{n,k} \theta^n / R(\theta, \eta), \quad k \in \mathbb{N}_0, \quad P\{Y = k\} = \eta^k \sum_{n \geq k} \check{\omega}_{n,k} \frac{\theta^n}{n!} / \check{R}(\theta, \eta), \quad k \in \mathbb{N}_0, \\ \text{pgf: } R(\theta, u\eta) / R(\theta, \eta), &\quad \text{pgf: } \check{R}(\theta, u\eta) / \check{R}(\theta, \eta), \\ P\{X - Y = k\} &= \zeta^k \sum_{n \geq k} \omega_{n,k} \theta^{n-k} / R(\theta, \eta), \quad P\{X - Y = k\} = \zeta^k \sum_{n \geq k} \frac{\check{\omega}_{n,k} \theta^{n-k}}{n!} / \check{R}(\theta, \eta), \\ \text{pgf: } R(\theta, u\zeta/\eta) / R(\theta, \zeta/\eta), &\quad \text{pgf: } \check{R}(\theta, u\zeta/\eta) / \check{R}(\theta, \zeta/\eta). \end{aligned}$$

In the last two lines $\zeta = \theta\eta$ and $k \in \mathbb{N}_0$.

Conditional distributions are similarly defined, for example

$$\begin{aligned} P\{Y = k | X = n\} &= \omega_{n,k} \eta^k / R(1, \eta), \quad n \geq k \geq 0, \quad P\{Y = k | X = n\} = \check{\omega}_{n,k} \eta^k / \check{R}(1, \eta), \quad n \geq k \geq 0. \\ \text{pgf: } R(1, u\eta) / R(1, \eta), &\quad \check{R}(1, u\eta) / \check{R}(1, \eta). \end{aligned}$$

6 Generalized Stirling numbers

Definition Generalized three-parameter Stirling numbers (G3SN), Hsu and Shiue (1998), is defined here based on the Relation 3, between oRA and eRA. The common pair of GF is

$$(\psi(t), \phi(t)) = \left((1 + \alpha t)^{\gamma/\alpha}, \frac{1}{\beta}((1 + \alpha t)^{\beta/\alpha} - 1) \right) =: \mathcal{S}(\alpha, \beta, \gamma), \quad \alpha\beta \neq 0.$$

Let $\beta \rightarrow 0$ and $\alpha \rightarrow 0$, and we obtain

$$\begin{aligned} \mathcal{S}(\alpha, 0, \gamma) &= \left((1 + \alpha t)^{\gamma/\alpha}, \frac{1}{\alpha} \log(1 + \alpha t) \right) = \mathcal{S}(\alpha, 0, 0) * \mathcal{S}(0, 0, \gamma), \quad \alpha \neq 0, \forall \gamma, \\ \mathcal{S}(0, \beta, \gamma) &= \left(e^{\gamma t}, \frac{1}{\beta} (e^{\beta t} - 1) \right) = \mathcal{S}(0, 0, \gamma) * \mathcal{S}(0, \beta, 0), \quad \beta \neq 0, \forall \gamma, \\ \mathcal{S}(0, 0, \gamma) &= (e^{\gamma t}, t), \quad \forall \gamma \neq 0. \\ \mathcal{S}(\alpha, \beta, \gamma)^{-1} &= \mathcal{S}(\beta, \alpha, -\gamma), \quad \forall \alpha, \beta, \gamma. \end{aligned}$$

Note that

$$\begin{aligned} [t^n] \psi(t) &= \binom{\gamma/\alpha}{n} \alpha^n \rightarrow \frac{\gamma^n}{n!}, (\alpha \rightarrow 0), \\ \left[\frac{t^n}{n!} \right] \psi(t) &= (\gamma|\alpha)_n \rightarrow \gamma^n, (\alpha \rightarrow 0), \\ [t^n] \phi(t) &= \binom{\beta/\alpha}{n} \frac{\alpha^n}{\beta} I[n > 0] \rightarrow \frac{(-\alpha)^{n-1}}{n} I[n > 0], (\beta \rightarrow 0), \\ \left[\frac{t^n}{n!} \right] \phi(t) &= (\beta - \alpha|\alpha)_{n-1} \rightarrow (-\alpha)^{n-1} (n-1)!, (\beta \rightarrow 0). \end{aligned}$$

By convention $(x)_{-1} = 0$.

Multiplication rules for the Bell polynomial case, $\gamma = 0$.

$$\begin{aligned} \mathcal{S}(\alpha, c, 0) * \mathcal{S}(c, \beta, 0) &= \mathcal{S}(\alpha, \beta, 0), \quad \forall \alpha, \beta, c, \quad (\text{including } c = 0), \\ \mathcal{S}(\alpha, \beta, 0) &= I, \quad \text{if } \alpha = \beta \neq 0, \\ \mathcal{S}(c, 0, 0) * \mathcal{S}(0, c, 0) &= I, \quad \text{if } c \neq 0. \\ \mathcal{S}(\alpha, c_1, 0) * \mathcal{S}(c_1, c_2, 0) * \mathcal{S}(c_2, \beta, 0) &= \mathcal{S}(\alpha, \beta, 0), \quad c_1 c_2 \neq 0, \end{aligned}$$

Multiplication rules for the general case, $\gamma \neq 0$.

$$\begin{aligned} \mathcal{S}(\alpha, c, \gamma_1) * \mathcal{S}(c, \beta, \gamma_2) &= \mathcal{S}(\alpha, \beta, \gamma_1 + \gamma_2), \forall \alpha, \beta, c, \gamma_1, \gamma_2, \\ \mathcal{S}(\alpha, 0, \gamma) * \mathcal{S}(0, \beta, 0) &= \mathcal{S}(\alpha, 0, 0) * \mathcal{S}(0, \beta, \gamma) = \mathcal{S}(\alpha, \beta, \gamma), \quad \alpha\beta \neq 0, \\ \mathcal{S}(\alpha, 0, 0) * \mathcal{S}(0, 0, \gamma) * \mathcal{S}(0, \beta, 0) &= \mathcal{S}(\alpha, \beta, \gamma), \quad \alpha\beta \neq 0. \\ \mathcal{S}(\alpha, c, 0) * \mathcal{S}(c, \beta, \gamma) &= \mathcal{S}(\alpha, \beta, \gamma), \quad \forall \alpha, \beta, c, \gamma, \\ \mathcal{S}(\alpha, c, \gamma) * \mathcal{S}(c, \beta, 0) &= \mathcal{S}(\alpha, \beta, \gamma), \quad \forall \alpha, \beta, c, \gamma. \\ \mathcal{S}(\alpha, 0, \gamma_1) * \mathcal{S}(0, \beta, \gamma_2) &= \mathcal{S}(\alpha, \beta, \gamma_1 + \gamma_2), \quad \forall \alpha, \beta, c, \gamma_1, \gamma_2, \\ \mathcal{S}(0, c, \gamma_1) * \mathcal{S}(c, 0, \gamma_2) &= \mathcal{S}(0, 0, \gamma_1 + \gamma_2), \quad \forall c, \gamma_1, \gamma_2. \end{aligned}$$

Table 3: G3SN arrays (eRA)

eRA $\check{S}_{n,k}(\alpha, \beta, \gamma)$	0	1	2	3	4
0	1	0	0	0	0
1	γ	1	0	0	0
2	$\gamma(\gamma - \alpha)$	$2\gamma - \alpha + \beta$	1	0	0
3	$(\gamma \alpha)_3$	$3\gamma^2 - 3\gamma(2\alpha - \beta) + (2\alpha - \beta)(\alpha - \beta)$	$3(\gamma - \alpha + \beta)$	1	0
4	$(\gamma \alpha)_4$	$(2\gamma - 3\alpha + \beta)(2\gamma^2 - 6\alpha\gamma + 2\beta\gamma + (2\alpha - \beta)(\alpha - \beta))$ $-2\gamma(3\alpha - \beta) + (2\alpha - \beta)(\alpha - \beta)$	$6\gamma^2 - 6\gamma(3\alpha - 2\beta)$ $+ (11\alpha - \beta)(\alpha - 7\beta)$	$4\gamma - 6\alpha + 6\beta$	1

$$\text{oRA: } S_{n,k}(\alpha, \beta, \gamma) = \frac{k!}{n!} \check{S}_{n,k}(\alpha, \beta, \gamma).$$

 Table 4: Basic properties of Stirling arrays, $\mathcal{S}(\alpha, \beta, \gamma)$.

Stirling pairs	oRA $S_{n,k}(\alpha, \beta, \gamma)$	eRA $\check{S}_{n,k}(\alpha, \beta, \gamma)$	note
$\mathcal{S}(\alpha, \beta, \gamma)$	$\frac{k!}{n!} \check{S}_{n,k}(\alpha, \beta, \gamma)$	$\frac{n!}{k!} S_{n,k}(\alpha, \beta, \gamma)$	$\forall \alpha, \beta, \gamma,$
$\mathcal{S}(\beta, \alpha, -\gamma)$	$(-1)^{n-k} S_{n,k}(\alpha, \beta, \gamma)$	$(-1)^{n-k} \check{S}_{n,k}(\alpha, \beta, \gamma)$	inverse of $\mathcal{S}(\alpha, \beta, \gamma)$
$\mathcal{S}(c\alpha, c\beta, c\gamma)$	$c^{n-k} S_{n,k}(\alpha, \beta, \gamma)$	$c^{n-k} \check{S}_{n,k}(\alpha, \beta, \gamma)$	
$\mathcal{S}(\alpha, \beta, \gamma) = ((1 + \alpha t)^{\gamma/\alpha}, \frac{1}{\beta}((1 + \alpha t)^{\beta/\alpha} - 1))$			
$\check{S}_{n,k}(\alpha, \beta, \gamma) = \sum_{n \geq m \geq l \geq k} \begin{bmatrix} n \\ m \end{bmatrix} (-\alpha)^{n-m} \begin{bmatrix} m \\ l \end{bmatrix} \gamma^{m-l} \begin{Bmatrix} l \\ k \end{Bmatrix} \beta^{l-k}.$			

Table 5: A list of G3SN arrays.

(α, β, γ)	$(\psi(t), \phi(t))$	oRA $[S_{n,k}(\alpha, \beta, \gamma)]$	eRA $[\check{S}_{n,k}(\alpha, \beta, \gamma)]$	note
$(1, -1, \gamma)$	$((1 + t)^\gamma, \frac{t}{1+t})$	$\binom{\gamma-k}{n-k}$	$\binom{n}{k} (\gamma - k)_{n-k}$	SS: $L_n^{(-\gamma-1)}(-x)$
$(1, -1, 1)$	$(1 + t, \frac{t}{1+t})$	$\binom{n-2}{k-2} (-1)^{n-k}, k \geq 2, \dagger$		
$(1, -1, -\gamma)$	$(\frac{1}{(1+t)^\gamma}, \frac{t}{1+t})$	$\binom{n+\gamma-1}{n-k} (-1)^{n-k}$	$\binom{n}{k} (\gamma + n - 1)_{n-k} (-1)^{n-k}$	SS: $L_n^{(\gamma-1)}(x)$
$(1, -1, -1)$	$(\frac{1}{1+t}, \frac{t}{1+t})$	$\binom{n}{k} (-1)^{n-k}$	$\frac{n!}{k!} \binom{n}{k} (-1)^{n-k}$	SS: $L_n^{(\gamma-1)}(-x)$
$(-1, 1, \gamma)$	$(\frac{1}{(1-t)^\gamma}, \frac{t}{1-t})$	$\binom{\gamma+n-1}{n-k}$	$\binom{n}{k} (\gamma + n - 1)_{n-k}$	SS: $L_n^{(\gamma-1)}(-x)$
$(-1, 1, 1)$	$(\frac{1}{1-t}, \frac{t}{1-t})$	$\binom{n}{k}$	$\frac{n!}{k!} \binom{n}{k}$	SS: $L_n^{(\gamma-1)}(x)$
$(-1, 1, -\gamma)$	$((1 - t)^\gamma, \frac{t}{1-t})$	$\binom{\gamma-m}{n-k} (-1)^{n-k}$	$\binom{n}{k} (\gamma - k)_{n-k} (-1)^{n-k}$	SS: $L_n^{(-\gamma-1)}(x)$
$(-1, 1, -1)$	$(1 - t, \frac{t}{1-t})$	$\binom{n-2}{k-2}, k \geq 2, \dagger$		
$(1, 1, \gamma)$	$((1 + t)^\gamma, t)$	$\binom{\gamma}{n-k}$	$\binom{n}{k} (\gamma)_{n-k}$	
$(1, 1, 1)$	$(1 + t, t)$	$I[k = n] + I[k = n - 1]$	$\frac{n!}{k!} (I[k = n] + I[k = n - 1])$	
$(1, 1, -\gamma)$	$(\frac{1}{(1+t)^\gamma}, t)$	$\binom{\gamma+n-k-1}{n-k} (-1)^{n-k}$	$\binom{n}{k} (\gamma + n - k - 1)_{n-k} (-1)^{n-k}$	
$(1, 1, -1)$	$(\frac{1}{1+t}, t)$	$(-1)^{n-k}$	$\frac{n!}{k!} (-1)^{n-k}$	
$(-1, -1, \gamma)$	$(\frac{1}{(1-t)^\gamma}, t)$	$\binom{-\gamma}{n-k} (-1)^{n-k} = \binom{\gamma+n-k-1}{n-k}$	$\binom{n}{k} (\gamma + n - k - 1)_{n-k}$	
$(-1, -1, 1)$	$(\frac{1}{1-t}, t)$	$I[n \geq k]$	$\frac{n!}{k!} I[n \geq k]$	
$(-1, -1, -\gamma)$	$((1 - t)^\gamma, t)$	$\binom{\gamma}{n-k} (-1)^{n-k}$	$\binom{n}{k} (\gamma)_{n-k} (-1)^{n-k}$	
$(-1, -1, -1)$	$(1 - t, t)$	$I[k = n] - I[k = n - 1]$	$\frac{n!}{k!} (I[k = n] + I[k = n - 1])$	
$(\alpha, \beta, 0)$	$(1, \frac{1}{\beta}((1 + \alpha t)^{\beta/\alpha} - 1))$	$B_{n,k}((\beta - \alpha \alpha)_{j-1})$	$\check{B}_{n,k}((\beta - \alpha \alpha)_{j-1})$	$\alpha \neq \beta,$
$(c, c, 0)$	$(1, t)$	$\frac{k!}{n!} I[k = n]$	$I[k = n]$	$\forall c \neq 0, \text{ identity}$
$(-c, c, 0)$	$(1, \frac{t}{1-ct})$	$\frac{n!}{(k-1)!} c^{n-k}$	$\frac{n!}{(k-1)!} c^{n-k}$	$\forall c, \text{ Lah numbers}$
$(\alpha, 0, 0)$	$(1, \frac{1}{\alpha} \log(1 + \alpha t))$	$\frac{k!}{n!} \begin{Bmatrix} n \\ k \end{Bmatrix} (-\alpha)^{n-k}$	$\begin{Bmatrix} n \\ k \end{Bmatrix} (-\alpha)^{n-k}$	Stirling num. 1st kind.
$(0, \beta, 0)$	$(1, \frac{1}{\beta}(e^{\beta t} - 1))$	$\frac{k!}{n!} \begin{Bmatrix} n \\ k \end{Bmatrix} \beta^{n-k}$	$\begin{Bmatrix} n \\ k \end{Bmatrix} \beta^{n-k}$	Stirling num. 2nd kind.
$(0, 0, \gamma)$	$(e^{\gamma t}, t)$	$\frac{1}{(n-k)!} \gamma^{n-k}$	$\begin{Bmatrix} n \\ k \end{Bmatrix} \gamma^{n-k}$	$\forall \gamma$

$$\text{oRA: } (\psi(t), \phi(t)) = (g(t), f(t)), \quad \text{eRA: } ((\psi(t), \phi(t)) = \check{g}(t), \check{f}(t)).$$

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