# Recurrence relations for connection coefficients between two families of orthogonal polynomials 

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#### Abstract

We describe a simple approach in order to build recursively the connection coefficients between two families of orthogonal polynomial solutions of second- and fourth-order differential equations.


Keywords: Orthogonal polynomials; Connection coefficients

## 1. The problem

Let us consider two families of orthogonal polynomials $\left\{P_{n}(x)\right\}_{n \in \mathbb{N}}$ and $\left\{Q_{n}(x)\right\}_{n \in \mathbb{N}}$ linked by the so-called [2] connection coefficients $C_{m}(n)$ given by the expansion of the $P_{n}$ family in terms of the $Q_{n}$ basis:

$$
\begin{equation*}
P_{n}(x)=\sum_{m=0}^{n} C_{m}(n) Q_{m}(x) . \tag{1}
\end{equation*}
$$

These connection coefficients play an important role in many situations of Pure and Applied Mathematics or in Mathematical Physics, where the nonnegative character of these coefficients has received particular attention [1, 2, 4, 9, 10].

The aim of this letter is to describe an elementary procedure in order to find recurrence relations, sometimes easy to solve, between the coefficients $C_{m}(n)$ when both $\left\{P_{n}(x)\right\}_{n \in \mathbb{N}}$ and $\left\{Q_{n}(x)\right\}_{n \in \mathbb{N}}$

[^0]families belong to the classical class of orthogonal polynomials: Jacobi (J), Bessel (B), Laguerre (L), Hermite (H). This gives an alternate way to be compared to a recent approach [5] restricted to Jacobi expansion. Moreover, extensions to nonclassical polynomials and some examples are given.

Let us write the recurrence relation, the structure relation and the second-order differential equation satisfied by the $P_{n}$ family in the following way:

$$
\begin{align*}
& x P_{n}=A_{n} P_{n+1}+B_{n} P_{n}+C_{n} P_{n-1} \quad\left(n \geqslant 0, P_{-1}=0 \text { and } P_{0}=1\right),  \tag{2}\\
& \sigma P_{n}^{\prime}=\alpha_{n} P_{n+1}+\beta_{n} P_{n}+\gamma_{n} P_{n-1} \quad\left(n \geqslant 0 \text { and } \alpha_{0}=\beta_{0}=\gamma_{0}=0\right),  \tag{3}\\
& L_{2}\left[P_{n}\right]:=\sigma P_{n}^{\prime \prime}+\tau P_{n}^{\prime}+\lambda_{n} P_{n}=0, \tag{4}
\end{align*}
$$

where the notation $P_{n}:=P_{n}(x), \sigma:=\sigma(x), \tau:=\tau(x)$ has been used. The coefficients $A_{n}, B_{n}, C_{n}, \alpha_{n}$, $\beta_{n}, \gamma_{n}$ are well known in the four classical cases [6]. Moreover, one has (see e.g. [6]): $\sigma_{\mathrm{J}}=1-x^{2}$, $\sigma_{\mathrm{B}}=x^{2}, \sigma_{\mathrm{L}}=x, \sigma_{\mathrm{H}}=1$ and the polynomial of first degree $\tau:=\tau(x)$ is given by $(\sigma \rho)^{\prime} / \rho$ where the weight $\rho:=\rho(x)$ is, respectively: $\rho_{\mathrm{J}}(x)=(1-x)^{\alpha}(1-x)^{\beta}(\alpha>-1, \beta>-1), \rho_{\mathrm{B}}(x)=x^{\alpha} \mathrm{e}^{-2 / x}$ $(\alpha \neq-2,-3, \ldots), \rho_{\mathrm{L}}(x)=x^{\alpha} \mathrm{e}^{-x}(\alpha>-1), \rho_{\mathrm{H}}(x)=\mathrm{e}^{-x^{2}}$. Finally, the eigenvalue $\lambda_{n}$ is computed from $2 \lambda_{n}=-n\left[2 \tau^{\prime}+(n-1) \sigma^{\prime \prime}\right]$.

The $\left\{Q_{n}(x)\right\}_{n \in \mathbb{N}}$ family, also classical, verify the relations (2), (3) and (4), called ( $\overline{2}$ ), ( $\overline{3}$ ) and ( $\overline{4}$ ) where the corresponding constants or polynomials will be denoted with an upper bar: $\left(\bar{A}_{n}, \bar{B}_{n}, \bar{C}_{n}\right)$, $\left(\bar{\alpha}_{n}, \bar{\beta}_{n}, \bar{\gamma}_{n}\right),\left(\bar{\sigma}, \bar{\tau}, \bar{\lambda}_{n}\right)$.

The link between the $P_{n}$ and the $Q_{m}$ given by (1) can easily be replaced by a linear relation involving only the $Q_{m}$ using (4):

$$
\begin{equation*}
\sigma \sum_{m=0}^{n}\left[C_{m}(n) Q_{m}^{\prime \prime}\right]+\tau \sum_{m=0}^{n}\left[C_{m}(n) Q_{m}^{\prime}\right]+\lambda_{n} \sum_{m=0}^{n}\left[C_{m}(n) Q_{m}\right]=0 . \tag{5}
\end{equation*}
$$

Notice that from (5) a linear system can be readily obtained by equating to zero the coefficients of $x^{k}, k=0,1, \ldots, n$. However, the coefficient of $x^{n}$ is

$$
C_{n}(n)\left\{\frac{n(n-1)}{2} \sigma^{\prime \prime}+n \tau^{\prime}+\lambda_{n}\right\}
$$

which is identically zero $(\equiv 0)$ (see the aforementioned expression for the eigenvalue $\lambda_{n}$ ). So, this linear system has $n+1$ unknowns (the $C_{m}(n)$ coefficients) and only $n$ equations. Of course, an equivalent linear system could also be obtained by expanding the polynomials $\sigma Q_{m}^{\prime \prime}(m=2, \ldots, n)$ and $\tau Q_{m}^{\prime}(m=1, \ldots, n)$ in the $Q_{m}$ basis, which should also have $n+1$ unknowns and only $n$ linearly independent equations at most.

However, these two linear systems are not very useful because they strongly depend on the coefficients of the polynomials $Q_{m}(m=0, \ldots, n)$. Instead of using them, let us proceed as follows:

Multiplication of (5) by $\bar{\sigma}$ and use of ( $\overline{4}$ ) and ( $\overline{3}$ ) gives

$$
\begin{align*}
& \sum_{m=0}^{n} C_{m}(n) \sigma\left[-\bar{\tau} Q_{m}^{\prime}-\bar{\lambda}_{m} Q_{m}\right] \\
& \quad+\sum_{m=0}^{n} C_{m}(n) \tau\left[\bar{\alpha}_{m} Q_{m+1}+\bar{\beta}_{m} Q_{m}+\bar{\gamma}_{m} Q_{m-1}\right]+\lambda_{n}\left[\sum_{m=0}^{n} C_{m}(n) \bar{\sigma} Q_{m}\right]=0 \tag{6}
\end{align*}
$$

where, as pointed out in (2), $Q_{-1}=0$. Since multiplication of (5) by a polynomial does not increase the number of linearly independent equations (at most $n$ in (5)), it turns out that (6) gives rise to a linear system which contains at most $n$ linearly independent equations.

Next, multiplication of (6) by $\bar{\sigma}$ together with the use of ( $\overline{3}$ ) allows to eliminate in (6) the term depending on $Q_{m}^{\prime}$. The last step consists to expand the remaining terms of type $\sigma \bar{\tau} Q_{k}$ $(k=1, \ldots, n+1), \bar{\sigma}^{2} Q_{k}(k=0, \ldots, n), \sigma \bar{\sigma} Q_{k}(k=2, \ldots, n)$ and $\tau \bar{\sigma} Q_{k}(k=0,1, \ldots, n+1)$ in linear combination of $Q_{m}$ (with constants coefficients) by using ( $\overline{2}$ ) repetitively.

After this process (6) reduces to

$$
\begin{equation*}
\sum_{m=0}^{N} M_{m}\left[C_{0}(n), \ldots, C_{n}(n)\right] Q_{m}(x)=0 \tag{7}
\end{equation*}
$$

where the notation

$$
\begin{aligned}
N=\max \{ & n+\operatorname{deg}(\sigma)+\operatorname{deg}(\bar{\sigma}), n+2 \operatorname{deg}(\bar{\sigma}) \\
& n+1+\operatorname{deg}(\tau)+\operatorname{deg}(\bar{\sigma}), n+1+\operatorname{deg}(\sigma)+\operatorname{deg}(\bar{\tau})\} \quad(N \geqslant n+1)
\end{aligned}
$$

has been used.
Finally, from (7) we deduce the linear system we are looking for:

$$
\begin{equation*}
M_{m}\left[C_{0}(n), \ldots, C_{n}(n)\right]=0, \quad m=0,1, \ldots, N \tag{8}
\end{equation*}
$$

It is easy to prove that the range of its coefficient matrix is greater than or equal to $n$. On the other other hand, since (7) is obtained from (6) by multiplying the latter equation by a polynomial, the number of linearly independent equations in (8) has to be the same as in (6), i.e. $n$ at most. So, choosing in (8) $n$ equations, the connection coefficients $C_{m}(n)$ can be obtained in terms of one of them which is arbitrary. Clearly, the remaining equations $(N-n>0)$ are then satisfied identically.

The relations (8) contains (linearly) several $C_{i}(n)$ depending essentially on the degrees of $\sigma$ and $\bar{\sigma}$. In the most complicated situations: connection between two Jacobi, between Jacobi and Bessel or between Bessel and Jacobi, $\sigma$ and $\bar{\sigma}$ are polynomials of second degree and the structure relation (3) contains effectively three terms; so expansions of type $\bar{\sigma}^{2} Q_{m}$ mixes already $Q_{m+4}, Q_{m+3}, \ldots, Q_{m-4}$. In these three situations the relations (6) link (in general and for $n \geqslant 8$ ) from $Q_{m+4}$ to $Q_{m-4}$ which means that the collecting polynomials $Q_{m}$ in (7) give a relation of type (maximum)

$$
\begin{equation*}
M_{m}\left[C_{m+4}, \ldots, C_{m-4}\right]=0 \tag{9}
\end{equation*}
$$

which is valid for $n$ greater than or equal to the number of initial conditions needed to start the recursion ( $n \geqslant 8$ in this case). Notice that when $n$ is less than the number of initial conditions, system (8) also gives the solution, but not in a recurrent way.

The starting values of the recurrence (9) can be computed from the linear system (8). Choosing $C_{n}(n)$ as the arbitrary coefficient, the simplest way is to consider as initial conditions $C_{n}(n), C_{n-1}(n), \ldots, C_{n-7}(n)$, where $C_{i}(n)(i=n-7, \ldots, n-1)$ are determined from (8) in terms of $C_{n}(n)$. This is always possible due to the lower triangular structure of the corresponding coefficients matrix. Then, $C_{n}(n)$, which only depends on the relative normalization of the two families $\left\{P_{n}(x)\right\}_{n \in \mathbb{N}}$ and $\left\{Q_{n}(x)\right\}_{n \in \mathbb{N}}$, can be easily obtained by identification of the highest power in the expansion (1).

## 2. Extensions

The approach we have just described can also be applied in more general situations. For semi-classical orthogonal $P_{n}$ and $Q_{n}$, (3) should be replaced by

$$
\begin{equation*}
\sigma P_{n}^{\prime}=\sum_{k=n-s-1}^{n+t-1} \beta_{k, n} P_{k} \tag{10}
\end{equation*}
$$

and $(\overline{3})$ by the analogous formula $(\overline{10})$. In the latter equation, $t$ is the degree (which can be arbitrary in the semi-classical class) of the polynomial $\sigma:=\sigma(x)$ and $s$ is an integer characterizing the class. Moreover, (4) and ( $\overline{4}$ ) are still valid for the semi-classical families except that the coefficients of $P_{n}^{\prime \prime}, P_{n}^{\prime}$ and $P_{n}$ (respectively $Q_{n}$ ) are now polynomials of fixed degree but with coefficients depending on $n$.

If $P_{n}$ belongs to the Laguerre $-H a h n$ class and $Q_{n}$ is semi-classical this technique also works. In this case, $P_{n}$ is solution of a fourth-order differential equation replacing (4). But to save computations it is possible to use equivalently a differential relation [7] which is very simple for instance for the associated polynomials $P_{n-1}^{(1)}(x)$ of the $P_{n}(x)$ ( $P_{n}$ being classical):

$$
L_{2}^{*}\left[P_{n-1}^{(1)}(x)\right]=\left(\sigma^{\prime \prime}-2 \tau^{\prime}\right) P_{n}^{\prime}(x)
$$

with $L_{2}^{*}$ the adjoint operator of $L_{2}$ :

$$
L_{2}^{*}[z(x)]=[\sigma z(x)]^{\prime \prime}-[\tau z(x)]^{\prime}+\lambda_{n} z(x)
$$

and

$$
\begin{equation*}
P_{n-1}^{(1)}(x)=\frac{1}{c_{0}} \int_{I} \frac{P_{n}(x)-P_{n}(s)}{x-s} \rho(s) \mathrm{d} s \quad \text { with } c_{0}=\int_{I} \rho(s) \mathrm{d} s \tag{11}
\end{equation*}
$$

$I$ being the orthogonality interval (the unit circle in the Bessel case).

## 3. Examples

These techniques are very easily implemented in any symbolic language in both cases: semiclassical in semi-classical and Laguerre-Hahn is semi-classical. Just as an example we derive, using Mathematica [11], the recurrence relations for the connection coefficients in the expansions:

$$
P_{n-1}^{(1)}(x)=\sum_{m=0}^{n-1} C_{m}(n-1) P_{m}(x)
$$

where the monic $P_{m}$ belongs to any classical family, taking care of the initial conditions matching properly the polynomials for $m=n-1$. For instance, from the definition (11), keeping $P_{n-1}^{(1)}(x)$ also monic, we obtain immediately that $C_{n-1}(n-1)=1$.

Now we give the results for the Hermite case which coincides with a known result [3] and for the Laguerre and Bessel ( $\alpha=0$ ) cases, which seems to be new. These coefficients $C_{m}(n-1)$ are denoted in the Mathematica algorithms by CA [m,n-1] (for connection associated).

### 3.1. Connection coefficients between first associated Hermite and Hermite

Let us consider the expansion for monic polynomials:

$$
H_{n-1}^{(1)}(x)=\sum_{m=0}^{n-1} \mathbf{C A}[m,-1+n] H_{m}(x)
$$

Then, by using the technique described above, the recurrence relation for the connection coefficients CA $[m, n-1]$ is a two-term one given by

$$
2(1+m+n) \mathbf{C A}[m,-1+n]+2(1+m)(2+m) \mathbf{C A}[2+m,-1+n]=0
$$

with $m=n-3, n-4, \ldots, 2,1,0$ and the initial conditions

$$
\begin{aligned}
& \mathbf{C A}[-1+n,-1+n]=1, \\
& \mathbf{C A}[-2+n,-1+n]=0 .
\end{aligned}
$$

As pointed out in the paragraph after (9), this recursion is valid only when $n \geqslant 2$.

### 3.2. Connection coefficients between first associated Laguerre and Laguerre

Let us consider the expansion for monic polynomials:

$$
\left[L_{n-1}^{(a)}\right]^{(1)}(x)=\sum_{m=0}^{n-1} \mathrm{CA}[m,-1+n] L_{m}^{(a)}(x)
$$

Then, by using the technique described above, the recurrence relation for connection coefficients $\mathbf{C A}[m, n-1]$ becomes in this case a four-term one given by

$$
\begin{aligned}
(m & +n) \mathbf{C A}[-1+m,-1+n] \\
& +\left(1+a+3 m+a m+4 m^{2}+n+a n+2 m n\right) \mathbf{C A}[m,-1+n] \\
& +(1+m)(1+a+m)(4+5 m+n) \mathbf{C A}[1+m,-1+n] \\
& +2(1+m)(2+m)(1+a+m)(2+a+m) \mathbf{C A}[2+m,-1+n]=0
\end{aligned}
$$

where $m=n-3, n-4, \ldots, 2,1$ and the initial conditions are
$\mathbf{C A}[-3+n,-1+n]=8+2 a-10 n-a n+3 n^{2}$,
$\mathbf{C A}[-2+n,-1+n]=2-2 n$,
$\mathbf{C A}[-1+n,-1+n]=1$.
As pointed out in the paragraph after (9), this recursion is valid only when $n \geqslant 3$.

### 3.3. Connection coefficients between first associated Bessel and Bessel

Let us consider the expansion for monic polynomials:

$$
\left[B_{n-1}^{(0)}\right]^{(1)}(x)=\sum_{m=0}^{n-1} \mathrm{CA}[m,-1+n] B_{m}^{(0)}(x)
$$

Then, by using the technique described above, the recurrence relation for connection coefficients $\mathbf{C A}[m, n-1]$ is a five-term one given by

$$
\begin{aligned}
& (-2+m-n)(-1+m+n) \mathbf{C A}[-2+m,-1+n] \\
& \quad+4(1-m) \mathbf{C A}[-1+m,-1+n] \\
& \quad+\frac{2\left(-6+7 m+7 m^{2}+n+n^{2}\right)}{(-1+2 m)(3+2 m)} \mathbf{C A}[m,-1+n] \\
& \quad-\frac{4(2+m)}{(1+2 m)(3+2 m)} \mathbf{C A}[1+m,-1+n] \\
& \quad+\frac{(2+m-n)(3+m+n)}{(1+2 m)(3+2 m)^{2}(5+2 m)} \mathbf{C A}[2+m,-1+n]=0
\end{aligned}
$$

where $m=n-3, n-4, \ldots, 2$ and the initial conditions are

$$
\begin{aligned}
& \mathbf{C A}[-4+n,-1+n]=\frac{-11+12 n-4 n^{2}}{3(-5+2 n)(-1+2 n)} \\
& \mathbf{C A}[-3+n,-1+n]=\frac{9-16 n+8 n^{2}}{3(-3+2 n)(-1+2 n)}
\end{aligned}
$$

$$
\mathbf{C A}[-2+n,-1+n]=-1
$$

$$
\mathbf{C A}[-1+n,-1+n]=1
$$

As pointed out in the paragraph after (9), this recursion is valid only when $n \geqslant 4$.
Remark. In a recent work [8] we apply a similar technique to the linearization problems:

$$
P_{i}(x) P_{j}(x)=\sum_{k=0}^{i+j} L_{i, j, k} P_{k}(x)
$$

in which we examine the complexity of the recurrence relation between linearization coefficients as function of orthogonality restrictions. For the connection coefficients the three levels of assumptions are: without orthogonality in the $\left\{P_{n}(x)\right\}_{n \in \mathbb{N}}$ and $\left\{Q_{n}(x)\right\}_{n \in \mathbb{N}}$ families (arbitrary basis), the coefficients $C_{m}(n)$ are of course also arbitrary; with orthogonality in each family $\left\{P_{n}(x)\right\}_{n \in \mathbb{N}}$ and $\left\{Q_{n}(x)\right\}_{n \in \mathbb{N}}$, the recurrence relations for the $C_{m}(n)$ mixed both indices (from both recurrence relations satisfied by the $P_{n}$ and $Q_{n}$ families after multiplication of (1) by $x$ ); with semi-classical (classical) assumption, the recurrence relations for the $C_{m}(n)$ keeps the same argument $n$ (from the existence of the structure relation (3)).

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