

Journal of Computational and Applied Mathematics 62 (1995) 67-73

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Recurrence relations for connection coefficients between two families of orthogonal polynomials

A. Ronveaux^{a, *}, A. Zarzo^b, E. Godoy^c

^a Mathematical Physics, Facultés Universitaires Notre-Dame de la Paix, rue de Bruxelles 61, B-5000 Namur, Belgium ^b Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad Politécnica de Madrid, 28006 Madrid, Spain

^c Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad de Vigo, Apartado de correos 62, 36280 Vigo, Spain

Received 25 October 1993; revised 9 May 1994

Abstract

We describe a simple approach in order to build recursively the connection coefficients between two families of orthogonal polynomial solutions of second- and fourth-order differential equations.

Keywords: Orthogonal polynomials; Connection coefficients

1. The problem

Let us consider two families of orthogonal polynomials $\{P_n(x)\}_{n \in \mathbb{N}}$ and $\{Q_n(x)\}_{n \in \mathbb{N}}$ linked by the so-called [2] connection coefficients $C_m(n)$ given by the expansion of the P_n family in terms of the Q_n basis:

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x).$$
(1)

These connection coefficients play an important role in many situations of Pure and Applied Mathematics or in Mathematical Physics, where the nonnegative character of these coefficients has received particular attention [1, 2, 4, 9, 10].

The aim of this letter is to describe an elementary procedure in order to find recurrence relations, sometimes easy to solve, between the coefficients $C_m(n)$ when both $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$

* Corresponding author.

^{0377-0427/95/\$09.50 © 1995} Elsevier Science B.V. All rights reserved SSDI 0377-0427(94)00079-4

families belong to the classical class of orthogonal polynomials: Jacobi (J), Bessel (B), Laguerre (L), Hermite (H). This gives an alternate way to be compared to a recent approach [5] restricted to Jacobi expansion. Moreover, extensions to nonclassical polynomials and some examples are given.

Let us write the recurrence relation, the structure relation and the second-order differential equation satisfied by the P_n family in the following way:

$$xP_n = A_n P_{n+1} + B_n P_n + C_n P_{n-1} \quad (n \ge 0, P_{-1} = 0 \text{ and } P_0 = 1),$$
(2)

$$\sigma P'_n = \alpha_n P_{n+1} + \beta_n P_n + \gamma_n P_{n-1} \quad (n \ge 0 \text{ and } \alpha_0 = \beta_0 = \gamma_0 = 0), \tag{3}$$

$$L_2[P_n] := \sigma P_n'' + \tau P_n' + \lambda_n P_n = 0, \tag{4}$$

where the notation $P_n := P_n(x)$, $\sigma := \sigma(x)$, $\tau := \tau(x)$ has been used. The coefficients A_n , B_n , C_n , α_n , β_n , γ_n are well known in the four classical cases [6]. Moreover, one has (see e.g. [6]): $\sigma_J = 1 - x^2$, $\sigma_B = x^2$, $\sigma_L = x$, $\sigma_H = 1$ and the polynomial of first degree $\tau := \tau(x)$ is given by $(\sigma\rho)'/\rho$ where the weight $\rho := \rho(x)$ is, respectively: $\rho_J(x) = (1 - x)^{\alpha}(1 - x)^{\beta}$ ($\alpha > -1$, $\beta > -1$), $\rho_B(x) = x^{\alpha}e^{-2/x}$ ($\alpha \neq -2, -3, ...$), $\rho_L(x) = x^{\alpha}e^{-x}$ ($\alpha > -1$), $\rho_H(x) = e^{-x^2}$. Finally, the eigenvalue λ_n is computed from $2\lambda_n = -n[2\tau' + (n-1)\sigma'']$.

The $\{Q_n(x)\}_{n\in\mathbb{N}}$ family, also classical, verify the relations (2), (3) and (4), called ($\overline{2}$), ($\overline{3}$) and ($\overline{4}$) where the corresponding constants or polynomials will be denoted with an upper bar: $(\overline{A}_n, \overline{B}_n, \overline{C}_n)$, $(\overline{\alpha}_n, \overline{\beta}_n, \overline{\gamma}_n)$, $(\overline{\sigma}, \overline{\tau}, \overline{\lambda}_n)$.

The link between the P_n and the Q_m given by (1) can easily be replaced by a linear relation involving only the Q_m using (4):

$$\sigma \sum_{m=0}^{n} \left[C_m(n) Q_m'' \right] + \tau \sum_{m=0}^{n} \left[C_m(n) Q_m' \right] + \lambda_n \sum_{m=0}^{n} \left[C_m(n) Q_m \right] = 0.$$
(5)

Notice that from (5) a linear system can be readily obtained by equating to zero the coefficients of x^k , k = 0, 1, ..., n. However, the coefficient of x^n is

$$C_n(n)\left\{\frac{n(n-1)}{2}\,\sigma''+n\tau'+\lambda_n\right\}$$

which is identically zero ($\equiv 0$) (see the aforementioned expression for the eigenvalue λ_n). So, this linear system has n + 1 unknowns (the $C_m(n)$ coefficients) and only *n* equations. Of course, an equivalent linear system could also be obtained by expanding the polynomials $\sigma Q''_m$ (m = 2, ..., n) and $\tau Q'_m$ (m = 1, ..., n) in the Q_m basis, which should also have n + 1 unknowns and only *n* linearly independent equations at most.

However, these two linear systems are not very useful because they strongly depend on the coefficients of the polynomials Q_m (m = 0, ..., n). Instead of using them, let us proceed as follows: Multiplication of (5) by $\bar{\sigma}$ and use of (4) and (3) gives

$$\sum_{m=0}^{n} C_{m}(n)\sigma\left[-\bar{\tau}Q'_{m}-\bar{\lambda}_{m}Q_{m}\right] + \sum_{m=0}^{n} C_{m}(n)\tau\left[\bar{\alpha}_{m}Q_{m+1}+\bar{\beta}_{m}Q_{m}+\bar{\gamma}_{m}Q_{m-1}\right] + \lambda_{n}\left[\sum_{m=0}^{n} C_{m}(n)\bar{\sigma}Q_{m}\right] = 0$$
(6)

68

where, as pointed out in (2), $Q_{-1} = 0$. Since multiplication of (5) by a polynomial does not increase the number of linearly independent equations (at most n in (5)), it turns out that (6) gives rise to a linear system which contains at most n linearly independent equations.

Next, multiplication of (6) by $\bar{\sigma}$ together with the use of (3) allows to eliminate in (6) the term depending on Q'_m . The last step consists to expand the remaining terms of type $\sigma \bar{\tau} Q_k$ (k = 1, ..., n + 1), $\bar{\sigma}^2 Q_k$ (k = 0, ..., n), $\sigma \bar{\sigma} Q_k$ (k = 2, ..., n) and $\tau \bar{\sigma} Q_k$ (k = 0, 1, ..., n + 1) in linear combination of Q_m (with constants coefficients) by using (2) repetitively.

After this process (6) reduces to

$$\sum_{m=0}^{N} M_m [C_0(n), \dots, C_n(n)] Q_m(x) = 0$$
⁽⁷⁾

where the notation

$$N = \max \{ n + \deg(\sigma) + \deg(\bar{\sigma}), n + 2\deg(\bar{\sigma}), n + 1 + \deg(\sigma) + \deg(\bar{\tau}) \} (N \ge n + 1)$$
$$(N \ge n + 1)$$

has been used.

Finally, from (7) we deduce the linear system we are looking for:

$$M_m[C_0(n), \dots, C_n(n)] = 0, \quad m = 0, 1, \dots, N.$$
(8)

It is easy to prove that the range of its coefficient matrix is greater than or equal to n. On the other other hand, since (7) is obtained from (6) by multiplying the latter equation by a polynomial, the number of linearly independent equations in (8) has to be the same as in (6), i.e. n at most. So, choosing in (8) n equations, the connection coefficients $C_m(n)$ can be obtained in terms of one of them which is arbitrary. Clearly, the remaining equations (N - n > 0) are then satisfied identically.

The relations (8) contains (linearly) several $C_i(n)$ depending essentially on the degrees of σ and $\bar{\sigma}$. In the most complicated situations: connection between two Jacobi, between Jacobi and Bessel or between Bessel and Jacobi, σ and $\bar{\sigma}$ are polynomials of second degree and the structure relation (3) contains effectively three terms; so expansions of type $\bar{\sigma}^2 Q_m$ mixes already $Q_{m+4}, Q_{m+3}, \dots, Q_{m-4}$. In these three situations the relations (6) link (in general and for $n \ge 8$) from Q_{m+4} to Q_{m-4} which means that the collecting polynomials Q_m in (7) give a relation of type (maximum)

$$M_m[C_{m+4}, \dots, C_{m-4}] = 0 \tag{9}$$

which is valid for *n* greater than or equal to the number of initial conditions needed to start the recursion $(n \ge 8$ in this case). Notice that when *n* is less than the number of initial conditions, system (8) also gives the solution, but not in a recurrent way.

The starting values of the recurrence (9) can be computed from the linear system (8). Choosing $C_n(n)$ as the arbitrary coefficient, the simplest way is to consider as initial conditions $C_n(n), C_{n-1}(n), \ldots, C_{n-7}(n)$, where $C_i(n)$ $(i = n - 7, \ldots, n - 1)$ are determined from (8) in terms of $C_n(n)$. This is always possible due to the lower triangular structure of the corresponding coefficients matrix. Then, $C_n(n)$, which only depends on the relative normalization of the two families $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$, can be easily obtained by identification of the highest power in the expansion (1).

2. Extensions

The approach we have just described can also be applied in more general situations. For semi-classical orthogonal P_n and Q_n , (3) should be replaced by

$$\sigma P'_n = \sum_{k=n-s-1}^{n+t-1} \beta_{k,n} P_k$$
(10)

and $(\overline{3})$ by the analogous formula $(\overline{10})$. In the latter equation, t is the degree (which can be arbitrary in the semi-classical class) of the polynomial $\sigma := \sigma(x)$ and s is an integer characterizing the class. Moreover, (4) and ($\overline{4}$) are still valid for the semi-classical families except that the coefficients of P''_n , P'_n and P_n (respectively Q_n) are now polynomials of fixed degree but with coefficients depending on n.

If P_n belongs to the Laguerre-Hahn class and Q_n is semi-classical this technique also works. In this case, P_n is solution of a fourth-order differential equation replacing (4). But to save computations it is possible to use equivalently a differential relation [7] which is very simple for instance for the associated polynomials $P_{n-1}^{(1)}(x)$ of the $P_n(x)$ (P_n being classical):

 $L_{2}^{*}[P_{n-1}^{(1)}(x)] = (\sigma'' - 2\tau')P_{n}'(x)$

with L_2^* the adjoint operator of L_2 :

$$L_2^*[z(x)] = [\sigma z(x)]'' - [\tau z(x)]' + \lambda_n z(x)$$

and

$$P_{n-1}^{(1)}(x) = \frac{1}{c_0} \int_I \frac{P_n(x) - P_n(s)}{x - s} \rho(s) \, \mathrm{d}s \quad \text{with } c_0 = \int_I \rho(s) \, \mathrm{d}s, \tag{11}$$

I being the orthogonality interval (the unit circle in the Bessel case).

3. Examples

These techniques are very easily implemented in any symbolic language in both cases: semiclassical in semi-classical and Laguerre–Hahn is semi-classical. Just as an example we derive, using Mathematica [11], the recurrence relations for the connection coefficients in the expansions:

$$P_{n-1}^{(1)}(x) = \sum_{m=0}^{n-1} C_m(n-1) P_m(x)$$

where the monic P_m belongs to any classical family, taking care of the initial conditions matching properly the polynomials for m = n - 1. For instance, from the definition (11), keeping $P_{n-1}^{(1)}(x)$ also monic, we obtain immediately that $C_{n-1}(n-1) = 1$.

Now we give the results for the Hermite case which coincides with a known result [3] and for the Laguerre and Bessel ($\alpha = 0$) cases, which seems to be new. These coefficients $C_m(n-1)$ are denoted in the Mathematica algorithms by CA[m, n-1] (for connection associated).

70

3.1. Connection coefficients between first associated Hermite and Hermite

Let us consider the expansion for monic polynomials:

$$H_{n-1}^{(1)}(x) = \sum_{m=0}^{n-1} \mathbf{CA}[m, -1 + n] H_m(x)$$

Then, by using the technique described above, the recurrence relation for the connection coefficients CA[m, n-1] is a two-term one given by

$$2(1 + m + n)CA[m, -1 + n] + 2(1 + m)(2 + m)CA[2 + m, -1 + n] = 0$$

with m = n - 3, n - 4, ..., 2, 1, 0 and the initial conditions

$$CA[-1 + n, -1 + n] = 1,$$

$$CA[-2 + n, -1 + n] = 0.$$

As pointed out in the paragraph after (9), this recursion is valid only when $n \ge 2$.

3.2. Connection coefficients between first associated Laguerre and Laguerre

Let us consider the expansion for monic polynomials:

$$[L_{n-1}^{(a)}]^{(1)}(x) = \sum_{m=0}^{n-1} \mathbf{CA}[m, -1 + n] L_m^{(a)}(x).$$

Then, by using the technique described above, the recurrence relation for connection coefficients CA[m, n-1] becomes in this case a four-term one given by

$$(m + n)\mathbf{CA}[-1 + m, -1 + n]$$

+ (1 + a + 3m + am + 4m² + n + an + 2mn)\mathbf{CA}[m, -1 + n]
+ (1 + m)(1 + a + m)(4 + 5m + n)\mathbf{CA}[1 + m, -1 + n]
+ 2(1 + m)(2 + m)(1 + a + m)(2 + a + m)\mathbf{CA}[2 + m, -1 + n] = 0,

where m = n - 3, n - 4, ..., 2, 1 and the initial conditions are

$$CA[-3 + n, -1 + n] = 8 + 2a - 10n - an + 3n^{2},$$

 $CA[-2 + n, -1 + n] = 2 - 2n,$
 $CA[-1 + n, -1 + n] = 1.$

As pointed out in the paragraph after (9), this recursion is valid only when $n \ge 3$.

3.3. Connection coefficients between first associated Bessel and Bessel

Let us consider the expansion for monic polynomials:

$$[B_{n-1}^{(0)}]^{(1)}(x) = \sum_{m=0}^{n-1} \mathbf{CA}[m, -1 + n] B_m^{(0)}(x).$$

Then, by using the technique described above, the recurrence relation for connection coefficients CA[m, n-1] is a five-term one given by

$$(-2 + m - n)(-1 + m + n)\mathbf{CA}[-2 + m, -1 + n] + 4(1 - m)\mathbf{CA}[-1 + m, -1 + n] + \frac{2(-6 + 7m + 7m^2 + n + n^2)}{(-1 + 2m)(3 + 2m)}\mathbf{CA}[m, -1 + n] - \frac{4(2 + m)}{(1 + 2m)(3 + 2m)}\mathbf{CA}[1 + m, -1 + n] + \frac{(2 + m - n)(3 + m + n)}{(1 + 2m)(3 + 2m)^2(5 + 2m)}\mathbf{CA}[2 + m, -1 + n] = 0,$$

where m = n - 3, n - 4, ..., 2 and the initial conditions are

$$CA[-4 + n, -1 + n] = \frac{-11 + 12n - 4n^2}{3(-5 + 2n)(-1 + 2n)},$$

$$CA[-3 + n, -1 + n] = \frac{9 - 16n + 8n^2}{3(-3 + 2n)(-1 + 2n)},$$

$$CA[-2 + n, -1 + n] = -1,$$

$$CA[-1 + n, -1 + n] = 1.$$

As pointed out in the paragraph after (9), this recursion is valid only when $n \ge 4$.

Remark. In a recent work [8] we apply a similar technique to the linearization problems:

$$P_i(x)P_j(x) = \sum_{k=0}^{i+j} L_{i,j,k}P_k(x)$$

in which we examine the complexity of the recurrence relation between linearization coefficients as function of orthogonality restrictions. For the connection coefficients the three levels of assumptions are: without orthogonality in the $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$ families (arbitrary basis), the coefficients $C_m(n)$ are of course also arbitrary; with orthogonality in each family $\{P_n(x)\}_{n\in\mathbb{N}}$ and $\{Q_n(x)\}_{n\in\mathbb{N}}$, the recurrence relations for the $C_m(n)$ mixed both indices (from both recurrence relations satisfied by the P_n and Q_n families after multiplication of (1) by x); with semi-classical (classical) assumption, the recurrence relations for the $C_m(n)$ keeps the same argument n (from the existence of the structure relation (3)).

Acknowledgements

The computer part of this work was done at the E.T.S.I. of the University of Vigo (Spain). A.R. thanks the Dept. of Math. for the kind invitation and for the financial support.

References

- [1] R. Askey, Orthogonal expansions with positive coefficients, Proc. Amer. Math. Soc. 26 (1965) 1191-1194.
- [2] R. Askey, Orthogonal polynomials and special functions, in: Proc. Regional Conf. in Appl. Math., Vol. 21 (SIAM, Philadelphia, PA, 1975).
- [3] R. Askey and J. Wimp, Associated Laguerre and Hermite polynomials, Proc. Royal Soc. Edinburgh Sect. A 96 (1985) 15-37.
- [4] G. Gasper, Positivity and special functions, in: R. Askey, Ed., Theory and Applications of Special Functions (Academic Press, New York, 1975) 375-433.
- [5] S. Lewanowicz, Quick construction of recurrence relations for the Jacobi coefficients, J. Comput. Appl. Math. 43 (1992) 355-372.
- [6] A. Nikiforov and V.B. Uvarov, Eléments de la Théorie des Fonctions Spéciales (Mir, Moscow, 1976).
- [7] A. Ronveaux, 4th order differential equations for numerator polynomials, J. Phys. A 21 (1988) L749-L753.
- [8] A. Ronveaux, M.N. Hounkonnou and S. Belmehdi, Generalized linearization problems, J. Phys. A, in print.
- [9] R. Szwarc, Connection coefficients of orthogonal polynomials, Canad. Math. Bull. 35 (4) (1992) 548-556.
- [10] R. Szwarc, Linearization and connection coefficients of orthogonal polynomials, Monatsh. Math. 113 (1992) 319-329.
- [11] Wolfram Research, Inc., Mathematica version 2.2 (Wolfram Research, Champaign, IL, 1993).