# Generalization of the Genocchi Numbers to their 

$q$-analogue

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#### Abstract

In the study of functions, it is often useful to derive a more generalized form of a given function and study it in order to shed new light on the original function, which is a special case of the object under study. One way in which to construct such generalizations is through the use of $q$-series. In this note, we will discuss some of the tools necessary for constructing these $q$-analogues of classical functions, their purpose, and then demonstrate one such construction on the Genocchi numbers and its close relative, the Euler numbers.

Two methods of generation for the Genocchi numbers will be given, and a verification of the relationship between the Genocchi numbers and the Euler numbers will be discussed in each case. Following that, the generalization to a $q$-analogue of each series will be discussed and the preservation of the relationship between the two series will be verified.


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## 1 Background: Building Blocks

The study of the $q$-Genocchi numbers, $q$-Euler numbers, and similar generalized sequences is a complex task which requires at least a basic familiarity with a number of concepts. In this section, we will be providing the tools needed to understand these sequences and begin working with them and the interesting properties which they possess.

The first of these concepts is that of the $q$-series. These series are built out of a generalization of classical continuous calculus, beginning with its basic component, the derivative. Once the $q$-derivative is constructed, further constructions in the realm of $q$-calculus (shortened from "quantum calculus", so named for its discrete nature) can be made, such as $q$-integrals, the $q$-binomial theorem, and $q$-analogues of various classically defined functions, sequences, and series. The idea behind using these $q$-series is that by generalizing continuous calculus and creating a discrete analogue of it, we are able to find connections between classical calculus and more discrete fields such as combinatorics and number theory. These generalized, discrete formulae also provide a number of insights into their continuous counterparts and how they might relate to more discrete applications, given that the continuous versions of these formulae are merely special cases of the general formulae.

## $1.1 \quad q$-Derivative

In classical, continuous calculus, the differential quotient is one of the fundamental concepts from which much of the field is derived. This quotient is defined as

$$
D f(x)=\frac{d f}{d x}=\lim _{x \rightarrow x_{0}} \frac{f(x)-f\left(x_{0}\right)}{x-x_{0}}
$$

This quotient is comprised of the differential operators $d f$ and $d x$, which are defined as $f(x)-f\left(x_{0}\right)$ and $\left(x-x_{0}\right)$, respectively. So, to build the $q$-derivative, we must first begin with a $q$-differential operator.

### 1.1.1 $\quad q$-Differential Operator

To construct such an analogue to the operator, take $x=q x_{0}$, where $q \neq 1$. With the introduction of this new variable $q$, we will no longer be looking at the previous limit, when $x$ approaches $x_{0}$. So, we then define the $q$-differential operator as follows:

$$
d_{q} f(x)=f(q x)-f(x)
$$

So, for instance, we have the following differential which will be important to the definition of the $q$-derivative:

$$
d_{q} x=(q-1) x
$$

One thing that is important to note about this operator that distinguishes it from the normal differential operator is that it is asymmetric when applied to the multiplication of two functions. In other words, we examine $d_{q}(f(x) g(x))$. Applying the
previously stated definition of the operator, we get

$$
d_{q}(f(x) g(x))=f(q x) g(q x)-f(x) g(x) .
$$

If we then add and subtract the term $f(q x) g(x)$ so as not to change the end value of the right side of the equation, we get

$$
d_{q}(f(x) g(x))=f(q x) g(q x)-f(q x) g(x)+f(q x) g(x)-f(x) g(x)
$$

Then, finally, factoring out like terms and recognizing the $q$-differential operators remaining, we are left with

$$
d_{q}(f(x) g(x))=f(q x) d_{q} g(x)+g(x) d_{q} f(x) .
$$

### 1.1.2 $\quad q$-Derivative

Now, with our desired analogue to the differential operator, we are now ready to define the $q$-derivative. The definition is fairly straightforward, as with the classical case:

$$
\begin{equation*}
D_{q} f(x)=\frac{d_{q} f(x)}{d_{q} x}=\frac{f(q x)-f(x)}{(q-1) x} \tag{1}
\end{equation*}
$$

In order for (1) to be a true generalization of classical calculus, what we want is for the original differential quotient to be a special case of this more general quotient,
and in fact, it is easily seen that

$$
\lim _{q \rightarrow 1} D_{q} f(x)=\frac{d f(x)}{d x}=D f(x)
$$

Furthermore, just as the continuous derivative does in classical calculus, this $q$ derivative functions as a linear operator. In other words:

$$
\begin{aligned}
D_{q}(a f(x)+b g(x)) & =\frac{d_{q}(a f(x)+b g(x))}{d_{q} x} \\
& =\frac{a f(q x)+b g(q x)-a f(x)-b g(x)}{(q-1) x} \\
& =\frac{a(f(q x)-f(x))}{(q-1) x}+\frac{b(g(q x)-g(x))}{(q-1) x} \\
& =a D_{q} f(x)+b D_{q} g(x)
\end{aligned}
$$

A useful example of applying the $q$-derivative in that it will introduce another tool that will be used frequently later is to compute the $q$-derivative of $f(x)=x^{n}$, where $n \in \mathbb{N}, n>0$. In doing so, we get

$$
\begin{aligned}
D_{q} x^{n} & =\frac{(q x)^{n}-x^{n}}{(q-1) x} \\
& =\frac{q^{n}-1}{q-1} x^{n-1} \\
& =[n] x^{n-1}
\end{aligned}
$$

Here, the notation $[n]$ is used to mean "the $q$-analogue of $n$ ", in the sense that as $q \rightarrow 1,[n] \rightarrow n$. This is a common concept that can be found in $q$-analogues of a number of functions, and will be seen repeatedly in later topics.

Two other useful examples of using the $q$-derivative is to take the $q$-derivative of the product and quotient of two arbitrary functions, $f(x)$ and $g(x)$. For the former, we have

$$
\begin{aligned}
D_{q}(f(x) g(x)) & =\frac{d_{q}(f(x) g(x))}{(q-1) x} \\
& =\frac{f(q x) d_{q} g(x)+g(x) d_{q} f(x)}{(q-1) x}
\end{aligned}
$$

which then, upon algebraic rearrangement, gives us the general product rule for $q$-derivatives,

$$
\begin{equation*}
D_{q}(f(x) g(x))=f(q x) D_{q} g(x)+g(x) D_{q} f(x) \tag{2}
\end{equation*}
$$

Interestingly, if we swap $f(x)$ and $g(x)$ on the left side of (2), we get an alternate yet equivalent formulation,

$$
\begin{equation*}
D_{q}(g(x) f(x))=f(x) D_{q} g(x)+g(q x) D_{q} f(x) \tag{3}
\end{equation*}
$$

In order to derive the quotient rule for $q$-derivatives, we begin by differentiating

$$
g(x) \cdot \frac{f(x)}{g(x)}=f(x)
$$

using (2). The result we obtain is

$$
g(q x) D_{q}\left(\frac{f(x)}{g(x)}\right)+\frac{f(x)}{g(x)} D_{q} g(x)=D_{q} f(x)
$$

which then gives us a quotient rule,

$$
\begin{equation*}
D_{q}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x) D_{q} f(x)-f(x) D_{q} g(x)}{g(x) g(q x)} \tag{4}
\end{equation*}
$$

Alternately, using (3) to calculate the derivative, we obtain

$$
g(x) D_{q}\left(\frac{f(x)}{g(x)}\right)+\frac{f(q x)}{g(q x)} D_{q} g(x)=D_{q} f(x),
$$

which also gives a quotient rule,

$$
\begin{equation*}
D_{q}\left(\frac{f(x)}{g(x)}\right)=\frac{g(q x) D_{q} f(x)-f(q x) D_{q} g(x)}{g(x) g(q x)} . \tag{5}
\end{equation*}
$$

Both (4) and (5) are valid quotient rules for the $q$-derivative, and their use typically depends on which is more useful for the specific functions $f(x)$ and $g(x)$ that are being considered.

### 1.2 The $q$-Binomial Theorem

Another primary object in the $q$-universe is the $q$-binomial theorem. It is in defining this analogue of the binomial theorem that the $q$-series itself appears, a concept that will appear a number of times in later topics. However, before delving into
the $q$-binomial theorem itself, it is necessary to take a brief detour and discuss a few components which will show themselves in the construction of the $q$-binomial theorem.

### 1.2.1 Taylor's Theorem

Taylor's theorem is a common tool used in the study of calculus to approximate a function about a given point using a polynomial whose coefficients are related to the derivatives of the function to be approximated. One specific use to be considered in the context of this note is its use in deriving a form of the binomial theorem which can be readily generalized to a $q$-analogue, which will be discussed later. The statement of Taylor's theorem is as follows:

Let $n \geq 0$ be an integer, and let $f$ be a function that is $n$ times differentiable on the closed interval $[a, x]$ and $n+1$ times differentiable on the open interval ( $a, x$ ). Then

$$
\begin{equation*}
f(x)=f(a)+\frac{f^{\prime}(a)}{1!}(x-a)+\frac{f^{\prime \prime}(a)}{2!}(x-a)^{2}+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^{n}+R_{n}(x), \tag{6}
\end{equation*}
$$

where $a$ is the point around which $f(x)$ is being approximated, and $R_{n}(x)$ denotes the remainder term, which specifies the difference between the polynomial, known as the Taylor approximation, and the original function being approximated. For values of $x$ arbitrarily close to $a$, this term approaches zero. For example, the $n$ th-order Taylor approximation of $f(x)=e^{x}$ about $x=0$ is

$$
e^{x} \approx 1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots+\frac{x^{n}}{n!}
$$

where in this particular case, the remainder term is implicit in the approximation symbol, rather than the definite equality described in (6).

### 1.2.2 Gamma Functions

The Gamma function was introduced in a series of letters between Leonhard Euler and Christian Goldbach in the year 1729. It came about as a solution to a problem many mathematicians were pondering at the time: a way to interpolate the factorials. What is meant by "interpolation" is best shown via example.

Take the sequences of sums, $1,1+2,1+2+3, \cdots$. These sums are known as the triangular numbers, with its general term denoted $T_{n}$. The value of the $n$th triangular number is rather well-known to be $T_{n}=\frac{1}{2} n(n+1)$. While this formula is commonly used for integer $n$ values, what makes it act as a true interpolation of the triangular numbers is that it is defined for not only integer values, but fractions and negative values as well. So, for instance, even though we would rarely ask what the negative second triangular number is in terms of partial sums of the integral sequence, we can still calculate the value of $T_{-2}$ to be $T_{-2}=\frac{1}{2}(-2)(-1)=1$. In this way, the original concept of the triangular numbers has its scope broadened, and with the inclusion of nonintegral values, we are able to get a sense of what is happening between the commonly known triangular numbers of integral order.

This is precisely what Goldbach and Euler were attempting to do with the factorials. Most people who are familiar with the definition of the factorial will be able to tell you that the value of $3!=6$ or that $5!=120$, but many of them will be stymied when asked to provide the value of $(5 / 3)$ ! or $(-1 / 2)$ !. This is where the gamma
function comes into play.
Euler, who frequently experimented with infinite products of numbers, happened to notice that the infinite product

$$
\begin{equation*}
\prod_{i=1}^{\infty}\left(\frac{i+1}{i}\right)^{n} \frac{i}{n+i}=n! \tag{7}
\end{equation*}
$$

whenever $n$ is a positive integer. This equality isn't very difficult to verify, and can be done by canceling all the common factors out of the terms in the numerators and denominators on the left hand side. What was particularly of interest to Euler, though was that upon entering the value $n=\frac{1}{2}$ into the product, the result was that he found

$$
\left(\frac{1}{2}\right)!=\frac{\sqrt{\pi}}{2}
$$

after recognizing the relationship between the resulting product on the left side and a product whose value was determined by John Wallis:

$$
\frac{\pi}{4}=\frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdots
$$

The fact that integral values of $n$ resulted in integral outputs from his product, but occasionally outputs containing multiples of $\pi$ encouraged Euler to search for a transformation that would express his product as an integral. After a good deal of
work, described in detail in [1], Euler came to the conclusion that

$$
\begin{equation*}
n!=\int_{0}^{1}(-\log x)^{n} d x \tag{8}
\end{equation*}
$$

This was the result he wanted; an integral that, for positive integer values of $n$ would give him the factorials, but also had meaning for negative and nonintegral values of $n$.

The modern day definition of the gamma function, however, was not introduced until Legendre noticed that (8) could be modified to

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{\infty} e^{-t} t^{z-1} d t \tag{9}
\end{equation*}
$$

This is the form in which the gamma function is typically considered today, with the integral being divergent for nonpositive integral values of $z$, having the specific values

$$
\Gamma(1)=1, \quad \Gamma(2)=1
$$

and possessing the property that

$$
\begin{equation*}
\Gamma(z+1)=z! \tag{10}
\end{equation*}
$$

To verify this property, we begin with an intermediate proposition, which will then
be extended to give us the result (10). This proposition is

$$
\begin{equation*}
\Gamma(z+1)=z \Gamma(z) . \tag{11}
\end{equation*}
$$

To prove this proposition, we begin with the definition (9). Setting $u=e^{-t}$ and $d v=t^{z-1} d t$, we perform integration by parts on the integral. The result of this integration is that

$$
\Gamma(z)=\left.\left(\frac{1}{z}\right) e^{-t} t^{z}\right|_{0} ^{\infty}-\left(\frac{1}{z}\right) \int_{0}^{\infty}-e^{-t} t^{z} d t
$$

where the $1 / z$ term is able to be removed from the integral due to its being constant with respect to the integrated variable, $t$. We are then able to factor out that $1 / z$ term from both addends, giving us

$$
\Gamma(z)=\frac{1}{z}\left(\left.e^{-t} t^{z}\right|_{0} ^{\infty}+\int_{0}^{\infty} e^{-t} t^{z} d t\right)
$$

The left addend, upon evaluation at 0 and $\infty$, disappears, leaving us then with

$$
\Gamma(z)=\frac{1}{z} \int_{0}^{\infty} e^{-t} t^{z} d t
$$

This integral on the right side is equivalent to $\Gamma(z+1)$, so by multiplying through by $z$ on both sides of the equation, we have our relationship (11).

So, now that we have this recursive relationship between successive values of $z$ used as inputs into the gamma function, we may note that applying the definition
recursively demonstrates the following behavior:

$$
\Gamma(z+1)=z \Gamma(z)=z(z-1) \Gamma(z-1)=z(z-1)(z-2) \Gamma(z-2)=\cdots
$$

Thus, if we continue to recursively apply this definition until we get to the known value of $\Gamma(1)$ as the only gamma term left on the right side, we will be left with

$$
\Gamma(z+1)=z(z-1)(z-2) \cdots 2 \cdot 1 \cdot 1=z!
$$

which is precisely the relationship we wanted. It is the gamma function's ability to represent factorials that allows it to relate to a wide variety of special functions that utilize either entire or partial factorials, such as Gauss's hypergeometric series.

### 1.2.3 Pochhammer Symbols

The Pochhammer symbol $(a)_{n}$, sometimes referred to the $a$-shifted factorial, and other times as the rising factorial, is defined as follows:

$$
\begin{equation*}
(a)_{n}=a(a+1)(a+2) \cdots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)} \tag{12}
\end{equation*}
$$

where $n=1,2, \cdots$, and $\Gamma(x)$ is the gamma function described in the previous section. This relationship with the gamma function is easily verifiable using (10), giving that

$$
\frac{\Gamma(a+n)}{\Gamma(a)}=\frac{(a+n-1)!}{(a-1)!}=(a+n-1)(a+n-2) \cdots(a+1)(a)=(a)_{n}
$$

The Pochhammer symbol's related object in the $q$-realm, the $q$-shifted factorial (also referred to simply as a $q$-series), is defined as

$$
(a ; q)_{n}= \begin{cases}1 & n=0  \tag{13}\\ (1-a)(1-a q)\left(1-a q^{2}\right) \cdots\left(1-a q^{n-1}\right) & n=1,2, \ldots\end{cases}
$$

The objects defined in (12) and (13), in addition to playing a critical role in the definition of the $q$-binomial theorem, compose the hypergeometric series and $q$ hypergeometric series respectively, and so their relationships are important to note in considering these functions, which will be discussed later, as well.

### 1.2.4 $\quad q$-Binomial Theorem

Now, having defined and discussed all the tools we need, we can construct the $q$ analogue of the binomial theorem. First, let us begin with the geometric series,

$$
\begin{equation*}
\sum_{n=0}^{\infty} z^{n}=(1-z)^{-1} \tag{14}
\end{equation*}
$$

where $|z|<1$. From this series, we would like to generalize the expression on the right side of the equation, examinine

$$
f(z)=(1-z)^{-a} .
$$

From this expression, we may apply Taylor's Theorem and use mathematical induction to see that the $n$th derivative,

$$
f^{(n)}(z)=\left.(a)_{n}(1-z)^{-a-n}\right|_{z=0}=(a)_{n}
$$

where $(a)_{n}$ is the pochhammer symbol as defined in the previous section. What this then allows us to do is to extend (14) to

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{(a)_{n}}{n!} z^{n}=(1-z)^{-a} \tag{15}
\end{equation*}
$$

(15) is sometimes referred to as the binomial theorem, because if $a=-m$ is a negative integer, and $z=-(x / y)$, then we get the common statement of the binomial theorem in the following way:

$$
\sum_{n=0}^{\infty} \frac{(-m)_{n}}{n!}\left(\frac{-x}{y}\right)^{n}=\left(1+\frac{-x}{y}\right)^{m}
$$

Multiplying both sides by $y^{m}$ then gives the statement we seek:

$$
\sum_{n=0}^{\infty}\binom{m}{n} x^{n} y^{m-n}=(x+y)^{m}
$$

Now, if we do as is commonly done in the study of $q$-series and assume $0<q<1$, we see that by l'Hôpital's rule,

$$
\lim _{q \rightarrow 1} \frac{1-q^{a}}{1-q}=a
$$

So, by extending this concept, we see that

$$
\lim _{q \rightarrow 1} \frac{\left(1-q^{a}\right)\left(1-q^{a+1}\right) \cdots\left(1-q^{a+n-1}\right)}{(1-q)\left(1-q^{2}\right) \cdots\left(1-q^{n}\right)}=\frac{(a)_{n}}{n!} .
$$

Additionally, it can be shown that

$$
\frac{\left(1-q^{a}\right)\left(1-q^{a+1}\right) \cdots\left(1-q^{a+n-1}\right)}{(1-q)\left(1-q^{2}\right) \cdots\left(1-q^{n}\right)}=\frac{(a ; q)_{n}}{(q ; q)_{n}},
$$

where $(a ; q)_{n}$ is the $q$-shifted factorial as described in the previous section. So, combining the facts that this quotient can be written in terms of $q$-shifted factorials and that its limit as $q$ approaches 1 is the general coefficient from the alternate statement of the binomial theorem, we then get the statement of the $q$-binomial theorem:

$$
\begin{equation*}
f(a, z)=\sum_{n=0}^{\infty} \frac{(a ; q)_{n}}{(q ; q)_{n}} z^{n}=\frac{(a x ; q)_{\infty}}{(x ; q)_{\infty}}, \tag{16}
\end{equation*}
$$

where $\frac{(a x ; q)_{\infty}}{(x ; q)_{\infty}}$ is a quotient of infinite $q$-shifted factorials whose limit as $q$ approaches 1 is $(1-z)^{-a}$.

## $1.3 \quad q$-Hypergeometric Series

To get an idea of the sort of manipulation that is intended to be done with the Genocchi and Euler sequences, a useful example is Gauss's hypergeometric series. It is a comparatively simple case which primarily utilizes the definition of the $q$ shifted factorial mentioned previously, and simple techniques to generate and verify
the generalization.

### 1.3.1 Hypergeometric Series

Gauss's hypergeometric series is defined as

$$
\begin{equation*}
F(a, b ; c ; z)=\sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{n!(c)_{n}} z^{n} \tag{17}
\end{equation*}
$$

where $c \neq 0,-1,-2, \cdots$. With this series being defined in terms of shifted factorials (with $n$ ! simply being the factorial shifted by $a=0$ ), it makes sense for its $q$-analogue to be defined in terms of $q$-shifted factorials. In fact, this turns out to be the case, as Heine defined the $q$-hypergeometric series as

$$
\begin{equation*}
\Phi(\alpha, \beta ; \gamma ; q, z)=\sum_{n=0}^{\infty} \frac{\left(q^{\alpha} ; q\right)_{n}\left(q^{\beta} ; q\right)_{n}}{(q ; q)_{n}\left(q^{\gamma} ; q\right)_{n}} z^{n} \tag{18}
\end{equation*}
$$

To verify that this generalized formula does indeed result in the classical hypergeometric series when the limit as $q$ approaches 1 is taken, we simply take that limit upon the general term of the series. So, we're looking at

$$
\lim _{q \rightarrow 1} \frac{\left(q^{\alpha} ; q\right)_{n}\left(q^{\beta} ; q\right)_{n}}{(q ; q)_{n}\left(q^{\gamma} ; q\right)_{n}}
$$

For simplicity's sake, let's begin by recognizing that this can be considered as the product of $\frac{\left(q^{\alpha} ; q\right)_{n}}{(q ; q)_{n}}$ and $\frac{\left(q^{\beta} ; q\right)_{n}}{\left(q^{\gamma} ; q\right)_{n}}$, and simply consider the first half, as the procedure
of evaluating the second half will be identical. So, we're looking at

$$
\lim _{q \rightarrow 1} \frac{\left(q^{\alpha} ; q\right)_{n}}{(q ; q)_{n}}
$$

Expanding out the $q$-shifted factorials, we see that we have

$$
\lim _{q \rightarrow 1} \frac{\left(1-q^{\alpha}\right)\left(1-q^{\alpha+1}\right) \cdots\left(1-q^{\alpha+n-1}\right)}{(1-q)\left(1-q^{2}\right) \cdots\left(1-q^{n}\right)}
$$

which, when we take the limit given, we find the result to be $\frac{(\alpha)_{n}}{n!}$. Similarly, if we take the limit $\lim _{q \rightarrow 1} \frac{\left(q^{\beta} ; q\right)_{n}}{\left(q^{\gamma} ; q\right)_{n}}$, we will find

$$
\lim _{q \rightarrow 1} \frac{\left(q^{\beta} ; q\right)_{n}}{\left(q^{\gamma} ; q\right)_{n}}=\frac{(\beta)_{n}}{(\gamma)_{n}}
$$

So, recombining the two portions, we see then that the limit as $q$ approaches 1 of (18) is, in fact, (17).

In this way, by taking the limit that is presupposed about $q$-series in general, we can verify that a function in the $q$-realm is, in fact, a true generalization of a classical function. This is the first step required of understanding and studying any $q$-analogue of a function.

## 2 Genocchi and Euler Numbers

The classical Genocchi numbers are defined in a number of ways. The way in which it is defined is often determined by which sorts of applications they are intended to be
used for. The Genocchi numbers have wide-ranging applications from number theory and combinatorics to numerical analysis and other fields of applied mathematics. In this note, two definitions of the Genocchi numbers will be given: the generating function definition, which is the most commonly used definition, and a Pascal-type triangle definition, first given by Philipp Ludwig von Seidel, and discussed in [3]. The main focus will then be to discuss the primary relationship between these Genocchi numbers and another sequence, the Euler numbers.

### 2.1 Generating Function

The generating function definition of the Genocchi numbers is by far the most commonly used definition when discussing and studying the sequence. This definition is as follows:

$$
\begin{equation*}
G(t)=\frac{2 t}{e^{t}+1}=\sum_{n=0}^{\infty} G_{n} \frac{t^{n}}{n!} \tag{19}
\end{equation*}
$$

The first few terms of this sequence are

$$
G_{1}=1, \quad G_{2}=-1, \quad G_{4}=1, \quad G_{6}=-3, \quad G_{8}=17,
$$

where $G_{3}=G_{5}=\cdots=G_{2 n-1}=0$.
This generating function definition gives rise to a number of interesting properties. One in particular describes its primary relationship to a very similar sequence of numbers, the Euler numbers. As might be expected, it has a similar generating
function definition:

$$
\begin{equation*}
E(t)=\frac{2}{e^{t}+1}=\sum_{n=o}^{\infty} E_{n} \frac{t^{n}}{n!} \tag{20}
\end{equation*}
$$

This sequence has opening terms

$$
E_{1}=\frac{1}{2}, \quad E_{3}=\frac{1}{4}, \quad E_{5}=\frac{1}{2}, \quad E_{7}=\frac{17}{8}, \quad E_{9}=\frac{31}{2}
$$

where $E_{2 m}=0, m \geq 1$.
It is easy to see that these two generating functions are incredibly similar. In fact, the only difference between the two is the factor of $t$ upon which the left side of the equation for the generating function for the Genocchi numbers is scaled by in comparison to that of the Euler numbers. As such, the two sequences can also be expected to have a relatively simply described relationship between them, which is as follows:

$$
\begin{equation*}
G_{2 m}=2 m E_{2 m-1} \tag{21}
\end{equation*}
$$

In this relation, the restriction to even-indexed Genocchi numbers is placed due to the fact that all odd-indexed Genocchi numbers (and also all even-indexed Euler numbers) have a value of zero under this definition. It is possible to alter the relationship slightly to have $G_{n}=n E_{n-1}$, but in this case, the zero-valued numbers in the sequence are considered trivial and thus omitted in the statement of the relation. There are, however, other instances in which including all elements of the
sequence is useful, and so sometimes the altered relationship is used. Additionally, there are other methods of generating the Genocchi numbers which do not include the zero-valued terms, and in which the absolute value of each term is considered. These methods are typically used where applications in fields such as combinatorics are concerned.

Verifying that (21) holds true between the sequences generated by (19) and (20) is relatively simple, given their close relationship. Beginning with the generating functions as above, if we divide both sides of (19) by the factor of $t$ to get the power series on the right side of each equation equal, we get the equation

$$
\sum_{n=0}^{\infty} G_{n} \frac{t^{n-1}}{n!}=\sum_{n=0}^{\infty} E_{n} \frac{t^{n}}{n!}
$$

With two power series in terms of $t$ equated, their individual terms with identical powers of $t$ (specifically the coefficients of those terms) must be equal as well. In order to consider terms with the same power of $t$ in both power series, we set $n=2 m$ on the left and $n=2 m-1$ on the right. Then, the arbitrary term we are examining is

$$
G_{2 m} \frac{t^{2 m-1}}{(2 m)!}=E_{2 m-1} \frac{t^{2 m-1}}{(2 m-1)!}
$$

Multiplying the term on the right side of the equation by $\frac{2 m}{2 m}$, we can then cancel the $\frac{t^{2 m-1}}{(2 m)!}$ terms, leaving us with precisely (21).

### 2.2 Seidel Triangle Generation

Another method of generating the Genocchi numbers that was introduced in the late 1800s by Philipp Ludwig von Seidel is via a Pascal-type triangle. This is one such method that, as mentioned previously, disregards the zero-valued terms and takes the absolute value of each of the nonzero terms. As such, it makes it very appealing for use in combinatorial applications.

The idea behind this definition, as in Pascal's triangle, is to utilize a recursive relationship given some initial conditions to generate the Genocchi numbers. The terms in the triangle are denoted $g_{i, j}$ where $i, j \geq 1$, with $i$ and $j$ denoting the term's position in the triangle. The recursion is as follows:

$$
\begin{cases}g_{2 i+1, j}=g_{2 i+1, j-1}+g_{2 i, j} & j=1,2, \cdots, i+1 \\ g_{2 i, j}=g_{2 i, j+1}+g_{2 i-1, j} & j=i, i-1, \cdots, 1\end{cases}
$$

where $g_{1,1}=g_{2,1}=1$.
In essence, what ends up happening is a sort of alternating addition up and down each column, resulting in the chart described in Table 1.

The result, then, upon construction of this table, is that

$$
\begin{equation*}
g_{2 n-1, n}=\left|G_{2 n}\right| . \tag{22}
\end{equation*}
$$

With no explicitly defined triangle to generate the Euler numbers in the same


Table 1: Seidel's Triangle generation of the Genocchi numbers
fashion as the Seidel triangle generates the Genocchi numbers, it is difficult to prove the relation (21) directly. However, one can make an empirical verification by directly applying the formula to the first few terms of the Genocchi numbers and checking that it does, in fact, give the first terms of the Euler sequence.

In order to perform this empirical verification, we begin by combining (21) and (22) to get a formula useful for directly relating the Genocchi numbers as entries in the Seidel triangle to the Euler numbers:

$$
g_{2 n-1, n}=2 n E_{2 n-1} .
$$

Then, since we are looking at moving from the Genocchi numbers to the Euler numbers given the generation of the Genocchi numbers, it is useful to rearrange this formula into the following form:

$$
\begin{equation*}
E_{2 n-1}=\frac{g_{2 n-1, n}}{2 n} \tag{23}
\end{equation*}
$$

Now, we may begin using (23) to compare the entries in the Seidel triangle to the list of the first few entries of the Euler numbers given above.

For example, begin by setting $n=1$. Then, (23) would become

$$
E_{1}=\frac{g_{1,1}}{2}=\frac{1}{2} .
$$

Similarly, for $n=2$, we would have

$$
E_{3}=\frac{g_{3,2}}{4}=\frac{1}{4}
$$

and for $n=3$,

$$
E_{5}=\frac{g_{5,3}}{6}=\frac{3}{6}=\frac{1}{2} .
$$

This process can be continued for further values of $n$, simply applying (23) to the values in the Seidel triangle and the values in the sequence of Euler numbers, verifying for each $n$ value that the relation does in fact hold. It is by no means an explicit proof of the identity, but unless a similar recursive Pascal-type relation is described for the Euler numbers, such a proof using the Seidel triangle is difficult, at best. The generation of such a recursive relation for the Euler numbers is difficult even simply considering the first few terms of the sequence, because we've got $E_{1}=E_{5}=$ $1 / 2$. Between these two terms, the sequence dips below $1 / 2$, with $E_{3}=1 / 4$, and then afterwards rises higher, with $E_{7}=17 / 8$ and $E_{9}=31 / 2$. On the other hand, when looking at the absolute values of the Genocchi numbers as are given in the

Seidel triangle, the sequence is strictly increasing. Thus, using an additive recursive definition similar to that of the Pascal triangle is possible. The Euler numbers are not strictly increasing, and thus such a definition is at the very least not as straightforward as for the Genocchi numbers.

## $3 \quad q$-Genocchi and $q$-Euler Numbers

Now, having discussed the classical Genocchi numbers and Euler numbers and two ways of defining them, we are ready to generalize them to their $q$-analogues, generating the $q$-Genocchi numbers in the same ways as in the previous section.

### 3.1 Generating Function

In generalizing the generating functions of the Genocchi numbers and Euler numbers to their respective $q$-analogues, it is useful to, rather than define the generating function for the Genocchi numbers first, and base the Euler numbers' generating function definition off that, to do it the other way around. So, we will be defining the generating function for the $q$-Euler numbers first, and then relate the generating function of the $q$-Genocchi numbers to it.

It is important to note that, even when deriving a $q$-analogue of the Genocchi and Euler numbers via generating functions, there are a number of different possibilities. In [2], one such analogue was mentioned, attributed to Leonard Carlitz, with the purpose of distinguishing the author's analogue from the previously known one. Thus, there is by no means one concrete $q$-analogue of a given function, series, etc.

This can be understood as a consequence of the fact that the analogue's limit as $q$ approaches 1 is the original function is a sufficient condition for the analogue to be valid.

We will be considering the generating function considered in [2],

$$
F_{q}(t)=[2]_{q} e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{(-1)^{j}}{1+q^{j+1}}\left(\frac{1}{1-q}\right) \frac{t^{j}}{j!}=\sum_{n=0}^{\infty} E_{n, q} \frac{t^{n}}{n!}
$$

where it is noted that in this instance, the notation $E_{n, q}$ is used to replace $E_{q}^{n}$ symbolically. After verifying that $\lim _{q \rightarrow 1} E_{n, q}=E_{n}$ by deriving the expression for $E_{n, q}$

$$
\begin{equation*}
E_{n, q}=\frac{[2]_{q}}{(1-q)^{n}} \sum_{l=0}^{n}\binom{n}{l} \frac{(-1)^{l}}{1+q^{l+1}}, \tag{24}
\end{equation*}
$$

and taking the aforementioned limit on that expression, a task which can easily be performed (and is encouraged) for the first few values of $n$, a more accessible generating function definition is defined for the $q$-Euler numbers, derived from the previous definition given:

$$
\begin{equation*}
F_{q}(t)=[2]_{q} \sum_{n=0}^{\infty}(-1)^{n} q^{n} e^{[n]_{q} t}=\sum_{n=0}^{\infty} E_{n, q} \frac{t^{n}}{n!} . \tag{25}
\end{equation*}
$$

From the generating function (25) for the $q$-Euler numbers, the Genocchi numbers can then be defined similarly as with the generating functions of the classical Euler and Genocchi numbers by simply multiplying through the left side by a factor of $t$ :

$$
\begin{equation*}
G_{q}(t)=[2]_{q} t \sum_{n=0}^{\infty}(-1)^{n} q^{n} e^{[n]_{q} t}=\sum_{n=0}^{\infty} G_{n, q} \frac{t^{n}}{n!} \tag{26}
\end{equation*}
$$

With these two definitions and their being related similarly as to the generating functions of the classical Euler and Genocchi numbers, it is not surprising that (21) is identical in the $q$-realm:

$$
\begin{equation*}
G_{2 m, q}=2 m E_{2 m-1, q} \tag{27}
\end{equation*}
$$

The proof of (27) is also identical to the proof of (21). In spite of the complicated nature of the left expression in both (25) and (26), the fact that the only difference between the two is a factor of $t$ allows us to divide through the power series on the right side of (26), giving

$$
G_{q}(x)=[2]_{q} \sum_{n=0}^{\infty}(-1)^{n} q^{n} e^{[n]_{q} t}=\sum_{n=0}^{\infty} G_{n, q} \frac{t^{n-1}}{n!}
$$

Thus, we've got the equation of the two power series

$$
\sum_{n=0}^{\infty} G_{n, q} \frac{t^{n-1}}{n!}=\sum_{n=0}^{\infty} E_{n, q} \frac{t^{n}}{n!}
$$

and by equating like terms in each power series as we did in the classical case, we get the relationship (27).

### 3.2 Seidel Triangle Generation

In addition to referencing the original Seidel triangle for the Genocchi numbers in [3], a $q$-analogue for the construction is given. It is from that $q$-analogue that a number of applications to combinatorics and applied mathematics are given in that
paper, but for the purposes of this note, we will just discuss the generation and its relationship to the original Seidel triangle.

As with any $q$-analogue, the goal is to create a construction that will produce polynomials in $q$ such that, as $q$ approaches 1 , the result will be precisely the values given in the original. Naturally, for a structured object such as the Seidel triangle, the structure of the construction will need to be preserved as well. Hence, as was the case with the original Seidel triangle, the $q$-analogue of the construction is given as a recursive definition, similar to Pascal's triangle:

$$
\begin{cases}g_{2 i+1, j}(q)=g_{2 i+1, j-1}(q)+q^{j-1} g_{2 i, j}(q) & j=1,2, \cdots, i+1 \\ g_{2 i, j}(q)=g_{2 i, j+1}(q)+q^{j-1} g_{2 i-1, j}(q) & j=i, i-1, \cdots, 1\end{cases}
$$

with $g_{1,1}(q)=g_{2,1}(q)=1$.
The primary difference between this $q$-analogue of Seidel's triangle and its original is the multiplication by a factor of $q_{j-1}$ of the second component in each addition. The result is that the sort of addition up and down each column that could be performed in the original Seidel triangle is altered somewhat, but when calculating new values for the triangle, it is still useful to follow such a path.

Once again, as with the original Seidel triangle for the classical Genocchi numbers, we have that

$$
G_{2 n}(q)=g_{2 n-1, n}(q)
$$

| 11 | $\begin{array}{cc} 1 & q \\ 1 & 1+q \\ \hline \end{array}$ | $\begin{gathered} 1+q+q^{2} \\ 1+q+q^{2} \\ 1+q \\ \hline \end{gathered}$ | $\begin{gathered} q^{2}+q^{3}+q^{4} \\ q+2 q^{2}+2 q^{3}+q^{4} \\ 1+2 q+2 q^{2}+2 q^{3}+q^{4} \end{gathered}$ | $\begin{gathered} 1+2 q+3 q^{2}+4 q^{3}+4 q^{4}+2 q^{5}+q^{6} \\ 1+2 q+3 q^{2}+4 q^{3}+4 q^{4}+2 q^{5}+q^{6} \\ 1+2 q+3 q^{2}+4 q^{3}+3 q^{4}+q^{5} \\ 1+2 q+2 q^{2}+2 q^{3}+q^{4} \end{gathered}$ | 4 3 2 1 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 12 | $3 \quad 4$ | 5 | 6 | 7 | $i / j$ |

Table 2: $q$-analogue of Seidel's triangle

Further, since all the entries into the table are simple polynomials in terms of $q$, it is easy to verify that, in fact, for all the values of the table's entries, $\lim _{q \rightarrow 1} g_{i, j}(q)=g_{i, j}$, where in this statement, $g_{i, j}$ refers to the entries in the original Seidel triangle.

One major difficulty in making any sort of purely mathematical statements regarding the comparison of the $q$-Genocchi numbers and $q$-Euler numbers using this definition of the former is that the only restriction on the $q$-polynomials that are achieved is that their limit as $q$ approaches 1 is its corresponding value in the classical sequence. What this allows for is a large number of possible polynomials that fit that description, and in this case, it turns out that the $q$-polynomials acquired by applying the generating function definition of the $q$-Genocchi numbers are very different from the polynomials acquired using the $q$-analogue of Seidel's triangle.

While this in itself may be considered a minor difficulty, what makes the sort of comparison described in section 2.2 to the $q$-Euler numbers is that there is no such tabular definition of the $q$-Euler numbers to be found in the literature on the topic. With the classical sequences, it was possible to make the comparisons because for
each definition, there was one definite value that was the $n$th Genocchi or $n$th Euler number. With the possibility of multiple polynomials and quotients of polynomials that can each legitimately qualify as $q$-analogues of these sequences, care must be taken to examine polynomials generated in the same fashion when trying to make any sort of statements about the polynomials acquired as values for each of the generalized sequences. Thus, with no analogous definition to the $q$-Seidel triangle for the $q$-Euler numbers, a direct verification of (27) using this tabular definition is not possible, given that our only tool for determining the values of the $q$-Euler numbers is (24), and the polynomials resulting from that expression are related to the values of the $q$-Genocchi numbers generated by the generating function definition, which are in turn vastly different from the polynomials generated by the Seidel's triangle recursion.

## 4 Conclusion

Using techniques similar to those described in this work, a wide variety of sequences, series, and functions can be generalized to their $q$-analogues. Typically, performing these kinds of generalizations will utilize a number of varying tools, the exact nature of which depend on the type of function being generalized.

While it is indeed true that working with these functions and their generalizations frequently serves to shed light on their individual natures, perhaps even more interesting is the ability to observe the intricate interplay between various mathematical entities, such as the gamma functions, $q$-series, and the Binomial theorem.

By combining these tools in different ways, just as much is learned about them as is learned about the objects they are being used to manipulate.

Further, it is hoped that this work may be used as a sort of guide to understanding the concept of $q$-analogues and the techniques that are required to find such analogues. Even merely between the two sequences considered within these pages, the classical Genocchi numbers and Euler numbers, there are a number of known identities which can be verified for their different $q$-analogues. This sort of study may prove to be very rich in the understanding of these sequences and in becoming familiar with how the tools that are used are combined to perform such verifications.

## References

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