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INVERSE RELATIONS AND COMBINATORIAL IDENTITIES

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1. Introduction. The inverse relations considered here are typified by

$$
\begin{aligned}
y^n &= (x+1)^n = \sum \binom{n}{k} x^k, \qquad n = 0, 1, \cdots \\
x^n &= (y-1)^n = \sum (-1)^{n+k} \binom{n}{k} y^k
\end{aligned}
\tag{1}
$$

or in a form more suggestive of the Jacobian injunction "always invert," by

$$
\begin{aligned}
a_n(x) &= (1+x)^n = \sum \binom{n}{k} x^k \\
x^n &= \sum (-1)^{n+k} \binom{n}{k} a_k(x).
\end{aligned}
\tag{1a}
$$

Such relations occur frequently in combinatorial analysis in a variety of contexts. Each pair is associated with an identity, such as, in the present instance,

$$
\delta_{nm} = \sum (-1)^{k+m} \binom{n}{k} \binom{k}{m}
\tag{2}
$$

with δ_{nm} the Kronecker delta. (The sum, here as above, is taken over the full range of nonzero values of the summand, with the convention that $\binom{n}{k} \equiv 0$, $k<0$, and need not be indicated.) As will appear, these orthogonal combinatorial identities have wider implication than the associated pairs of relations from which they proceed. In particular, they imply other pairs of relations and other identities, and thus provide a guide line through the forest of these prolific entities. Unfortunately the guide is weak since what emerges is the usual embarrassment of riches, with open paths in many directions.

The object of this paper is to assemble a variety of old and new results on the subjects of the title. A study of the relations in equations (1), which despite appearances are worth extended attention, sets the stage for Stirling numbers, for relations associated with Legendre and Chebyshev polynomials, and for other results.

2. The simplest inverse relations. It is convenient to begin with the pair of relations of equations (1) or (1a). Equation (2) follows from substitution of either into the other and equating coefficients of powers of the variable x or y. Since it is an orthogonality on the coefficients it follows that (1) and (1a) may be replaced by

$$
a_n = \sum \binom{n}{k} b_k, \qquad b_n = \sum (-1)^{n+k} \binom{n}{k} a_k.
\tag{1b}
$$

Moreover, sign factors may be moved at pleasure; a more symmetrical form is

$$a_n = \sum (-1)^k \binom{n}{k} b_k, \qquad b_n = \sum (-1)^k \binom{n}{k} a_k. \tag{1c}$$

Further, the orthogonality may be taken as associated with matrix multiplication; the coefficients of the relations are associated with a matrix and its inverse. Thus equations (1b) are associated with matrices $B = \{b_{ij}\}$ and $B^{-1} = \{\beta_{ij}\}$, where

$$b_{ij} = \binom{i}{j}, \qquad \beta_{ij} = (-1)^{i+j}\binom{i}{j};$$

in fact, by (2), $BB^{-1} = B^{-1}B = I$, with I the identity matrix. Both matrices are of triangular form, of finite or infinite extent. Thus the familiar arrays of coefficients

$$\begin{array}{cccc} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \\ & \cdots & & \end{array} \qquad \begin{array}{cccc} 1 & & & \\ -1 & 1 & & \\ 1 & -2 & 1 & \\ -1 & 3 & -3 & 1 \\ & \cdots & & \end{array}$$

ordinarily not regarded as mathematical objects become so by adding the brackets or braces indicating matrices.

Next, $\delta_{nm} = \delta_{mn}$, so (2) is the same as

$$\delta_{nm} = \sum (-1)^{m+k} \binom{k}{n}\binom{m}{k} \tag{2a}$$

implying

$$a_n = \sum \binom{k}{n} b_k, \qquad b_n = \sum (-1)^{n+k} \binom{k}{n} a_k \tag{3}$$

the relations connecting binomial moments a_n, and probabilities b_n, and usually not related to (1). The arrays of coefficients are now

$$\begin{array}{cccc} 1 & 1 & 1 & 1\cdots \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -1\cdots \\ & 1 & -2 & 3 \\ & & 1 & -3 \\ & & & 1 \end{array}$$

The triangle of coefficients has been rotated.

Next,

$$\delta_{nm} = \delta_{n+p,m+p} = \sum (-1)^{n+k} \binom{n+p}{k+p} \binom{k+p}{m+p} \tag{2b}$$

which implies

$$a_n = \sum \binom{n+p}{k+p} b_k, \qquad b_n = \sum (-1)^{n+k} \binom{n+p}{k+p} a_k. \tag{4}$$

Equations (4) have been used by L. Carlitz in [2]. It is worth noticing that the arrays of these coefficients are those for $p=0$ with the first p columns removed; thus for $p=1$, they are

$$\begin{array}{cccc} 1 & & & \\ 2 & 1 & & \\ 3 & 3 & 1 & \\ 4 & 6 & 4 & 1 \\ & \cdots & & \end{array} \qquad \begin{array}{cccc} 1 & & & \\ -2 & 1 & & \\ 3 & -3 & 1 & \\ -4 & 6 & -4 & 1 \\ & \cdots & & \end{array}$$

These several points show the orthogonality condition, equation (2), to be prolific in consequences. Another variation follows from the matrix equation: $BB^{-1}=I$, which implies $(BB^{-1})^p = B^pB^{-p} = I$. Writing $B^p = \{b_{ij}(p)\}$, we find that

$$b_{ij}(p) = \sum b_{ik}(p-1)\binom{k}{j} = p^{i+j}\binom{i}{j}, \qquad p = \pm 1, \pm 2, \cdots.$$

Hence no essentially new pair of inverse relations appears.

Turn now to numbers introduced by I. Lah [6]; they are defined by

$$(-x)_n = \sum L_{nk}(x)_k = (-x)(-x-1)\cdots(-x-n+1),$$

whose inverse is $(x)_n = \sum L_{nk}(-x)_k$ so that $\delta_{nm} = \sum L_{nk}L_{km}$. But (see problem 16 of Chapter 2 of [8])

$$L_{nk} = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}$$

and

$$\begin{aligned} \delta_{nm} &= \sum (-1)^{n+k} \frac{n!}{k!}\binom{n-1}{k-1}\frac{k!}{m!}\binom{k-1}{m-1} \\ &= \frac{n!}{m!} \sum (-1)^{n-1+k}\binom{n-1}{k}\binom{k}{m-1} \end{aligned}$$

or, using (2) $\delta_{nm} = (n!/m!)\delta_{n-1,m-1}$. This points the way to other modifications of (2), yielding more inverse relations; these are given $\delta_{nm} = \delta_{nm}F(n, m)$ if $F(n, n) = 1$, $F(n, m) \neq \infty$ for the range of n and m in question.

3. Stirling numbers. The Stirling numbers of first kind, $s(n, k)$, and second kind, $S(n, k)$, are usually defined by the inverse relations

$$(x)_n = \sum s(n, k)x^k, \qquad x^n = \sum S(n, k)(x)_k \tag{5}$$

with $(x)_n = x(x-1) \cdots (x-n+1)$. Hence, as is well known,

$$\delta_{nm} = \sum s(n, k)S(k, m) = \sum S(n, k)s(k, m) \tag{6}$$

and hence, as above,

$$a_n = \sum s(n, k)b_k, \qquad b_n = \sum S(n, k)a_k. \tag{5a}$$

All the parallel implications of (2) hold equally. The pair of relations, similar to (3),

$$a_n = \sum s(k, n)b_k, \qquad b_n = \sum S(k, n)a_k,$$

suggest a kind of moment, unfortunately nonexistent in probability and statistics. The numbers associated with the powers of matrices have been studied, however, by E. T. Bell in [1].

One example of further possibilities now open is as follows. Take the basic pair as

$$a_n(x) = \sum S(n, k)x^k, \qquad x^n = \sum s(n, k)a_k(x) \tag{5b}$$

and write

$$a_n(x; 1) - a_{n-1}(x; 1) = a_n(x) \tag{7}$$

so that

$$\begin{aligned} a_n(x; 1) &= a_n(x) + a_{n-1}(x) + \cdots + a_0(x) = \sum_{k=0}^{n} x^k \sum_{m=0}^{n} S(m, k) \\ x^n &= \sum s(n, k)[a_k(x; 1) - a_{k-1}(x; 1)] \\ &= \sum [s(n, k) - s(n, k+1)]a_k(x; 1) \end{aligned} \tag{8}$$

is a new pair. Since, with a prime denoting a derivative, it follows from $S(n, k) = S(n-1, k-1) + kS(n-1, k)$ that

$$a_n(x) = xa_{n-1}(x) + xa'_{n-1}(x), \qquad n = 1, 2, \cdots$$

it follows by (7) that

$$a_n(x; 1) = 1 + xa_{n-1}(x; 1) + xa'_{n-1}(x; 1), \qquad n = 1, 2, \cdots \tag{9}$$

which implies a simple recurrence relation for the coefficients.

This is readily generalized by writing

$$a_n(x; j) - a_{n-1}(x; j) = a_n(x; j-1), \qquad j = 1, 2, \cdots.$$

Then it is found similarly that $a_0(x; j) = 1$,

$$a_n(x;j) = \binom{n+j-1}{n} + xa_{n-1}(x;j) + xa'_{n-1}(x;j), \qquad n = 1, 2, \cdots. \tag{10}$$

Thus if

$$a_n(x;j) = \sum_{k=0} a_{nk}(j)x^k, \qquad j = 1, 2, \cdots \tag{11}$$

with $a_{n0}(j) = \binom{n+j-1}{n}$, $a_{00}(j) = 1$, $a_{nk}(j) = ka_{n-1,k}(j) + a_{n-1,k-1}(j)$, then

$$x^n = \sum_{k=0} b_{nk}(j)a_k(x;j) \qquad j = 1, 2, \cdots, \tag{12}$$

where

$$b_{nk}(j) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} s(n, k+i).$$

Similar results follow from the introduction of polynomials

$$\begin{aligned} \alpha_n(x; 1) + \alpha_{n-1}(x; 1) &= a_n(x) \\ \alpha_n(x; j) + \alpha_{n-1}(x; j) &= \alpha_n(x; j-1), \qquad j = 2, 3, \cdots. \end{aligned} \tag{13}$$

Indeed

$$\alpha_n(x;j) = (-1)^n\binom{n+j-1}{n} + x\alpha_{n-1}(x;j) + x\alpha_n'(x;j) \tag{14}$$

while $\beta_{nk}(j) = \sum_{i=0}^{j} \binom{j}{i} s(n, k+i)$, where

$$x^n = \sum_{k=0} \beta_{nk}(j)\alpha_k(x;j). \tag{15}$$

Similar developments may be obtained for polynomials associated with $b_n(x) = \sum s(n, k)x^k$.

4. Chebyshev polynomials. The Chebyshev polynomials $T_n(x) = \cos n\theta$, $\theta = \cos^{-1}x$, are associated with a pair of inverse relations which may be written as follows

$$a_n = \sum \binom{n}{k} b_{n-2k}, \qquad b_n = \sum (-1)^k \frac{n}{n-k}\binom{n-k}{k} a_{n-2k}. \tag{16}$$

It is assumed that both a_n and b_n are null for negative indices. The orthogonality they imply is

$$\begin{aligned} \delta_{m0} &= \sum (-1)^{m+j}\binom{n}{j}\frac{n-2j}{n-m-j}\binom{n-m-j}{m-j} \\ &= \sum (-1)^j \binom{n-2j}{m-j}\frac{n}{n-j}\binom{n-j}{j}. \end{aligned} \tag{17}$$

Indeed, if $a_n = \sum_{2k\leq n} a_{nk}b_{n-2k}$, $b_n = \sum_{2k\leq n} b_{nk}a_{n-2k}$ then

$$\delta_{m0} = \sum a_{nj}b_{n-2j,m-j} = \sum a_{n-2j,m-j}b_{nj}$$

with j not greater than the smaller of m and $n/2$.

The first half of (17) has been proved simply and directly by H. W. Gould [4]. A direct proof of (16) may be given as follows. Write

$$b_n(x) = \sum_{2k\leq n} (-1)^k \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}. \tag{18}$$

Then $b_0(x)=1$, $b_1(x)=x=xb_0(x)$, $b_2(x)=x^2-2$. For $n=3, 4, \cdots$, it is easy to show that

$$b_n(x) = xb_{n-1}(x) - b_{n-2}(x), \qquad n = 3, 4, \cdots \tag{19}$$

since

$$\frac{n}{n-k}\binom{n-k}{k} = \binom{n-k}{k} + \binom{n-k-1}{k-1}$$

or

$$xb_n(x) = b_{n+1}(x) + b_{n-1}(x), \qquad n = 2, 3, \cdots \tag{19a}$$

while $xb_1(x)=b_2(x)+2b_0(x)$, $xb_0(x)=b_1(x)$. If $x^n = \sum a_{nk}b_{n-2k}(x)$ then equation (19a) and its initial modifications imply

$$\begin{aligned} a_{nk} &= a_{n-1,k} + a_{n-1,k-1}, \qquad k < [n/2] \\ a_{2n,n} &= 2a_{2n-1,n-1} \\ a_{2n+1,n} &= a_{2n,n} + a_{2n,n-1} \end{aligned}$$

with brackets indicating integral part. These are the familiar recurrences for binomial coefficients and, along with boundary conditions, they prove (16).

It is worth noting that the alternative procedure of working from $a_n(x) = \sum_{2k\leq n} \binom{n}{k} x^{n-2k}$ is not so simple.

The "rotated" form of (16) is

$$a_n = \sum_{k=0} \binom{n+2k}{k} b_{n+2k}, \qquad b_n = \sum_{k=0} (-1)^k \frac{n+2k}{n+k}\binom{n+k}{k} a_{n+2k}. \tag{20}$$

If new polynomials $b_n(x; 1)$ are defined by

$$b_n(x; 1) - b_{n-2}(x; 1) = b_n(x) \tag{21}$$

with $b_n(x)$ the polynomial defined above (equation (18)), then

$$b_0(x; 1) = b_0(x) = 1,\ b_1(x; 1) = b_1(x) = x,\ b_2(x; 1) = b_2(x) + b_0(x) = x^2 - 1$$

and, since

$$\frac{n}{n-k}\binom{n-k}{k} = \binom{n-k}{k} + \binom{n-k-1}{k-1}, \tag{22}$$

$$b_n(x;1) = \sum_{2k\leq n} (-1)^k \binom{n-k}{k} x^{n-2k}$$

and

$$\begin{aligned} x^n &= \sum \binom{n}{k} b_{n-2k}(x) = \sum \binom{n}{k} [b_{n-2k}(x;1) - b_{n-2-2k}(x;1)] \\ &= \sum \left[\binom{n}{k} - \binom{n}{k-1}\right] b_{n-2k}(x;1). \end{aligned} \tag{23}$$

The polynomial $b_n(x;\ 1)$ is the Chebyshev polynomial $U_n(x/2)$ where $U_n(x) = \sin(n+1)\theta/\sin\theta$, $\cos\theta = x$. Extension of (21) to

$$b_n(x;j) - b_{n-2}(x;j) = b_n(x;j-1), \qquad j = 2, 3, \cdots$$

leads to nothing interesting. On the other hand, $\beta_n(x;\ 1)+\beta_{n-2}(x;\ 1) = b_n(x)$ yields the pair

$$a_n = \sum \binom{n+1}{k} b_{n-2k}, \quad b_n = \sum (-1)^k \frac{n+1}{n+1-k}\binom{n+1-k}{k} a_{n-2k}, \tag{24}$$

whose orthogonality relation is just (17) with n replaced by $n+1$. Thus (24) may be generalized to

$$\begin{aligned} a_n &= \sum \binom{n+p}{k} b_{n-2k} \\ b_n &= \sum (-1)^k \frac{n+p}{n+p-k}\binom{n+p-k}{k} a_{n-2k}, \qquad p = 0, 1, 2, \cdots. \end{aligned} \tag{25}$$

Returning to (17), rewritten as

$$\begin{aligned} \delta_{m0} &= \sum (-1)^{m+j}\binom{n}{j}\left[\binom{n-m-j}{m-j} + \binom{n-m-j-1}{m-j-1}\right] \\ &= \sum (-1)^j \binom{n-2j}{m-j}\left[\binom{n-j}{j} + \binom{n-j-1}{j-1}\right] \end{aligned} \tag{17a}$$

two identities may be screened out, namely

$$1 = \sum_{j=0}^{m} (-1)^{m+j}\binom{n}{j}\binom{n-m-j}{m-j} \tag{26}$$

$$\binom{2m-1}{m} = \sum_{j=0} (-1)^{m+j+1}\binom{n}{j}\binom{n-m-j-1}{m-j-1}. \tag{27}$$

The Chebyshev inverse relations given in (16) are an instance of the following, due to H. W. Gould [4]

$$
\begin{aligned}
f(a) &= \sum_{k=0} \binom{a}{k} F(a+bk-k) \\
F(a) &= \sum_{k=0} (-1)^k \frac{a}{a+bk}\binom{a+bk}{k} f(a+bk-k).
\end{aligned} \tag{28}
$$

For $b=-2$, in present notation (28) may be written

$$
a_n = \sum_{3k\leq n} \binom{n}{k} b_{n-3k}, \qquad b_n = \sum_{3k\leq n} (-1)^k \frac{n}{n-2k}\binom{n-2k}{k} a_{n-3k}
$$

and the corresponding orthogonal relation may be written

$$
\begin{aligned}
\delta_{m0} &= \sum_{j=0}^{m} (-1)^{m+j}\binom{n}{j}\left[\binom{n-2m-j}{m-j} + 2\binom{n-2m-j-1}{m-j-1}\right] \\
&= \sum_{j=0} (-1)^j \binom{n-3j}{m-j}\left[\binom{n-2j}{j} + 2\binom{n-2j-1}{j-1}\right].
\end{aligned}
$$

Then, if $f_{nm} = \sum_{j=0}^{m} (-1)^{m+j}\binom{n}{j}\binom{n-2m-j}{m-j}$ with $f_{n0}=1$, $f_{n1}=2$, it is found by recurrence that

$$
f_{nm} = f_{n-1,m} = f_{0m} = (-1)^m \binom{-2m}{m} = \binom{3m-1}{m}.
$$

Hence,

$$
\begin{aligned}
\binom{3m-1}{m} &= \sum_{0}^{m} (-1)^{m+j}\binom{n}{j}\binom{n-2m-j}{m-j} \\
&= 2\sum_{0}^{m-1} (-1)^{m+j+1}\binom{n}{j}\binom{n-2m-j-1}{m-j-1}.
\end{aligned} \tag{29}
$$

It is worth noting that the first form of (29) may also be written

$$
\frac{2}{3}\binom{3m}{m} = \sum_{0}^{m} (-1)^{m+j}\binom{3m}{m}\binom{m}{j}\binom{n}{3m}\binom{n-j}{2m}^{-1}
$$

or

$$
\frac{2}{3}\binom{n}{3m}^{-1} = \sum_{0}^{m} (-1)^{m+j}\binom{m}{j}\binom{n-j}{2m}^{-1}
$$

an instance of the protean character of binomial identities.

5. Associated Legendre polynomials. If $P_n(x)$ is a Legendre polynomial, the associated polynomial in question is

$$q_n(x) = (1 - x)^n P_n\left(\frac{1+x}{1-x}\right) = \sum \binom{n}{k}^2 x^k. \tag{30}$$

What is its inverse? More precisely, if

$$x^n = \sum (-1)^{n+k} \beta_{nk} q_k(x) \tag{31}$$

what are the coefficients β_{nk}?

Of the many recurrences for $q_n(x)$, the following (cf. problem 15 of Chapter 7 of [8]) is apt for present purposes:

$$q_n'(x) + xq_n''(x) = n^2 q_{n-1}(x)$$

with primes denoting derivatives. By (31)

$$nx^{n-1} + (n-1)x^{n-1} = \sum (-1)^{n+k}\beta_{nk}[q_k'(x) + xq_k''(x)] = \sum (-1)^{n+k}\beta_{nk}k^2 q_{k-1}(x)$$

or $\sum (-1)^{n-1+k}\beta_{n-1,k}n^2 q_k(x) = \sum (-1)^{n+k}\beta_{nk}k^2 q_{k-1}(x)$. Hence,

$$\beta_{nk} = (n/k)^2 \beta_{n-1,k-1} = \binom{n}{k}^2 \beta_{n-k,0} = \binom{n}{k}^2 \beta_{n-k}$$

with the last a definition. Using this in (31) yields

$$x^n = \sum (-1)^{n+k} \binom{n}{k}^2 \beta_{n-k} q_k(x). \tag{31a}$$

Then $\beta_0 = 1$ and since $q_k(0) = 1$

$$0 = \sum (-1)^{n+k} \binom{n}{k}^2 \beta_{n-k}, \tag{32}$$

a recurrence which may be taken as a definition of the numbers. The first few values are

n	0	1	2	3	4	5	6	7
β_n	1	1	3	19	211	3651	90921	3091513

It is tempting to suppose that (32) may be replaced by some simpler linear or quasi-linear recurrence like

$$\sum_{j=0}^{k} A_j(n)\beta_{n+j} = 0,$$

where the $A_j(n)$ are polynomials in n, but Professor Carlitz has proved (private communication) that the latter is impossible. The relations $q_{2n+1}(-1) = 0$ and $q_{2n}(-1) = (-1)^n\binom{2n}{n}$ lead, however, to

$$1 = \sum (-1)^k \beta_{n-2k} \binom{n}{2k}^2 \binom{2k}{k}. \tag{33}$$

The numbers β_n appear in other notations in L. Carlitz [3], where they arise in the expansion

$$\frac{1}{J_0(2\sqrt{z})} = \sum_{n=0} \beta_n \frac{z^n}{n!^2}$$

with $J_0(z)$ the Bessel function. Carlitz also gives the inverse relations

$$(34) \qquad w_n(x) = \sum \binom{n}{k}^2 \beta_k(-x)^{n-k}, \qquad x^n = \sum (-1)^k \binom{n}{k}^2 w_k(x)$$

which however have the same orthogonality as (30) and (31a), namely

$$(35) \qquad \delta_{nm} = \sum (-1)^{k+m} \binom{n}{k}^2 \binom{k}{m}^2 \beta_{k-m} = \sum (-1)^{k+m} \binom{n}{k}^2 \binom{k}{m}^2 \beta_{n-k}.$$

To find a pair of inverses associated with Legendre polynomials involving only binomial coefficients, consider $r_n(x) = (1+x)^n q_n(x/(1+x)) = P_n(1+2x)$. Then

$$(36) \qquad \begin{aligned} r_n(x) &= \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k} x^k = \sum_{k=0}^{n} \binom{n+k}{2k} s_k(x), \\ s_k(x) &= \binom{2k}{k} x^k. \end{aligned}$$

For simplicity, this may be examined for the related function

$$(37) \qquad \rho_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k} x^k.$$

The recurrence for $\rho_n(x)$ is found to be

$$\rho_n(x) = (2+x)\rho_{n-1}(x) - \rho_{n-2}(x), \qquad n = 2, 3, \cdots$$

while $\rho_1(x) = (2+x)\rho_0(x) - 1 = 1+x$. Using these as before, it is found that

$$(38) \qquad x^n = \sum (-1)^{n+k} \frac{2k+1}{n+k+1} \binom{2n}{n+k} \rho_k(x)$$

which is the inverse of (37).

6. Generating functions. Exponential generating functions lead directly to a number of inverse relations with binomial coefficients. The simplest pair, Eq. (1b), is equivalent to $\exp xa = \exp x(b+1)$, $a^n \equiv a_n$, $b^n \equiv b_n$, $\exp x(a-1) = \exp xb$, with a, b umbral or Blissard variables. More generally the inverse relations

$$(39) \qquad a_n = \sum \binom{n}{k} c_{n-k} b_k, \qquad b_n = \sum \binom{n}{k} \gamma_{n-k} a_k$$

are equivalent to $\exp xa = \exp(b+c)$, $\exp xb = \exp(a+\gamma)$, and it is necessary that $\exp(c+\gamma)=1$. Of course, a, b, c, γ are all umbral.

As a first example, take $\exp xc=(e^t-1)t^{-1}$ so that $c^n \equiv c_n=(n+1)^{-1}$. Then $\exp x\gamma = t(e^t-1)^{-1}=\exp Bt$ with B_n a Bernoulli number (in the even suffix notation). Then

$$a_n = \sum \binom{n}{k} (n-k+1)^{-1} b_k, \qquad b_n = \sum \binom{n}{k} B_{n-k} a_k \tag{40}$$

are inverse relations.

Next, consider $2 \exp xa = \exp x(b+1) + \exp x(b-1) = (e^x+e^{-x}) \exp xb$. Then $\exp x\gamma = 2(e^x+e^{-x})^{-1} = \exp xE$ with E_n a Euler number ($E_{2n+1}=0$), and

$$a_n = \sum \binom{n}{2k} b_{n-2k}, \qquad b_n = \sum \binom{n}{2k} E_{2k} a_{n-2k} \tag{41}$$

is an inverse pair. Its "rotated" form is

$$a_n = \sum \binom{n+2j}{2j} b_{n+2j}, \qquad b_n = \sum \binom{n+2j}{2j} E_{2j} a_{n+2j}. \tag{42}$$

For sums having only odd binomial coefficients the generating function relation is $\exp xa = \frac{1}{2}(e^x-e^{-x}) \exp xb = (1/2x)(e^x-e^{-x})x \exp xb$ and

$$\exp x\gamma = \frac{2x}{e^x - e^{-x}} = \exp xd,$$

the last in a notation convenient for present purposes. The numbers d_n apparently have no patronymic. The inverse relations are

$$a_n = \sum \binom{n}{2k+1} b_{n-2k-1}, \qquad nb_{n-1} = \sum \binom{n}{2k} d_{2k} a_{n-2k}. \tag{43}$$

Now turn to an instance of the Lagrange theorem in the form

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} D^{n-1}[f'(x)e^{nx}]_{x=0} \tag{44}$$

with $w=ze^{-z}$, $D=d/dx$, and the prime denoting a derivative. Then, if $f(z) = \exp zb$, $b^n \equiv b_n$

$$a_n = D^{n-1}[f'(x)e^{nx}]_{x=0} = b(b+n)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k} b_{k+1} = \sum_{k=0}^{n} \binom{n}{k} n^{n-1-k} k b_k.$$

But, directly from (44),

$$\exp zb \equiv b_0 + \sum_{n=1}^{\infty} \frac{z^n e^{-nz}}{n!} a_n,$$

which implies $b_n = \sum (-1)^{n+k}\binom{n}{k}k^{n-k}a_k$. The inverse pair

$$(45) \qquad a_n = \sum \binom{n}{k} n^{n-1-k}kb_k, \qquad b_n = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}a_k$$

is an instance of the Abel inverses given by H. W. Gould [5]. Some examples of its use are as follows.

Take $f(z) = (1-z)^{-1}$, so that $b_n = n!$. But, with $S(n, k)$ a Stirling number of the second kind, Δ the difference operator,

$$n! = n!S(n, n) = \Delta^n 0^n = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}k^k$$

and by (45)

$$(46) \qquad n^n = \sum \binom{n}{k} n^{n-1-k}k\cdot k!$$

a relation appearing in [9].

Next, take $f(z) = e^{-z}(1-x)^{-1} = \exp zD$, with $D_n = \Delta^n 0!$, a displacement number (=subfactorial). Then $f'(z) = ze^{-z}(1-z)^{-2}$ and

$$a_n = \sum \binom{n-1}{k} D^k[x(1-x)^{-2}]D^{n-1-k}(e^{(n-1)x})$$

with both derivatives evaluated at $x=0$. Thus

$$a_n = \sum \binom{n-1}{k} k\cdot k!(n-1)^{n-1-k} = (n-1)^n$$

the last by use of (46). Hence, by (45)

$$(47) \quad (n-1)^n = \sum \binom{n}{k} n^{n-1-k}kD_k, \qquad D_n = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}(k-1)^k.$$

The first of these appeared in [9], the second is due to H. J. Ryser [10].

Next take $f(z) = \exp xz$, so that $b_n = x^n$; then (45) becomes

$$(48) \quad a_n \equiv a_n(x) = \sum \binom{n}{k} n^{n-1-k}kx^k, \qquad x^n = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}a_k(x).$$

These are actually relations for enumerating cycle-free mappings or labeled forests of rooted trees; the coefficient of x^k in $a_n(x)$ is the number of forests with n labeled points and k rooted trees, which is to say that $a_n(x)$ is the enumerator of labeled rooted forests with n labeled points by number of rooted trees. The reader may be reminded that $R(y)$, the enumerator of rooted trees with all points labeled by number of points, satisfies the equation $ze^{-z} = y$ with $z = R(y)$, and if

$$a(x, y) = \sum_{n=1}^{\infty} a_n(x)y^n/n!$$

then $a(x, y) = \exp xR(y)$. Now

$$a(1, y) = \sum a_n(1)y^n/n! = \exp R(y) = y^{-1}R(y) = \sum_1^{\infty} (n + 1)^{n-1}y^n/n!.$$

Hence by (48)

$$(49)\quad (n + 1)^{n-1} = \sum \binom{n}{k} n^{n-1-k}k, \qquad 1 = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}(k + 1)^{k-1}.$$

The corresponding enumeration for labeled forests of (free) trees goes as follows. First, if $A_n(x)$ is the enumerator of forests of trees with n labeled points by number of trees, then

$$(50)\quad \begin{aligned} A(x, y) &= \sum_{n=1} A_n(x)y^n/n! = \exp x[R(y) - R^2(y)/2] \\ &= \exp x(z - z^2/2), \qquad z = R(y), \qquad ze^{-z} = y \end{aligned}$$

and if $\exp x(z - z^2/2) = \sum B_n(x)z^n/n!$ then

$$(51)\quad B_n(x) = \sum_{k=0} (-1)^k \frac{n!}{2^k k!(n - 2k)!} x^{n-k}.$$

Thus by (46)

$$(52)\quad A_n(x) = \sum \binom{n}{k} n^{n-1-k}kB_k(x), \qquad B_n(x) = \sum (-1)^{n+k}\binom{n}{k} k^{n-k}A_k(x).$$

The first of equations (52) is equivalent to a result of Alfred Rényi [7]; the second, its inverse, seems to be new. The result of Rényi just mentioned, in present notation, is as follows. Write

$$A_n(x) = \sum A_{nj}x^j, \qquad B_n(x) = \sum B_{nj}x^j;$$

then by the first of (52), and by (51)

$$(53)\quad A_{nj} = \sum \binom{n}{k} n^{n-1-k}kB_{kj} = \frac{1}{j!}\sum_{k=0}^{j} (-\tfrac{1}{2})^k \binom{n-1}{k+j-1} n^{n-j-k}(j + k)!$$

Thus $A_{n1} = n^{n-2}$ (the number of labeled trees with n points),

$A_{n2} = n^{n-4}(n - 1)(n + 6)/2$, $A_{n3} = n^{n-6}(n - 1)(n - 2)(n^2 + 13n + 60)/8$.

On the other hand, the second half of (52), or

$$A_{n+1}(x) = x\sum \binom{n}{k} T_{k+1}A_{n-k}(x), \qquad T_k = k^{k-2}$$

leads to

$$A_{nn} = 1$$

$$A_{n,n-1} = \binom{n}{2}$$

$$A_{n,n-2} = 3\binom{n+1}{4}$$

$$A_{n,n-3} = 15\binom{n+2}{6} + \binom{n}{4}$$

$$A_{n,n-6} = 105\binom{n+3}{8} + 15\binom{n+1}{6} + 5\binom{n}{5}$$

but there seems to be no simple general form.

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THE NUMBER OF PARTITIONS OF A SET

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Let S be a finite nonempty set with n elements. A *partition* of S is a family of disjoint subsets of S called "blocks" whose union is S. The number B_n of distinct partitions of S has been the object of several arithmetical and combinatorial investigations. The earliest occurrence in print of these numbers has never been traced; as expected, the numbers have been attributed to Euler, but an explicit reference to Euler has not been given, and Bell [7] doubts that it can