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Embedded Pascal Triangles and Its Application for Minimal Cut Sets of Fault Tree Analysis

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Abstract: In this paper, the author presents the Gu elph expansion, which is used for binomial expansion of the form $(\omega+\lambda 1)$ $(\omega+\lambda 2)$ $(\omega+\lambda 3)\ldots$ $(\omega+\lambda n)$. Such expansion can be extended to a generalized multinomial expansion of the form $(x1+x2+x3+\ldots\ xk)^n$, which is referred to as the Embedded Pascal's Triangles (EPTs) for polynomial expansion. Such triangles are the higher orders of the classical Pascal triangle representing the binomial expansion. An efficient algorithm for calculating the coefficients of such multinomial expansion (calculating the entries of the EPTs) and the corresponding monomials of these coefficients is highlighted. Such expansion could have many application varying from neutronics to possible genetics and genomics. In this paper, however, the author demonstrates the application of polynomial expansion via the EPTs expansion to identify the minimal cut sets (82 of them) for fault tree analysis of a system's vacuum loss fault, as an undesired event, in an engineering redundant system. The chosen example for analysis has been modeled by the multinomial expansion of $(x1+x2+x3+\ldots\ x18)^2$, to generate the required minimal cut sets.

Key words: Fault tree analysis . minimal cut set . embedded pascal's triangles . polynomial expansion . modeling

INTRODUCTION

A certain generalized binomial expansion that has previously been introduced by [1, 2] for polynomials of the form $(\omega+\lambda_1)$ $(\omega+\lambda_2)$ $(\omega+\lambda_3)\ldots$ $(\omega+\lambda_n)$ is presented in this paper. Such expansion is named as the Guelph expansion. The coefficients of the asscociated monomials of the expansion are determined from the roots $(-\lambda i's)$ of such polynomial as to be demonstrated. Furthermore, the Guelph expansion its self represents the inverse problem, that is knowing the coeffecients of the polynomial, one can determine the roots of such polynomial [3]. In the case of distinct λi's, the coefficients are obtained by summing all the combinations of the products of k of the λi's, k=0,…,n. The number of such combinations is determined by the binomial coefficient, $T_{n,k} = \binom{n}{k}$, where n represents the degree of the polynomial and k, the number of λi's involved. Tn,k is in fact the (k+1)th entry of the (n+1)th row of Pascal's triangle. A basic Pascal's triangle can be generated vertically, with the entries of row n+1 constituting the coefficients of the binomial expansion of $(x+y)^n$. The expansion of polynomials involving more than two variables raised to the power n, such as $(x_1+\ldots x_l)^n$ is then derived. As it turns out, the coefficients of the monomials contained in the resulting polynomial expansion can be determined in terms of the coefficients of the monomials included in the expansion of $(x_1+\ldots x_{l-1})^n$ and the rows of Pascal's triangles of successive orders expanded horizontally, hence the concept of Embedded Pascal's Triangles (EPT's) [4]. A recursive algorithm is derived for generating the coefficients of such expansions [5]. Many applications for the EPT's can be explored, in dice throwings, that is, throwing a dice with six faces represented by x1,x2,..x6, where the power n represents the number of throwings of the dice. In such experiments one can determine from the coeffecients of the EPT's expansion the probabilities of the appearings of the respected faces due to n throwings. In genetics, one can demonstrate the probabilities of the combinations of n outcomes of boys, x and girls, y, from the coefficients of the expansion $(x+y)^n$. Also, it is appealing to explore the possible application of the EPT's expansion for genomic

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studies, via the expansion of $(A+T+C+G)^n$ representing an isolated DNA strip of length n structred with the known A, T, C, G nucletoids.

The author, in this paper, studies the EPT's expansion for application in engineering safety analysis, namely, identifying the minimal cut sets (82 of them) for fault tree analysis of a system's vacuum loss fault, as an undesired event, in an engineering redundant system [6]. The chosen example for analysis has been modeled by the multinomial expansion of $(x1+x2+x3+\ldots x18)^2$ to generate the required minimal cut sets.

THEORATICAL ANALYSIS

Guelph expansion and the Pascal triangle: The Guelph expansion is a generalization of a binomial expansion of the form:

$$\prod_{i=1}^{n}(\omega+\lambda_i) = (\omega+\lambda_1)(\omega+\lambda_2)(\omega+\lambda_3)....(\omega+\lambda_n) \tag{1}$$

To derive the expansion as given in [1, 2], one starts with simple examples:
For n=1, one has

$$\prod_{i=1}^{1}(\omega+\lambda_i) = \omega+\lambda_1 \tag{2}$$

for n=2

$$\prod_{i=1}^{2}(\omega+\lambda_i) = (\omega+\lambda_1)(\omega+\lambda_2) = \omega^2 + (\lambda_1+\lambda_2)\omega + \lambda_1\lambda_2 \tag{3}$$

for n=3,

$$\prod_{i=1}^{3}(\omega+\lambda_i) = (\omega+\lambda_1)(\omega+\lambda_2)(\omega+\lambda_3) = \omega^3 + (\lambda_1+\lambda_2+\lambda_3)\omega^2 + (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\omega + \lambda_1\lambda_2\lambda_3 \tag{4}$$

and for a product of n terms, the following general formula can be deduced:

$$\prod_{i=1}^{n}(\omega+\lambda_i) = \sum_{k=0}^{n}\omega^{n-k}\sum_{T_{n,k}=\binom{n}{k}}\lambda_{\ldots}\overset{k}{\ldots}\ldots\lambda \tag{5}$$

This generalization of the binomial expansion was referred to as the Guelph expansion in [2]. The notation $\sum_{T_{n,k}=\binom{n}{k}}\lambda_{\ldots}\overset{k}{\ldots}\ldots\lambda$ can be defined as follows: For each possible combination of k elements of the set $\lambda_1\ldots\lambda_n$, take the product of those λi's and then add up these terms. The number of such combinations is the binomial coefficient which is given by

$$T_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{6}$$

For example, consider $\sum_{T_{4,3}=\binom{4}{3}}\lambda_{\ldots}\overset{3}{\ldots}\ldots\lambda$, in which case n=4 and k=3. Given that n= 4, one has the set λ_1, λ_2, λ_3, λ_4 and since k=3, three of the four elements are to be multiplied at a time giving $(\lambda_1\lambda_2\lambda_3),(\lambda_1\lambda_2\lambda_4),(\lambda_2\lambda_3\lambda_4),(\lambda_3\lambda_4\lambda_1)$, the number of possible combinations being $T_{4,3} = \binom{4}{3} = \frac{4!}{3!(1)!} = 4$. Thus,

$$\sum_{T_{4,3}=\binom{4}{3}=4}\lambda_{\ldots}\overset{3}{\ldots}\ldots\lambda = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1$$

Then according to Equation (5), one can use Guelph expansion to expand:

$$\prod_{i=1}^{4}(\omega+\lambda_i)=\sum_{k=0}^{4}\omega^{4-k}\sum_{T_{4,k}=\binom{4}{k}}\lambda_{\ldots}^{k}\ldots\lambda=\omega^4\sum_{T_{4,0}=\binom{4}{0}=1}\lambda_{\ldots}^{0}\ldots\lambda+\omega^3\sum_{T_{4,1}=\binom{4}{1}=4}\lambda_{\ldots}^{1}\ldots\lambda+\omega^2\sum_{T_{4,2}=\binom{4}{2}=6}\lambda_{\ldots}^{2}\ldots\lambda$$
$$+\omega^1\sum_{T_{4,3}=\binom{4}{3}=4}\lambda_{\ldots}^{3}\ldots\lambda+\omega^0\sum_{T_{4,4}=\binom{4}{4}=1}\lambda_{\ldots}^{4}\ldots\lambda$$

where

$$\sum_{T_{4,0}=\binom{4}{0}=1}\lambda_{\ldots}^{0}\ldots\lambda=1$$
$$\sum_{T_{4,1}=\binom{4}{1}=4}\lambda_{\ldots}^{1}\ldots\lambda=\lambda_1+\lambda_2+\lambda_3+\lambda_4$$
$$\sum_{T_{4,2}=\binom{4}{2}=6}\lambda_{\ldots}^{2}\ldots\lambda=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4$$
$$\sum_{T_{4,3}=\binom{4}{3}=4}\lambda_{\ldots}^{3}\ldots\lambda=\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_2\lambda_3\lambda_4+\lambda_3\lambda_4\lambda_1$$
$$\sum_{T_{4,4}=\binom{4}{4}=1}\lambda_{\ldots}^{4}\ldots\lambda=\lambda_1\lambda_2\lambda_3\lambda_4$$

that is

$$\prod_{i=1}^{4}(\omega+\lambda_i)=\omega^4+(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\omega^3+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\omega^2$$
$$+(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_2\lambda_3\lambda_4+\lambda_3\lambda_4\lambda_1)\omega+\lambda_1\lambda_2\lambda_3\lambda_4$$

Clearly, for single-valued λI's, this polynomial has the expansion,

$$(\omega+\lambda)^4=\omega^4+4\lambda\omega^3+6\lambda^2\omega^2+4\lambda^3\omega+\lambda^4$$

The coeffecients of the expansion (the binomial coeffecients) represents the number of possible combinations of the roots of the polynomial (that is non-combinations, single combinations, two combinations, three combinations, untill n-combinations. Such coefficients are actually the entries of the well known Pascal triangle. Table 1 presents the generation of Pascal triangle using the n's and k's of the binomial coefficient.

Embedded Pascal triangles (EPT's): The expansion given in Equation (5) is now extended to expressions consisting of powers of sums involving I terms. The resulting expansion is given as [4]:

$$\left(\sum_{i=1}^{I}x_i\right)^n=\sum_{\substack{k=0\ldots n\\ k'=0\ldots k\\ k''=0\ldots k'\\ k'''=0\ldots k''\\ \vdots\\ k^{(I-2)}=0\ldots k^{(I-3)}}}(T_{n,k}T_{k,k'}T_{k',k''}\ldots T_{k^{(I-3)},k^{(I-2)}})\,x_1^{n-k}x_2^{k-k'}x_3^{k'-k''}\ldots x_{I-1}^{k^{(I-3)}-k^{(I-2)}}x_I^{k^{(I-2)}} \tag{7}$$

Table 1: Pascal's triangle

$T_{n,k}=\binom{n}{k}=\frac{n!}{k!(n-k)!}$	k=0	k=1	k=2	k=3	k=4
n=0	1				
n=1	1	1			
n=2	1	2	1		
n=3	1	3	3	1	
n=4	1	4	6	4	1

where k, k', k'', k''',…,$k^{(I-2)}$ can take on the values that are specified under the summation sign. The explicit expansion specified by Equation (7) is obtained by successive application of the binomial expansion specified by Equation (5). For instance, when I=4, one would first expand $(x1+ (x2+x3+x4))^n$, then expand $(x2+(x3 +x4))^k$ and finally, $(x3+x4)^{k'}$, resulting in a representation expressed in terms of a triple sum involving the products ($T_{n,k}$ $T_{k,k'}$ $T_{k',k''}$). Table 2 [4] demonstrates the expansions of binomials, trinomials and quadronomials raised to powers n=0, 1, 2, 3 and 4. Although, the coefficients of the expansion can be calculated using the T's, one can deduce such triangle that comprises r0,r1,…,rc as its rows shall be referred to as a Pascal's triangle of order c and denoted Δc. Observe that the rows of embedded coefficients by inspections. Such inspection [5] can be developed by Letting rk denote a vector representing the $(k+1)^{th}$ row of Pascal's triangle, a Pascal's triangles of successive orders are being stacked horizontally. Table 2 illustrates how the rows of embedded Pascal's triangles are stacked in order to determine the coefficients of the expansions. Note that a basic Pascal's triangle is usually generated vertically and that the components of nth row are the coefficients of the binomial expansion of $(x+y)^n$. As seen from Table 2, when n is equal to 3 or 4, the expansions for trinomials and quadrinomials raised to the power n, that is, $(x+y+z)^n$ and $(x+y+z+w)^n$, are obtained by making use of stacked rows of Pascal's triangles. Consider for instance $(x+y+z+w)^3$.

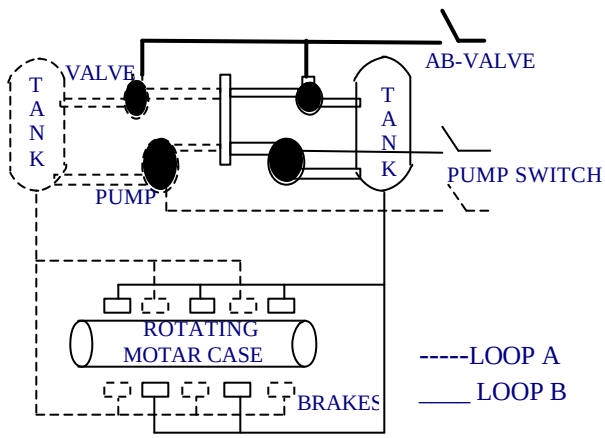

Fig. 1: Redundant mechanical system for vacuum pressure provision (Loops A and B) [6]

Table 2: Embedded Pascal's triangles and the determination of the coefficients of the polynomial expansions [4]

Expansion	Embedded Pascal's Triangles (white background) and Polynomial Coefficients of the expansion (shaded)
(x+y)^0	1
(x+y+z)^0	1
(x+y+z+w)^0	1
(x+y)^1	1 1
(x+y+z)^1	1 1 1
(x+y+z+w)^1	1 1 1 1
(x+y)^2	1 2 1
(x+y+z)^2	1 1 1 1 2 1
	1 2 2 1 2 1
(x+y+z+w)^2	1 1 1 1 1 1 1 1 2 1
	1 2 2 2 1 2 2 1 2 1
(x+y)^3	1 3 3 1
(x+y+z)^3	1 1 1 1 2 1 1 3 3 1
	1 3 3 3 6 3 1 3 3 1
(x+y+z+w)^3	1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 3 3 1
	1 3 3 3 3 6 6 3 6 3 1 3 3 3 6 3 1 3 3 1
(x+y)^4	1 4 6 4 1
(x+y+z)^4	1 1 1 1 2 1 1 3 3 1 1 4 6 4 1
	1 4 4 6 12 6 4 12 12 4 1 4 6 4 1
(x+y+z+w)^4	1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 3 3 1 1 1 1 1 2 1 1 3 3 1 1 4 6 4 1
	1 4 4 4 6 12 12 6 12 6 4 12 12 12 4 12 4 12 12 4 1 4 4 6 12 6 4 12 12 4 1 4 6 4 1

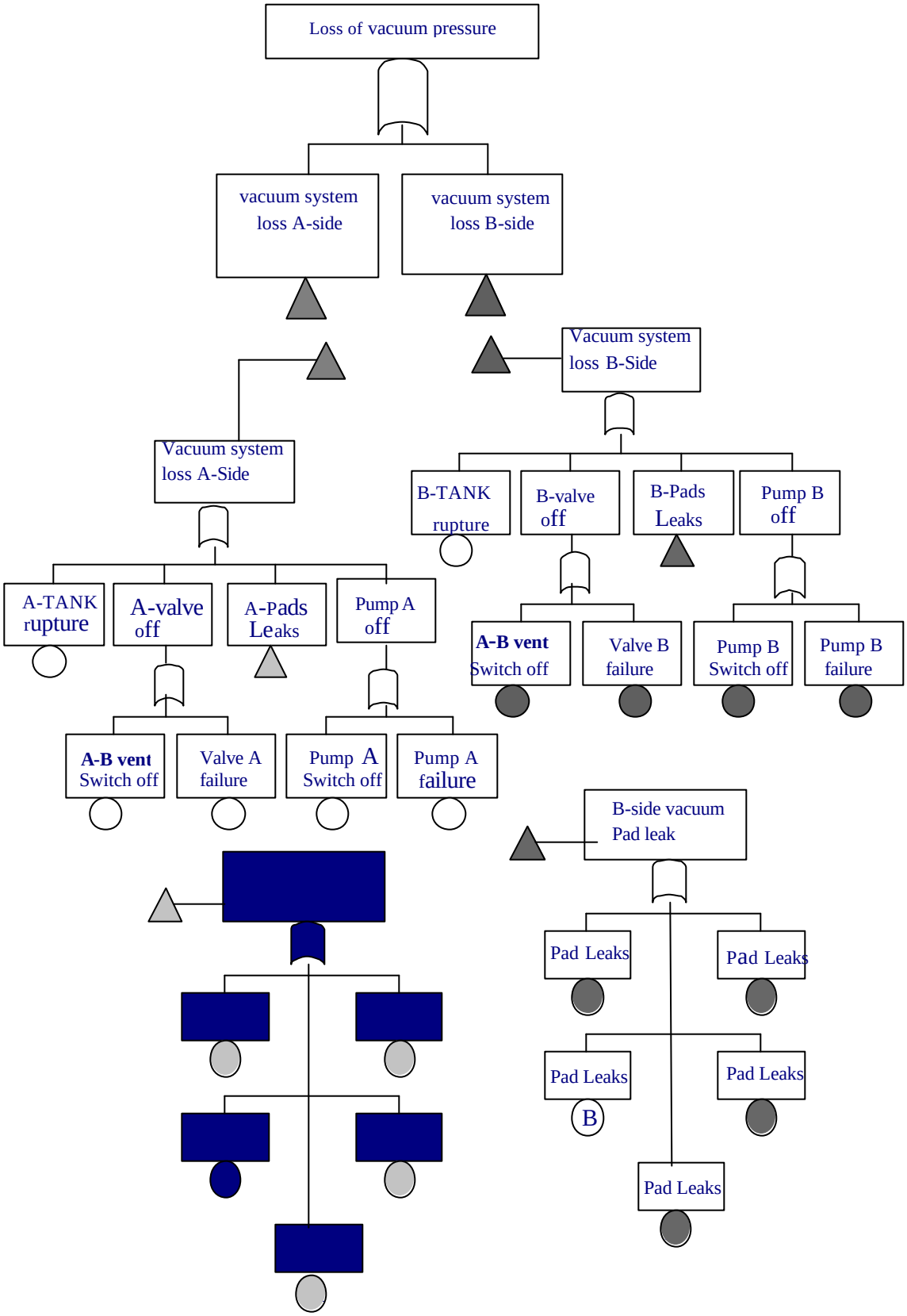

Fig. 2: Fault tree diagram for redundant system vacuum loss

Observe that the entries appearing on the upper line are (r0,r0,r1,r0,r1,r2,r0,r1,r2,r3)=u3 and that this vector contains the stacked rows of the embedded triangles, Δ0, Δ1, Δ2, Δ3. The coefficients of the monomials in the expansion of $(x+y+z+w)^3$, which appear on the lower line, are then obtained by multiplying the ten coefficients of the expansion of $(x+y+z)^3$ by the components of the ten corresponding subvectors comprising u3, that is, (1×r0, 3×r0, 3×r1, 3×r0, 6×r1, 3×r2, 1×r0, 3×r1, 3×r2,1×r3)= (1,3,3,3,3,6,6,3,6,3,1,3,3,3,6,3,1,3,3,1). An efficient algorithm [5] has been developed using mathematica software to expand such general multinomial expansion of the form given by Eq. (7).

APPLICATION OF EPT's FOR DETERMINING MINIMAL CUT SETS [6]

Mechanical system description: Figure 1 shows (www.fault-tree.net/presents-html/cutset/cut-20.htm) a redundant mechanical system (loops A and B) with two tanks, two pumps with their two associated switches, two valves with their single associated switch and two sets of pads (5 each) as breaks applied to a rotating motar case.
The fault tree analysis diagram representing the system's vacuum loss fault as an undesired event is shown in Fig. 2.

Modeling the minimal cut sets using EPT's polynomial expansion: Figure 2 presents the fault tree diagram leading to loss of vacuum pressure. There are many combinations of faults that cause the failure of the system. However, there are minimum cut sets which make the system in a failure (halts the system). Such sets are represented by combinations of 9 basic events from loop A denoted by the circles in the fault tree diagram) and 9 basic events of loop B plus a common event which is the failure of the AB-valve switch. This can be modelled by the multinomial expansion of 18 variables representing the basic events; 9 events for each loop and since a combination of two events (one from each loop) will suffice to bring the sytem into halt, one raises the multinomials to power of 2; that is: one has to expand the following multinomial:

$$(x1+x2+x3+\ldots x18)^2 \tag{8}$$

$1X1^2$+2X1X2+2X1X3+.....+2X1X6+2X1X7+2X1X8+2X1X9+.2X1X10+2X1X11+2X1X12+2X1X13+2X1X14+2X1X15+2X1X16+2X1X17+2X1X18+

$1X2^2$+2X2X3+2X2X4+.....+2X2X7+2X2X8+2X2X9+.2X2X10+2X2X11+2X2X12+2X2X13+2X2X14+2X2X15+2X2X16+2X2X17+2X2X18+

$1X3^2$+2X3X4+2X3X5+ .....+2X3X7+2X3X8+2X3X9+.2X3X10+2X3X11+2X3X12+2X3X13+2X3X14+2X3X15+2X3X16+2X3X17+2X3X18+

$1X4^2$+2X4X5+.....+2X4X8+2X4X9+.2X4X10+2X4X11+2X4X12+2X4X13+2X4X14+2X4X15+2X4X16+2X4X17+2X4X18+

$1X5^2$+2X5X6+.....+2X5X8+2X5X9+.2X5X10+2X5X11+2X5X12+2X5X13+2X5X14+2X5X15+2X5X16+2X5X17+2X5X18+

$1X6^2$+2X6X7+.....+2X6X9+2X6X10+2X6X11+2X6X12+2X6X13+2X6X14+2X6X15+2X6X16+2X6X17+2X6X18+

$1X7^2$+2X7X8+.....+2X7X9+.2X7X10+2X7X11+2X7X12+2X7X13+2X7X14+2X7X15+2X7X16+2X7X17+2X7X18+

$1X8^2$+2X8X9+ 2X8X10+2X8X11+2X8X12+2X8X13+2X8X14+2X8X15+2X8X16+2X8X17+2X8X18+

$1X9^2$+ 2X9X10+ 2X9X11+ 2X9X12+2 X9X13+2X9X14+2X9X15+2X9X16+2X9X17+2X9X18+

$1X10^2$+2X10X11+2X10X12+2X10X13+2X10X14+2X10X15+2X10X16+2X10X17+2X10X18+

$1X11^2$+2X11X12+2X11X13+2X11X14+2X11X15+2X11X16+2X11X17+2X11X18+

$1X12^2$+2X12X13+2X12X14+2X12X15+2X12X16+2X12X17+2X12X18+

$1X13^2$+2X13X14+2X13X15+2X13X16+2X13X17+2X13X18+

$1X14^2$+2X14X15+2X14X16+2X14X17+2X14X18+

$1X15^2$+2X15X16+2X15X17+2X15X18+

$1X16^2$+2X16X17+2X16X18+

$1X17^2$+2X17X18+

$1X18^2$

Fig. 3: Shaded combinations of the EPT's expansion represents the minimal cut sets

Table 3: Basic events and their variable names

Item	Pad 1	Pad 2	Pad 3	Pad 4	Pad 5	Pump	Pump switch	Tank	Valve	Valve switch
loopA	X1	X2	X3	X4	X5	X6	X7	X8	X9	X19
loopB	X10	X11	X12	X13	X14	X15	X16	X17	X18	

With the first 9 variables (x1,x2,….x9) represent the basic events of loop A and the other 9 variables (x9, x10, ….x18) represent the basic events of loop B and of course one adds the single basic event (failure of AB-common switch), Table 3 presents such variables. Of course one eliminates any combinations of basic events ocuuring in one loop, since that does not makes the system halts because vacuum pressure is still avaliable from the other loop.

Figure 3 presents the EPT's multinomial expansion of eq.(8), where the shaded area represents the minimial cut set for the combinations from loopA and loopB plus of course the single basic event represented by X19. In the expansion, one notices that the coefficient 2 of the expansion implies for example the combination X1X10 or X10X1 which are the same that is; the order is irrelevant. Counting the shaded terms, one finds them to be 81 sets and if we add the single basic event X19, the total adds up to 82 which represent the minimal cut sets for this problem.

CONCLUSION

In this paper, the author introduces what is called the Guelph expansion for special form of binomial expansion. The binomial coefficient Tn,k can generate the entries of the classical Pascal triangle. When an extension of the guelph expansion is applied to multinomial expansion, a multiplication of a sequence of binomial coefficients (T's) generates the entries of what is reffered to as Embedded Pascal Triangles. Such EPT's entries represent the coefficients of a multinomial expansions. An efficient algorithm is available for generating such coefficients. Although, the Guelph expansion has been used for neutronic studies, it also has another application for determining the roots of a known polynomial coefficients (whether the roots are real or imaginary) and also has the application of the inverse problem, that is, it can determine the polynomial coefficients if the roots of the polynomial were known. Also, the EPT's expansion has applications to dice throwing of multi-faces and also to genetics. A possible exploration to genomics is worth exploration especially for isolated DNA strips (nucletoids sequenceing analysis). In this paper, the author extends the EPT's for polynomial expansion to an application of engineering safety analysis, namely, identifying the minimal cut sets (82 of them) for fault tree analysis of a system's vacuum loss fault. It is recommended here that further explorations to other applications is worth trying in other fields.

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