Erratum

Feng Qi and Bai-Ni Guo

Alternative proofs of a formula for Bernoulli numbers in terms of Stirling numbers

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Abstract: In the paper, the authors provide four alternative proofs of an explicit formula for computing Bernoulli numbers in terms of Stirling numbers of the second kind.

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Feng Qi, Bai-Ni Guo: School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, P. R. China,
e-mail: qifeng618@gmail.com, bai.ni.guo@gmail.com

1 Introduction

It is well known that Bernoulli numbers B_k for $k \ge 0$ may be generated by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi. \tag{1.1}$$

In combinatorics, Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 0$ may be computed by

$$S(n,k) = \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n$$

and may be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}.$$

In [5, p. 536] and [6, p. 560], the following simple formula for computing Bernoulli numbers B_n in terms of Stirling numbers of the second kind $S(n,k)$ was incidentally obtained.

Theorem 1.1. *For $n \in \{0\} \cup \mathbb{N}$, we have*

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} S(n,k). \tag{1.2}$$

The aim of this paper is to provide four alternative proofs for the explicit formula (1.2).

2 Four alternative proofs of the formula (1.2)

Considering $S(0,0) = 1$, it is clear that the formula (1.2) is valid for $n = 0$. Further considering $S(n,0) = 0$ for $n \geq 1$, it is sufficient to show

$$B_n = \sum_{k=1}^{n} (-1)^k \frac{k!}{k+1} S(n,k), \quad n \geq 1.$$

First proof. It is listed in [1, p. 230, 5.1.32] that

$$\ln \frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} \,\mathrm{d}u. \tag{2.1}$$

Taking $a = 1$ and $b = 1 + x$ in (2.1) yields

$$\frac{\ln(1+x)}{x} = \int_0^\infty \frac{1 - e^{-xu}}{xu} e^{-u} \,\mathrm{d}u = \int_0^\infty \left(\int_{1/e}^1 t^{xu-1} \,\mathrm{d}t \right) e^{-u} \,\mathrm{d}u. \tag{2.2}$$

Replacing x by $e^x - 1$ in (2.2) results in

$$\frac{x}{e^x - 1} = \int_0^\infty \left(\int_{1/e}^1 t^{ue^x - u - 1} \,\mathrm{d}t \right) e^{-u} \,\mathrm{d}u. \tag{2.3}$$

In combinatorics, Bell polynomials of the second kind (also called partial Bell polynomials) $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ are defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^n i\ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}$$

for $n \geq k \geq 1$, see [4, p. 134, Theorem A]. They satisfy

$$\mathrm{B}_{n,k}(abx_1, ab^2 x_2, \dots, ab^{n-k+1} x_{n-k+1}) = a^k b^n \mathrm{B}_{n,k}(x_1, x_n, \dots, x_{n-k+1}) \tag{2.4}$$

and

$$\mathrm{B}_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1}) = S(n,k), \tag{2.5}$$

see [4, p. 135], where a and b are any complex numbers. The well-known Faà di Bruno formula may be described in terms of Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f \circ g(x) = \sum_{k=1}^{n} f^{(k)}(g(x)) \mathrm{B}_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)), \tag{2.6}$$

see [4, p. 139, Theorem C].

Applying in (2.6) the function $f(y) = t^y$ and $y = g(x) = ue^x - u - 1$ gives

$$\frac{\mathrm{d}^n t^{ue^x}}{\mathrm{d}x^n} = \sum_{k=1}^{n} (\ln t)^k t^{ue^x} \mathrm{B}_{n,k}(\overbrace{ue^x, ue^x, \dots, ue^x}^{n-k+1}). \tag{2.7}$$

Making use of the formulas (2.4) and (2.5) in (2.7) reveals

$$\frac{\mathrm{d}^n t^{ue^x}}{\mathrm{d}x^n} = t^{ue^x} \sum_{k=1}^{n} S(n,k) u^k (\ln t)^k e^{kx}. \tag{2.8}$$

Differentiating n times on both sides of (2.3) and considering (2.8), we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{e^x - 1}\right) = \sum_{k=1}^{n} S(n,k) e^{kx} \int_0^\infty u^k \left(\int_{1/e}^{1} (\ln t)^k t^{ue^x - u - 1} \,\mathrm{d}t \right) e^{-u} \,\mathrm{d}u. \tag{2.9}$$

On the other hand, differentiating n times on both sides of (1.1) gives

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{e^x - 1}\right) = \sum_{k=n}^{\infty} B_k \frac{x^{k-n}}{(k-n)!}. \tag{2.10}$$

Equating (2.9) and (2.10) and taking the limit $x \to 0$, we deduce

$$\begin{aligned} B_n &= \sum_{k=1}^{n} S(n,k) \int_0^\infty u^k \left(\int_{1/e}^{1} \frac{(\ln t)^k}{t} \,\mathrm{d}t \right) e^{-u} \,\mathrm{d}u \\ &= \sum_{k=1}^{n} \frac{(-1)^k}{k+1} S(n,k) \int_0^\infty u^k e^{-u} \,\mathrm{d}u \\ &= \sum_{k=1}^{n} \frac{(-1)^k k!}{k+1} S(n,k). \end{aligned}$$

The first proof of Theorem 1.1 is complete. □

Second proof. In the book [2, p. 386] and in the papers [3, p. 615] and [11, p. 885], it was given that

$$\frac{\ln b - \ln a}{b - a} = \int_0^1 \frac{1}{(1-t)a + tb}\,\mathrm{d}t,$$

where $a, b > 0$ and $a \neq b$. Replacing a by 1 and b by e^x yields

$$\frac{x}{e^x - 1} = \int_0^1 \frac{1}{1 + (e^x - 1)t}\,\mathrm{d}t.$$

Applying the functions $f(y) = \frac{1}{y}$ and $y = g(x) = 1 + (e^x - 1)t$ in the formula (2.6) and simplifying by (2.4) and (2.5) give

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{e^x - 1}\right) &= \int_0^1 \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{1}{1 + (e^x - 1)t}\right]\mathrm{d}t \\
&= \int_0^1 \sum_{k=1}^{n} (-1)^k \frac{k!}{[1 + (e^x - 1)t]^{k+1}} \mathrm{B}_{n,k}(\overbrace{te^x, te^x, \dots, te^x}^{n-k+1})\,\mathrm{d}t \\
&= \sum_{k=1}^{n} (-1)^k k! \int_0^1 \frac{t^k}{[1 + (e^x - 1)t]^{k+1}} \mathrm{B}_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1})\,\mathrm{d}t \\
&\to \sum_{k=1}^{n} (-1)^k k! \int_0^1 t^k \mathrm{B}_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1})\,\mathrm{d}t, \quad x \to 0 \\
&= \sum_{k=1}^{n} (-1)^k k! S(n,k) \int_0^1 t^k\,\mathrm{d}t \\
&= \sum_{k=1}^{n} (-1)^k \frac{k!}{k+1} S(n,k).
\end{aligned}$$

On the other hand, taking the limit $x \to 0$ in (2.10) leads to

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{e^x - 1}\right) = \sum_{k=n}^{\infty} B_k \frac{x^{k-n}}{(k-n)!} \to B_n, \quad x \to 0.$$

The second proof of Theorem 1.1 is thus complete. □

Third proof. Let $CT[f(x)]$ be the coefficient of x^0 in the power series expansion of $f(x)$. Then

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k\frac{k!}{k+1}S(n,k) &= \sum_{k=1}^{n}(-1)^k CT\left[\frac{n!}{x^n}\frac{(e^x-1)^k}{k+1}\right]\\ &= n!CT\left[\frac{1}{x^n}\sum_{k=1}^{\infty}(-1)^k\frac{(e^x-1)^k}{k+1}\right]\\ &= n!CT\left[\frac{1}{x^n}\frac{\ln[1+(e^x-1)]-(e^x-1)}{e^x-1}\right]\\ &= n!CT\left[\frac{1}{x^n}\frac{x}{e^x-1}\right]\\ &= B_n.\end{aligned}$$

Thus, the formula (1.2) follows. □

Fourth proof. It is clear that the equation (1.1) may be rewritten as

$$\frac{\ln[1+(e^x-1)]}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}. \tag{2.11}$$

Differentiating n times on both sides of (2.11) and taking the limit $x\to 0$ reveal

$$\begin{aligned}B_n &= \lim_{x\to 0}\sum_{k=n}^{\infty}B_k\frac{x^{k-n}}{(k-n)!}\\ &= \lim_{x\to 0}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\ln[1+(e^x-1)]}{e^x-1}\right)\\ &= \lim_{x\to 0}\sum_{k=1}^{n}\left[\frac{\ln(1+u)}{u}\right]^{(k)}\mathrm{B}_{n,k}(\overbrace{e^x,e^x,\dots,e^x}^{n-k+1}),\quad u=e^x-1\\ &= \lim_{x\to 0}\sum_{k=1}^{n}\left[\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{u^{\ell-1}}{\ell}\right]^{(k)}\mathrm{B}_{n,k}(\overbrace{e^x,e^x,\dots,e^x}^{n-k+1})\\ &= \lim_{x\to 0}\sum_{k=1}^{n}\left[\sum_{\ell=k+1}^{\infty}(-1)^{\ell-1}\frac{(\ell-1)!}{(\ell-k-1)!\ell}u^{\ell-k-1}\right]\mathrm{B}_{n,k}(\overbrace{e^x,e^x,\dots,e^x}^{n-k+1})\\ &= \sum_{k=1}^{n}\lim_{u\to 0}\left[\sum_{\ell=k+1}^{\infty}(-1)^{\ell-1}\frac{(\ell-1)!}{(\ell-k-1)!\ell}u^{\ell-k-1}\right]\lim_{x\to 0}\mathrm{B}_{n,k}(\overbrace{e^x,e^x,\dots,e^x}^{n-k+1})\\ &= \sum_{k=1}^{n}(-1)^k\frac{k!}{k+1}\mathrm{B}_{n,k}(\overbrace{1,1,\dots,1}^{n-k+1})\\ &= \sum_{k=1}^{n}(-1)^k\frac{k!}{k+1}S(n,k).\end{aligned}$$

The fourth proof of Theorem 1.1 is thus complete. □

Remark 2.1. In [9, p. 1128, Corollary], among other things, it was found that

$$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k\sum_{i=1}^{k-1} \frac{A_{2(k-i)}}{2(k-i)+1}$$

for $k \in \mathbb{N}$, where A_m is defined by

$$\sum_{m=1}^{n} m^k = \sum_{m=0}^{k+1} A_m n^m.$$

It was listed in [6, p. 559] and recovered in [8, Theorem 2.1] that

$$\left(\frac{1}{e^x-1}\right)^{(k)} = (-1)^k \sum_{m=1}^{k+1} (m-1)!S(k+1,m)\left(\frac{1}{e^x-1}\right)^m \tag{2.12}$$

for $k \in \{0\} \cup \mathbb{N}$. In [8, Theorem 3.1], by the identity (2.12), it was obtained that

$$\begin{aligned} B_{2k} = 1 &+ \sum_{m=1}^{2k-1} \frac{S(2k+1,m+1)S(2k,2k-m)}{\binom{2k}{m}} \\ &- \frac{2k}{2k+1}\sum_{m=1}^{2k} \frac{S(2k,m)S(2k+1,2k-m+1)}{\binom{2k}{m-1}}, \quad k \in \mathbb{N}. \end{aligned}$$

In [12, Theorem 1.4], among other things, it was presented that

$$B_{2k} = \frac{(-1)^{k-1}k}{2^{2(k-1)}(2^{2k}-1)} \sum_{i=0}^{k-1}\sum_{\ell=0}^{k-i-1} (-1)^{i+\ell}\binom{2k}{\ell}(k-i-\ell)^{2k-1}, \quad k \in \mathbb{N}.$$

Remark 2.2. The identities in (2.12) have been generalized and applied in [7, 13].

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