# Generalized Bell polynomials and the combinatorics of Poisson central moments

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**Abstract**

We introduce a family of polynomials that generalizes the Bell polynomials, in connection with the combinatorics of the central moments of the Poisson distribution. We show that these polynomials are dual of the Charlier polynomials by the Stirling transform, and we study the resulting combinatorial identities for the number of partitions of a set into subsets of size at least 2.

## 1 Introduction

The moments of the Poisson distribution are well-known to be connected to the combinatorics of the Stirling and Bell numbers. In particular the Bell polynomials $B_n(\lambda)$ satisfy the relation

$$B_n(\lambda) = E_\lambda[Z^n], \qquad n \in \mathbb{N}, \tag{1.1}$$

where $Z$ is a Poisson random variable with parameter $\lambda > 0$, and

$$B_n(1) = \sum_{c=0}^{n} S(n, c) \tag{1.2}$$

is the Bell number of order $n$, i.e. the number of partitions of a set of $n$ elements. In this paper we study the central moments of the Poisson distribution, and we show that they can be expressed using the number of partitions of a set into subsets of size at least 2, in connection with an extension of the Bell polynomials.

Consider the above mentioned Bell (or Touchard) polynomials $B_n(\lambda)$ defined by the exponential generating function

$$e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(\lambda), \tag{1.3}$$

$\lambda, t \in \mathbb{R}$, cf. e.g. §11.7 of [4], and given by the Stirling transform

$$B_n(\lambda) = \sum_{c=0}^{n} \lambda^c S(n,c), \tag{1.4}$$

where

$$S(n,c) = \frac{1}{c!} \sum_{l=0}^{c} (-1)^{c-l} \binom{c}{l} l^n \tag{1.5}$$

denotes the Stirling number of the second kind, i.e. the number of ways to partition a set of $n$ objects into $c$ non-empty subsets, cf. § 1.8 of [7], Proposition 3.1 of [3] or § 3.1 of [6], and Relation (1.2) above.

In this note we define a two-parameter generalization of the Bell polynomials, which is dual to the Charlier polynomials by the Stirling transform. We study the links of these polynomials with the combinatorics of Poisson central moments, cf. Lemma 3.1, and as a byproduct we obtain the binomial identity

$$S_2(m,n) = \sum_{k=0}^{n} (-1)^k \binom{m}{k} S(m-k, n-k), \tag{1.6}$$

where $S_2(n,a)$ denotes the number of partitions of a set of size $n$ into $a$ subsets of size at least 2, cf. Corollary 3.2 below, which is the binomial dual of the relation

$$S(m,n) = \sum_{k=0}^{n} \binom{m}{k} S_2(m-k, n-k),$$

cf. Proposition 3.3 below.

We proceed as follows. Section 2 contains the definition of our extension of the Bell polynomials. In Section 3 we study the properties of the polynomials using the Poisson central moments, and we derive Relation (1.6) as a corollary. Finally in Section 4 we state the connection between these polynomials and the Charlier polynomials via the Stirling transform.

# 2 An extension of the Bell polynomials

We let $(B_n(x,\lambda))_{n\in\mathbb{N}}$ denote the family of polynomials defined by the exponential generating function

$$e^{ty-\lambda(e^t-t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(y,\lambda), \qquad \lambda, y, t \in \mathbb{R}.. \tag{2.1}$$

Clearly from (1.3) and (2.1), the definition of $B_n(x,\lambda)$ generalizes that of the Bell polynomials $B_n(\lambda)$, in that

$$B_n(\lambda) = B_n(\lambda, -\lambda), \qquad \lambda \in \mathbb{R}. \tag{2.2}$$

When $\lambda > 0$, Relation (2.1) can be written as

$$e^{ty} E_\lambda[e^{t(Z-\lambda)}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(y, -\lambda), \qquad y, t \in \mathbb{R},$$

which yields the relation

$$B_n(y, -\lambda) = E_\lambda[(Z + y - \lambda)^n], \qquad \lambda, y \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{2.3}$$

which is analog to (1.1), and shows the following proposition.

**Proposition 2.1** *For all $n \in \mathbb{N}$ we have*

$$B_n(y, \lambda) = \sum_{k=0}^{n} \binom{n}{k} (y - \lambda)^{n-k} \sum_{i=0}^{k} \lambda^i S(k, i), \qquad y, \lambda \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{2.4}$$

*Proof.* Indeed, by (2.3) we have

$$\begin{aligned}
B_n(y, -\lambda) &= E_\lambda[(Z + y - \lambda)^n], \\
&= \sum_{k=0}^{n} \binom{n}{k} (y - \lambda)^{n-k} E_\lambda[Z^k] \\
&= \sum_{k=0}^{n} \binom{n}{k} (y - \lambda)^{n-k} B_k(\lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k} (y - \lambda)^{n-k} \sum_{i=0}^{k} \lambda^i S(k, i), \qquad y, \lambda \in \mathbb{R}.
\end{aligned}$$

□

# 3 Combinatorics of the Poisson central moments

As noted in (1.1) above, the connection between Poisson moments and polynomials is well understood, however the Poisson central moments seem to have received less attention.

In the sequel we will need the following lemma, which expresses the central moments of a Poisson random variable using the number $S_2(n, b)$ of partitions of a set of size $n$ into $b$ subsets with no singletons.

**Lemma 3.1** *Let $Z$ be a Poisson random variable with intensity $\lambda > 0$. We have*

$$B_n(0, -\lambda) = E_\lambda[(Z - \lambda)^n] = \sum_{a=0}^{n} \lambda^a S_2(n, a), \qquad n \in \mathbb{N}. \tag{3.1}$$

*Proof.* We start by showing the recurrence relation

$$E_\lambda[(Z-\lambda)^{n+1}] = \lambda \sum_{i=0}^{n-1} \binom{n}{i} E_\lambda\left[(Z-\lambda)^i\right], \qquad n \in \mathbb{N}, \tag{3.2}$$

for $Z$ a Poisson random variable with intensity $\lambda$. We have

$$\begin{aligned}
E_\lambda[(Z-\lambda)^{n+1}] &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(k-\lambda)^{n+1} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}(k-\lambda)^n - \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(k-\lambda)^n \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}((k+1-\lambda)^n - (k-\lambda)^n) \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \sum_{i=0}^{n-1} \binom{n}{i} (k-\lambda)^i \\
&= \lambda e^{-\lambda} \sum_{i=0}^{n-1} \binom{n}{i} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (k-\lambda)^i \\
&= \lambda \sum_{i=0}^{n-1} \binom{n}{i} E_\lambda[(Z-\lambda)^i]..
\end{aligned}$$

Next, we show that the identity

$$E_\lambda[(Z-\lambda)^n] = \sum_{a=1}^{n-1} \lambda^a \sum_{0=k_1 \ll \cdots \ll k_{a+1}=n} \prod_{l=1}^{a} \binom{k_{l+1}-1}{k_l} \tag{3.3}$$

holds for all $n \geq 1$, where $a \ll b$ means $a < b-1$. Note that the degree of (3.3) in $\lambda$ is the largest integer $d$ such that $2d \leq n$, hence it equals $n/2$ or $(n-1)/2$ according to the parity of $n$.

Clearly, the identity (3.3) is valid when $n = 1$ and when $n = 2$. Assuming that it holds up to the rank $n \geq 2$, from (3.2) we have

$$\begin{aligned}
E_\lambda[(Z-\lambda)^{n+1}] &= \lambda \sum_{k=0}^{n-1} \binom{n}{k} E_\lambda\left[(Z-\lambda)^k\right] \\
&= \lambda + \lambda \sum_{k=1}^{n-1} \binom{n}{k} E_\lambda\left[(Z-\lambda)^k\right] \\
&= \lambda + \lambda \sum_{k=1}^{n-1} \binom{n}{k} \sum_{b=1}^{k-1} \lambda^b \sum_{0=k_1 \ll \cdots \ll k_{b+1}=k} \prod_{l=1}^{b} \binom{k_{l+1}-1}{k_l}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda + \lambda \sum_{k=1}^{n-1} \binom{n}{k} \sum_{b=2}^{k} \lambda^{b-1} \sum_{0=k_1 \ll \cdots \ll k_b = k} \prod_{l=1}^{b-1} \binom{k_{l+1}-1}{k_l} \\
&= \lambda + \lambda \sum_{k_b=1}^{n-1} \binom{n}{k_b} \sum_{b=2}^{k_b} \lambda^{b-1} \sum_{0=k_1 \ll \cdots \ll k_b} \prod_{l=1}^{b-1} \binom{k_{l+1}-1}{k_l} \\
&= \lambda + \lambda \sum_{k_b=1}^{n-1} \sum_{b=2}^{k_b} \lambda^{b-1} \sum_{0=k_1 \ll \cdots \ll k_b \ll k_{b+1} = n} \prod_{l=1}^{b} \binom{k_{l+1}-1}{k_l} \\
&= \lambda + \lambda \sum_{k_b=1}^{n} \sum_{b=2}^{k_b} \lambda^{b-1} \sum_{0=k_1 \ll \cdots \ll k_b \ll k_{b+1} = n} \prod_{l=1}^{b} \binom{k_{l+1}-1}{k_l} \\
&= \lambda + \sum_{b=2}^{n} \lambda^{b} \sum_{0=k_1 \ll \cdots \ll k_{b+1} = n+1} \prod_{l=1}^{b} \binom{k_{l+1}-1}{k_l} \\
&= \sum_{b=1}^{n} \lambda^{b} \sum_{0=k_1 \ll \cdots \ll k_{b+1} = n+1} \prod_{l=1}^{b} \binom{k_{l+1}-1}{k_l},
\end{aligned}
$$

and it remains to note that

$$
\sum_{0=k_1 \ll \cdots \ll k_{b+1} = n} \prod_{l=1}^{b} \binom{k_{l+1}-1}{k_l} = S_2(n,b) \tag{3.4}
$$

equals the number $S_2(n,b)$ of partitions of a set of size $n$ into $b$ subsets of size at least 2. Indeed, any contiguous such partition is determined by a sequence of $b-1$ integers $k_2, \ldots, k_b$ with $2b \leq n$ and $0 \ll k_2 \ll \cdots \ll k_b \ll n$ so that subset n$^o$ $i$ has size $k_{i+1} - k_i \geq 2$, $i = 1, \ldots, b$, with $k_{b+1} = n$, and the number of not necessarily contiguous partitions of that size can be computed inductively on $i = 1, \ldots, b$ as

$$
\binom{n-1}{n-1-k_b} \binom{k_b - 1}{k_b - 1 - k_{b-1}} \cdots \binom{k_2 - 1}{k_2 - 1 - k_1} = \prod_{l=1}^{b} \binom{k_{l+1}-1}{k_l}..
$$

For this, at each step we pick an element which acts as a boundary point in the subset n$^o$ $i$, and we do not count it in the possible arrangements of the remaining $k_{i+1} - 1 - k_i$ elements among $k_{i+1} - 1$ places. □

Lemma 3.1 and (3.4) can also be recovered by use of the cumulants $(\kappa_n)_{n \geq 1}$ of $Z - \lambda$, defined from the cumulant generating function

$$
\log E_\lambda[e^{t(Z-\lambda)}] = \lambda(e^t - 1) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!},
$$

i.e. $\kappa_1 = 0$ and $\kappa_n = \lambda$, $n \geq 2$, which shows that

$$
E_\lambda[(Z-\lambda)^n] = \sum_{a=1}^{n} \sum_{B_1, \ldots, B_a} \kappa_{|B_1|} \cdots \kappa_{|B_a|},
$$

where the sum runs over the partitions $B_1, \ldots, B_a$ of $\{1, \ldots, n\}$ with cardinal $|B_i|$ by the Faà di Bruno formula, cf. § 2.4 of [5]. Since $\kappa_1 = 0$ the sum runs over the partitions with cardinal $|B_i|$ at least equal to 2, which recovers

$$E_\lambda[(Z-\lambda)^n] = \sum_{a=1}^{n} \lambda^a S_2(n,a), \tag{3.5}$$

and provides another proof of (3.4). In addition, (3.2) can be seen as a consequence of a general recurrence relation between moments and cumulants, cf. Relation (5) of [8].

In particular when $\lambda = 1$, (3.1) shows that the central moment

$$B_n(0,-1) = E_1[(Z-1)^n] = \sum_{a=0}^{n} S_2(n,a) \tag{3.6}$$

is the number of partitions of a set of size $n$ into subsets of size at least 2, as a counterpart to (1.2).

By (2.3) we have

$$B_n(y,\lambda) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} E_\lambda[(Z-\lambda)^k] = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} B_k(0,-\lambda),$$

$y \in \mathbb{R}$, $\lambda > 0$, $n \in \mathbb{N}$, hence Lemma 3.1 shows that we have

$$B_n(y,\lambda) = \sum_{l=0}^{n} \binom{n}{l} y^{n-l} \sum_{c=0}^{l} \lambda^c S_2(l,c), \qquad \lambda, y \in \mathbb{R}, \quad n \in \mathbb{N}.. \tag{3.7}$$

As a consequence of Relations (2.4) and (3.7) we obtain the following binomial identity.

**Corollary 3.2** *We have*

$$S_2(n,c) = \sum_{k=0}^{c} (-1)^k \binom{n}{k} S(n-k, c-k), \qquad 0 \le c \le n. \tag{3.8}$$

*Proof.* By Relation (2.4) we have

$$\begin{aligned} B_n(y,\lambda) &= \sum_{k=0}^{n} \binom{n}{k} (y-\lambda)^k \sum_{i=0}^{n-k} \lambda^i S(n-k,i) \\ &= \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} y^l (-\lambda)^{k-l} \sum_{i=0}^{n-k} \lambda^i S(n-k,i) \\ &= \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{l} \binom{n-l}{n-k} y^l (-\lambda)^{k-l} \sum_{i=0}^{n-k} \lambda^i S(n-k,i) \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{n}\sum_{k=l}^{n}\binom{n}{l}\binom{n-l}{n-k}y^l(-\lambda)^{k-l}\sum_{i=0}^{n-k}\lambda^i S(n-k,i)\\
&= \sum_{l=0}^{n}\sum_{b=0}^{n-l}\binom{n}{l}\binom{n-l}{b}y^l(-\lambda)^{n-b-l}\sum_{i=0}^{b}\lambda^i S(b,i)\\
&= \sum_{l=0}^{n}\sum_{b=0}^{l}\binom{n}{l}\binom{l}{b}y^{n-l}(-\lambda)^{b}\sum_{i=0}^{l-b}\lambda^i S(l-b,i)\\
&= \sum_{l=0}^{n}\sum_{b=0}^{l}\binom{n}{l}\binom{l}{b}y^{n-l}(-\lambda)^{b}\sum_{c=b}^{l}\lambda^{c-b} S(l-b,c-b)\\
&= \sum_{l=0}^{n}\binom{n}{l}y^{n-l}\sum_{c=0}^{l}\lambda^c\sum_{b=0}^{c}(-1)^b\binom{l}{b}S(l-b,c-b), \qquad y,\lambda\in\mathbb{R},
\end{aligned}$$

and we conclude by Relation (3.7). □

As a consequence of (3.7) and (3.8) we have the identity

$$B_n(0,-\lambda) = E_\lambda\left[(Z-\lambda)^n\right] = \sum_{c=0}^{n}\lambda^c\sum_{a=0}^{c}(-1)^a\binom{n}{a}S(n-a,c-a),$$

for the central moments of a Poisson random variable $Z$ with intensity $\lambda > 0$.

The following proposition, which is the inversion formula of (3.8) has a natural interpretation by recalling that $S_2(m,b)$ is the number of partitions of a set of $m$ elements made of $b$ sets of cardinal greater or equal to 2, as will be seen in Proposition 3.4 below.

**Proposition 3.3** *We have the combinatorial identity*

$$S(n,b) = \sum_{l=0}^{b}\binom{n}{l}S_2(n-l,b-l), \qquad b,n\in\mathbb{N}. \tag{3.9}$$

*Proof.* By Relation (3.7) we have

$$\begin{aligned}
B_n(\lambda) &= B_n(\lambda,-\lambda)\\
&= \sum_{l=0}^{n}\binom{n}{l}\lambda^{n-l}\sum_{b=0}^{l}\lambda^b S_2(l,b)\\
&= \sum_{b=0}^{n}\lambda^b\sum_{l=0}^{b}\binom{n}{l}S_2(n-l,b-l),
\end{aligned}$$

and we conclude from (1.4). □

Relation (3.9) is in fact a particular case for $a=0$ of the identity proved in the next proposition, since $S(l-c,0) = \mathbf{1}_{\{l=c\}}$.

**Proposition 3.4** *For all $a, b, n \in \mathbb{N}$ we have*

$$\binom{a+b}{a} S(n, a+b) = \sum_{c=0}^{b} \sum_{l=c}^{n} \binom{n}{l} \binom{l}{c} S(l-c, a) S_2(n-l, b-c)..$$

*Proof.* The partitions of $\{1, \ldots, n\}$ made of $a+b$ subsets are labeled using all possibles values of $l \in \{0, 1, \ldots, n\}$ and $c \in \{0, 1, \ldots, l\}$, as follows. For every $l \in \{0, 1, \ldots, n\}$ and $c \in \{0, 1, \ldots, l\}$ we decompose $\{1, \ldots, n\}$ into

- a subset $(k_1, \ldots, k_l)$ of $\{1, \ldots, n\}$ with $\binom{n}{l}$ possibilities,
- $c$ singletons within $(k_1, \ldots, k_l)$, i.e. $\binom{l}{c}$ possibilities,
- a remaining subset of $(k_1, \ldots, k_l)$ of size $l - c$, which is partitioned into $a \in \mathbb{N}$ (non-empty) subsets, i.e. $S(l-c, a)$ possibilities, and
- a remaining set $\{1, \ldots, n\} \setminus (k_1, \ldots, k_l)$ of size $n - l$ which is partitioned into $b - c$ subsets of size at least 2, i.e. $S_2(n-l, b-c)$ possibilities.

In this process the $b$ subsets mentioned above were counted with their combinations within $a+b$ sets, which explains the binomial coefficient $\binom{a+b}{a}$ on the right-hand side. □

# 4 Stirling transform

In this section we consider the Charlier polynomials $C_n(x, \lambda)$ of degree $n \in \mathbb{N}$, with exponential generating function

$$e^{-\lambda t}(1+t)^x = \sum_{n=0}^{\infty} \frac{t^n}{n!} C_n(x, \lambda), \qquad x, t, \lambda \in \mathbb{R},$$

and

$$C_n(x, \lambda) = \sum_{k=0}^{n} x^k \sum_{l=0}^{k} \binom{n}{l} (-\lambda)^{n-l} s(k, l), \qquad x, \lambda \in \mathbb{R}, \tag{4.1}$$

cf. § 3.3 of [7], where

$$s(k, l) = \frac{1}{l!} \sum_{i=0}^{l} (-1)^i \binom{l}{i} (l-i)^k$$

is the Stirling number of the first kind, cf. page 824 of [1], i.e. $(-1)^{k-l} s(k, l)$ is the number of permutations of $k$ elements which contain exactly $l$ permutation cycles, $n \in \mathbb{N}$.

In the next proposition we show that the Charlier polynomials $C_n(x, \lambda)$ are dual to the generalized Bell polynomials $B_n(x - \lambda, \lambda)$ defined in (2.1) under the Stirling transform.

**Proposition 4.1** *We have the relations*

$$C_n(y,\lambda) = \sum_{k=0}^{n} s(n,k) B_k(y-\lambda,\lambda) \quad \textit{and} \quad B_n(y,\lambda) = \sum_{k=0}^{n} S(n,k) C_k(y+\lambda,\lambda),$$

$y, \lambda \in \mathbb{R}$, $n \in \mathbb{N}$.

*Proof.* For the first relation, for all fixed $y, \lambda \in \mathbb{R}$ we let

$$A(t) = e^{-\lambda t}(1+t)^{y+\lambda} = \sum_{n=0}^{\infty} \frac{t^n}{n!} C_n(y+\lambda,\lambda), \qquad t \in \mathbb{R},$$

with

$$A(e^t-1) = e^{t(y+\lambda)-\lambda(e^t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(y,\lambda), \qquad t \in \mathbb{R},$$

and we conclude from Lemma 4.2 below. The second part can be proved by inversion using Stirling numbers of the first kind, as

$$\begin{aligned}
\sum_{k=0}^{n} S(n,k) C_k(y+\lambda,\lambda) &= \sum_{k=0}^{n} \sum_{l=0}^{k} S(n,k) s(k,l) B_l(y,\lambda) \\
&= \sum_{l=0}^{n} B_l(y,\lambda) \sum_{k=l}^{n} S(n,k) s(k,l) \\
&= B_n(y,\lambda),
\end{aligned}$$

from the inversion formula

$$\sum_{k=l}^{n} S(n,k) s(k,l) = \mathbf{1}_{\{n=l\}}, \qquad n, l \in \mathbb{N}, \tag{4.2}$$

for Stirling numbers, cf. e.g. page 825 of [1]. □

Next we recall the following lemma, cf. e.g. Relation (3) page 2 of [2], which has been used in Proposition 4.1 to show that the polynomials $B_n(y,\lambda)$ are connected to the Charlier polynomials.

**Lemma 4.2** *Assume that the function $A(t)$ has the series expansion*

$$A(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} a_k, \qquad t \in \mathbb{R}.$$

*Then we have*

$$A(e^t-1) = \sum_{k=0}^{n} \frac{t^k}{k!} b_k, \qquad t \in \mathbb{R},$$

*with*

$$b_n = \sum_{k=0}^{n} a_k S(n,k), \qquad n \in \mathbb{N}.$$

Finally we note that from (2.4) we have the relation

$$B_n(y, y+\lambda) = \sum_{k=0}^{n} (y+\lambda)^k \sum_{l=k}^{n} \binom{n}{l} (-\lambda)^{n-l} S(l,k), \qquad y, \lambda \in \mathbb{R}, \quad n \in \mathbb{N},$$

which parallels (4.1).

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