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Diophantine approximations using Padé approximations

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Abstract

We show how Padé approximations are used to get Diophantine approximations of real or complex numbers, and so to prove the irrationality. We present two kinds of examples. First, we study two types of series for which Padé approximations provide exactly Diophantine approximations. Then, we show how Padé approximants to the asymptotic expansion of the remainder term of a value of a series also leads to Diophantine approximation. © 2000 Elsevier Science B.V. All rights reserved.

1. Preliminary

Definition 1 (*Diophantine approximation*). Let x a real or complex number and $(p_n/q_n)_n$ a sequence of $\mathbb{Q}$ or $\mathbb{Q}(i)$.

If $\lim_{n\to\infty}|q_nx-p_n|=0$ and $p_n/q_n \neq x$, $\forall n \in \mathbb{N}$, then the sequence $(p_n/q_n)_n$ is called a Diophantine approximation of x.

It is well known that Diophantine approximation of x proves the irrationality of x.

So, to construct Diophantine approximation of a number, a mean is to find rational approximation, for example with Padé approximation.

We first recall the theory of formal orthogonal polynomials and its connection with Padé approximation and ε-algorithm.

1.1. Padé approximants

Let h be a function whose Taylor expansion about $t=0$ is $\sum_{i=0}^{\infty} c_it^i$. The Padé approximant $[m/n]_h$ to h is a rational fraction $N_m(t)/D_n(t)$ whose Taylor series at $t=0$ coincides with that of h up to

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the maximal order, which is in general the sum of the degrees of numerator and denominator of the fraction, i.e,

$$\deg(N_m)\leqslant m, \qquad \deg(D_n)\leqslant n, \qquad D_n(t)h(t)-N_m(t)=\mathrm{O}(t^{m+n+1}), \quad t\to 0.$$

Note that the numerator N_m and the denominator D_n both depend on the index m and n.

The theory of Padé approximation is linked with the theory of orthogonal polynomials (see [10]): Let us define the linear functional c acting on the space $\mathscr{P}$ of polynomials as follows:

$$c:\mathscr{P}\to\mathbb{R} \quad (\text{or } \mathbb{C}),$$

$$x^i \to \langle c, x^i\rangle = c_i, \quad i=0,1,2,\ldots \text{ and if } p\in\mathbb{Z},$$

$$c^{(p)}:\mathscr{P}\to\mathbb{R} \quad (\text{or } \mathbb{C}),$$

$$x^i \to \langle c^{(p)}, x^i\rangle := \langle c, x^{i+p}\rangle = c_{i+p}, \quad i=0,1,2,\ldots \quad (c_i=0, i<0),$$

then the denominators of the Padé approximants $[m/n]$ satisfy the following orthogonality property:

$$\langle c^{(m-n+1)},\ x^i\tilde{D}_n(x)\rangle = 0, \quad i=0,1,2,\ldots,n-1,$$

where $\tilde{D}_n(x)=x^nD_n(x^{-1})$ is the reverse polynomial. Since the polynomials D_n involved in the expression of Padé approximants depend on the integers m and n, and since $\tilde{D}_n$ is orthogonal with respect to the shifted linear functional $c^{(m-n+1)}$, we denote

$$P_n^{(m-n+1)}(x)=\tilde{D}_n(x),$$

$$\tilde{Q}_n^{(m-n+1)}(x)=N_m(x).$$

If we set

$$R_{n-1}^{(m-n+1)}(t) := \left\langle c^{(m-n+1)}, \frac{P_n^{(m-n+1)}(x)-P_n^{(m-n+1)}(t)}{x-t}\right\rangle, \quad R_{n-1}^{(m-n+1)}\in\mathscr{P}_{n-1},$$

where $c^{(m-n+1)}$ acts on the letter x, then

$$N_m(t)=\left(\sum_{i=0}^{m-n} c_i t^i\right)\tilde{P}_n^{(m-n+1)}(t)+t^{m-n+1}\tilde{R}_{n-1}^{(m-n+1)}(t),$$

where $\tilde{R}_{n-1}^{(m-n+1)}(t)=t^{n-1}R_{n-1}^{(m-n+1)}(t^{-1})$, $\tilde{P}_n^{(m-n+1)}(t)=t^nP_n^{(m-n+1)}(t^{-1})$ and $\sum_{i=0}^{n-m} c_i t^i=0$, $n<m$.

The sequence of polynomials $(P_k^{(n)})_k$, of degree k, exists if and only if $\forall n\in\mathbb{Z}$, the Hankel determinant

$$H_k^{(n)} := \begin{vmatrix} c_n & \cdots & c_{n+k-1} \\ \cdots & \cdots & \cdots \\ c_{n+k-1} & \cdots & c_{n+2k-2} \end{vmatrix} \neq 0,$$

where $c_n=0$ if $n<0$.

In that case, we shall say that the linear functional c is completely definite. For the noncompletely definite case, the interested reader is referred to Draux [15].

For extensive applications of Padé approximants to Physics, see Baker's monograph [5].

If c admits an integral representation by a nondecreasing function α, with bounded variation

$$c_i = \int_{\mathbb{R}} x^i \mathrm{d}\alpha(x),$$

then the theory of Gaussian quadrature shows that the polynomials P_n orthogonal with respect to c, have all their roots in the support of the function α and

$$h(t) - [m/n]_h(t) = \frac{t^{m-n+1}}{(\tilde{P}_n^{(m-n+1)}(t))^2} c^{(m-n+1)} \left(\frac{(\tilde{P}_n^{(m-n+1)}(x))^2}{1-xt} \right)$$

$$= \frac{t^{m-n+1}}{(\tilde{P}_n^{(m-n+1)}(t))^2} \int_{\mathbb{R}} x^{m-n+1} \frac{(\tilde{P}_n^{(m-n+1)}(x))^2}{1-xt} \mathrm{d}\alpha(x). \tag{1}$$

Note that if $c_0 = 0$ then $[n/n]_h(t) = t[n-1/n]_{h/t}(t)$ and if $c_0 = 0$ and $c_1 = 0$, then $[n/n]_h(t) = t^2[n-2/n]_{h/t^2}(t)$.

Consequence: If α is a nondecreasing function on $\mathbb{R}$, then

$$h(t) \neq [m/n]_f(t) \quad \forall t \in \mathbb{C} - \mathrm{supp}(\alpha).$$

1.2. Computation of Padé approximants with ε-algorithm

The values of Padé approximants at some point of parameter t, can be recursively computed with the ε-algorithm of Wynn. The rules are the following:

$$\varepsilon_{-1}^{(n)} = 0, \ \varepsilon_0^{(n)} = S_n, \quad n = 0, 1, \ldots,$$

$$\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + \frac{1}{\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}}, \quad k, n = 0, 1, \ldots \quad \text{(rhombus rule)},$$

where $S_n = \sum_{k=0}^{n} c_k t^k$.

ε-values are placed in a double-entry array as following:

$$\begin{array}{ccccc}
\varepsilon_{-1}^{(0)} = 0 & & & & \\
 & \varepsilon_0^{(0)} = S_0 & & & \\
\varepsilon_{-1}^{(1)} = 0 & & \varepsilon_1^{(0)} & & \\
 & \varepsilon_0^{(1)} = S_1 & & \varepsilon_2^{(0)} & \\
\varepsilon_{-1}^{(2)} = 0 & & \varepsilon_1^{(1)} & & \varepsilon_3^{(0)} \\
 & \varepsilon_0^{(2)} = S_2 & & \varepsilon_2^{(1)} & & \ddots \\
\varepsilon_{-1}^{(3)} = 0 & & \varepsilon_1^{(2)} & \vdots & & & \ddots \\
\vdots & \varepsilon_0^{(3)} = S_3 & & \vdots & & & & \ddots
\end{array}$$

The connection between Padé approximant and ε-algorithm has been established by Shanks [26] and Wynn [35]:

Theorem 2. *If we apply ε-algorithm to the partial sums of the series* $h(t)=\sum_{i=0}^{\infty} c_i t^i$, *then*

$$\varepsilon_{2k}^{(n)} = [n+k/k]_h(t).$$

Many convergence results for ε-algorithm has been proved for series which are meromorphic functions in some complex domain, or which have an integral representation (Markov–Stieltjes function) (see [29,6,11] for a survey).

2. Diophantine approximation of sum of series with Padé approximation

Sometimes, Padé approximation is sufficient to prove irrationality of values of a series, as it can be seen in the following two results.

2.1. Irrationality of $\ln(1-r)$

We explain in the following theorem, how the old proof of irrationality of some logarithm number can be re-written in terms of ε-algorithm.

Theorem 3. *Let* $r=a/b,\ a\in\mathbb{Z},\ b\in\mathbb{N},\ b\neq 0$, *with* b.e.$(1-\sqrt{1-r})^2<1(\ln e=1)$ *Then ε-algorithm applied to the partial sums of* $f(r):=-\ln(1-r)/r=\sum_{i=0}^{\infty} r^i/(i+1)$ *satisfies that* $\forall n\in\mathbb{N},\ (\varepsilon_{2k}^{(n)})_k$ *is a Diophantine approximation of* $f(r)$.

Proof. From the connection between Padé approximation, orthogonal polynomials and ε-algorithm, the following expression holds:

$$\varepsilon_{2k}^{(n)} = \sum_{i=0}^{n} \frac{r^i}{i+1} + r^{n+1}\frac{\tilde{R}_{k-1}^{(n+1)}(r)}{\tilde{P}_k^{(n+1)}(r)} = \frac{N_{n+k}(r)}{\tilde{P}_k^{(n+1)}(r)},$$

where

$$\tilde{P}_k^{(n+1)}(t) = t^k P_k^{(n+1)}(t^{-1}) = \sum_{i=0}^{k} \binom{k}{k-i}\binom{k+n+1}{i}(1-t)^i$$

is the reversed shifted Jacobi polynomial on [0,1], with parameters $\alpha=0,\ \beta=n+1$, and $\tilde{R}_{k-1}^{(n+1)}(t)=t^{k-1}R_{k-1}^{(n+1)}(t^{-1})$ with $R_{k-1}^{(n+1)}(t)=\langle c^{(n+1)}, \frac{P_k^{(n+1)}(x)-P_k^{(n+1)}(t)}{x-t}\rangle(\langle c^{(n+1)}, x^i\rangle:=1/(n+i+2))$ (c acts on the variable x).

Since $\tilde{P}_k^{(n+1)}(t)$ has only integer coefficients, $b^k\tilde{P}_k^{(n+1)}(a/b)\in\mathbb{Z}$.

The expression of $R_{k-1}^{(n+1)}(t)$ shows that $d_{n+k+1}b^k\tilde{R}_{k-1}^{(n+1)}(a/b)\in\mathbb{Z}$, where $d_{n+k+1}:=\mathrm{LCM}(1,2,\ldots,n+k+1)$ (LCM means lowest common multiple).

We prove now that the sequence $(\varepsilon_{2k}^{(n)})_k$ is a Diophantine approximation of $\ln(1-a/b)$.

The proof needs asymptotics for d_{n+k+1}, for $\tilde{P}_k^{(n+1)}(a/b)$ and for $(\varepsilon_{2k}^{(n)} - f(r))$ when k tends to infinity. $d_n = \mathrm{e}^{n(1+\mathrm{o}(1))}$ follows from analytic number theory [1].

$\lim_k(\tilde{P}_k^{(n+1)}(x))^{1/k} = x(y+\sqrt{y^2-1})$, $x>1$, $y=2/x-1$, comes from asymptotic properties of Jacobi polynomials (see [30]), and $\lim_{k\to+\infty}(\varepsilon_{2k}^{(n)} - f(r))^{1/k} = (2/r-1-\sqrt{(2/r-1)^2-1})^2$ (error of Padé approximants to Markov–Stieltjes function).

So

$$\lim_{k\to+\infty} \sup \left| d_{n+k+1} b^k \tilde{P}_k^{(n+1)}(a/b) f(r) - d_{n+k+1} b^k N_{n+k}(a/b)\right|^{1/k}$$

$$\leqslant \lim_{k\to+\infty} \sup(d_{n+k+1})^{1/k} \limsup_k \left| b^k \tilde{P}_k^{(n+1)}(a/b)\right|^{1/k} \lim_{k\to+\infty} \sup \left|\varepsilon_{2k}^{(n)} + 1/r\ln(1-r)\right|^{1/k}$$

$$\leqslant \text{e.b.r.}(2/r-1+\sqrt{(2/r-1)^2-1})(2/r-1-\sqrt{(2/r-1)^2-1})^2$$

$$= \text{e.b.}(2/r-1-\sqrt{(2/r-1)^2-1}) = \text{e.b.}(1-\sqrt{1-r})^2 < 1$$

by hypothesis, which proves that

$$\forall n\in\mathbb{N},\quad \lim_{k\to+\infty}(d_{n+k+1} b^k \tilde{P}_k^{(n+1)}(a/b) f(r) - d_{n+k+1} b^k N_{n+k}(a/b)) = 0.$$

Moreover,

$$\varepsilon_{2k}^{(n)} + 1/r\ln(1-r) = -\frac{r^{2k+n+1}}{(\tilde{P}_k^{(n+1)}(r))^2}\int_0^1 \frac{(P_k^{(n+1)}(x))^2}{1-xr}(1-x)^{n+1}\,\mathrm{d}x \neq 0.$$

So the sequence $(\varepsilon_{2k}^{(n)})_k$ is a Diophantine approximation of $\ln(1-a/b)$, if b.e.$(1-\sqrt{1-a/b})^2<1$. □

2.2. Irrationality of $\sum t^n/w_n$

The same method as previously seen provides Diophantine approximations of $f(t) := \sum_{n=0}^{\infty} t^n/w_n$ when the sequence $(w_k)_k$ satisfies a second-order recurrence relation

$$w_{n+1} = sw_n - pw_{n-1}, \quad n\in\mathbb{N}, \tag{2}$$

where w_0 and w_{-1} are given in $\mathbb{C}$ and s and p are some complex numbers.

We suppose that $w_n \neq 0$, $\forall n\in\mathbb{N}$ and that the two roots of the characteristic equation $z^2 - sz + p = 0$, α and β satisfy $|\alpha| > |\beta|$.

So w_n admits an expression in term of geometric sequences: $w_n = A\alpha^n + B\beta^n$, $n\in\mathbb{N}$.

The roots of the characteristic equation are assumed to be of distinct modulus ($|\alpha|>|\beta|$), so there exists an integer r such that $|\alpha/\beta|^r > |B/A|$.

Lemma 4 (see [25]). *If α,β,A,B are some complex numbers, and $|\alpha|>|\beta|$, then the function*

$$f(t) := \sum_{k=0}^{\infty} \frac{t^k}{A\alpha^k + B\beta^k}$$

admits another expansion

$$f(t)=\sum_{k=0}^{r-1}\frac{t^k}{A\alpha^k+B\beta^k}-\frac{t^r}{A\alpha^r}\sum_{k=0}^{\infty}\frac{[(-B/A)(\beta/\alpha)^{r-1}]^k}{t/\alpha-(\alpha/\beta)^k},$$

where $r\in\mathbb{N}$ is chosen such that $|\alpha|^r|A|>|\beta|^r|B|$.

With the notations of Section 1.1, the Padé approximant $[n+k-1/k]_f$ is

$$[n+k-1/k]_f(t)=\frac{\tilde{Q}_k^{(n)}(t)}{\tilde{P}_k^{(n)}(t)},$$

where $\tilde{P}_k^{(n)}(t)=t^kP_k^{(n)}(t^{-1})$.

In a previous papers by the author [24,25], it has been proved that for all $n\in\mathbb{Z}$, the sequence of Padé approximants $([n+k-1/k])_k$ to f converges on any compact set included in the domain of meromorphy of the function f, with the following error term:

$$\forall t\in\mathbb{C}\setminus\{\alpha(\alpha/\beta)^j, j\in\mathbb{N}\},\ \forall n\in\mathbb{N},\quad \limsup_k|f(t)-[n+k-1/k]_f(t)|^{1/k^2}\leqslant\frac{\beta}{\alpha}, \tag{3}$$

where α and β are the two solutions of $z^2-sz+p=0$, $|\alpha|>|\beta|$.

Theorem 5. *If $\tilde{Q}_k^{(n)}(t)/\tilde{P}_k^{(n)}(t)$ denotes the Padé approximant $[n+k-1/k]_f$, then*

$$\text{(a)}\quad \tilde{P}_k^{(n)}(t)=\sum_{i=0}^{k}\binom{k}{i}_q q^{i(i-1)/2}(-t/\alpha)^i\prod_{j=1}^{i}\frac{A+Bq^{n+k-j}}{A+Bq^{n+2k-j}},$$

where

$$q:=\beta/\alpha,\ \binom{k}{i}_q:=\frac{(1-q^k)\dots(1-q^{k-i+1})}{(1-q)(1-q^2)\dots(1-q^i)},\quad 1\leqslant i\leqslant k \ (\textit{Gaussian binomial coefficient}),$$

$$\binom{k}{0}_q=1.$$

$$\text{(b)}\quad |\tilde{P}_k^{(n)}(t)-\prod_{j=0}^{k-1}(1-tq^j/\alpha)|\leqslant R|q|^k,\quad k\geqslant K_0$$

for some constant R independent of k and K_0 is an integer depending on A,B,q,n.

Moreover, if $s,p,w_{-1},w_0\in\mathbb{Z}(i)$, for all common multiple d_m of *$\{w_0,w_1,\dots,w_m\}$*

$$\text{(c)}\quad w_{n+k}\cdots w_{n+2k-1}\tilde{P}_k^{(n)}\in\mathbb{Z}(i)[t],\quad \forall n\in\mathbb{Z}/n+k-1\geqslant 0$$

and

(d) $d_{n+k-1}\ w_{n+k}\cdots w_{n+2k-1}\ \tilde{Q}_k^{(n)} \in \mathbb{Z}(i)[t], \quad \forall n \in \mathbb{Z}\ /n+k-1 \geqslant 0.$

Proof. (a) is proved in [16] and (b) is proved in [25]. (c) and (d) comes from expression (a). □

The expression of w_n is

$$w_n = A\alpha^n + B\beta^n.$$

If A or B is equal to 0 then $f(t)$ is a rational function, so without loss of generality, we can assume that $AB \neq 0$.

The degrees of $\tilde{Q}_k^{(n)}$ and $\tilde{P}_k^{(n)}$ are, respectively, $k+n-1$ and k, so if we take $t \in \mathbb{Q}(i)$ with $vt \in \mathbb{Z}(i)$, the above theorem implies that the following sequence:

$$e_{k,n} := f(t) \times v^{k'}\ d_{n+k-1} w_{n+k} \cdots w_{n+2k-1} \tilde{P}_k^{(n)}(t) - v^{k'}\ d_{n+k-1} w_{n+k} \cdots w_{n+2k-1} \tilde{Q}_k^{(n)}(t),$$

where $k' = \max\{n+k-1, k\}$ is a Diophantine approximation to $f(t)$, if

(i) $\forall n \in \mathbb{Z}, \lim_{k\to\infty} e_{k,n} = 0,$
(ii) $[n+k-1/k]_f(t) \neq [n+k/k+1]_f(t).$

For sake of simplicity, we only display the proof for the particular case $n=0$.

We set

$$e_k := e_{k,0}, \qquad \tilde{Q}_k := \tilde{Q}_k^{(0)} \quad \text{and} \quad \tilde{P}_k := \tilde{P}_k^{(0)}.$$

From the asymptotics given in (3), we get

$$\limsup_k |e_k|^{1/k^2} \leqslant \limsup_k \left| f(t) - \frac{\tilde{Q}_k(t)}{\tilde{P}_k(t)} \right|^{1/k^2} \limsup_k \left| v^k d_{k-1} w_k \cdots w_{2k-1} \tilde{P}_k(t) \right|^{1/k^2} \tag{4}$$

$$\leqslant |p| \limsup |\rho_{k-1}|^{1/k^2}, \tag{5}$$

where $\rho_k := d_k / \prod_{i=0}^k w_i$.

We will get $\lim_{k\to\infty} e_k = 0$ if the following condition is satisfied:

$$\limsup_{k\to\infty} |\rho_{k-1}|^{1/k^2} < 1/|p|.$$

Moreover, from the Christoffel–Darboux identity between orthogonal polynomials, condition (ii) is satisfied since the difference

$$\tilde{Q}_{k+1}(t)\tilde{P}_k(t) - \tilde{P}_{k+1}(t)\tilde{Q}_k(t) = t^{2k} \frac{(-1)^k}{A+B} \prod_{i=1}^k AB\, p^{2i-2} (\alpha^i - \beta^i)^2 \frac{w_{i-1}^2}{w_{2i-1}^2 w_{2i} w_{2i-2}^2}$$

is different from 0.

The following theorem is now proved.

Theorem 6. *Let f be the meromorphic function defined by the following series*:

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{w_n},$$

where $(w_n)_n$ *is a sequence of* $\mathbb{Z}(i)$ *satisfying a three-term recurrence relation*

$$w_{n+1} = s\, w_n - p\, w_{n-1}, \quad s, p \in \mathbb{Z}(i)$$

with the initial conditions: $w_{-1}, w_0 \in \mathbb{Z}(i)$. *If for each integer* m, *there exists a common multiple* d_m *for the numbers* $\{w_0, w_1, \dots, w_m\}$ *such that* ρ_m *defined by*

$$\rho_m := \frac{d_m}{\prod_{i=0}^{m} w_i}$$

satisfies the condition

$$\limsup_m |\rho_m|^{1/m^2} < 1/|p|, \tag{6}$$

then for $t \in \mathbb{Q}(i)$, $t \neq \alpha(\alpha/\beta)^j$, $j = 0, 1, 2, \dots$ *we have*

$$f(t) \notin \mathbb{Q}(i).$$

See [25] for application to Fibonacci and Lucas series. (If F_n and L_n are, respectively, Fibonacci and Lucas sequences, then $f(t) = \sum t^n/F_n$ and $g(t) = \sum t^n/L_n$ are not rational for all t rational, not a pole of the functions f or g, which is a generalization of [2].)

3. Diophantine approximation with Padé approximation to the asymptotic expansion of the remainder of the series

For sums of series f, Padé approximation to the function f does not always provide Diophantine approximation. Although the approximation error $|x - p_n/q_n|$ is very sharp, the value of the denominator q_n of the approximation may be too large such that $|q_n x - p_n|$ does not tend to zero when n tends to infinity.

Another way is the following.

Consider the series $f(t) = \sum_{i=0}^{\infty} c_i t^i = \sum_{i=0}^{n} c_i t^i + R_n(t)$. If, for some complex number t_0, we know the asymptotic expansion of $R_n(t_0)$ on the set $\{1/n^i, i = 1, 2, \dots\}$, then it is possible to construct an approximation of $f(t_0)$, by adding to the partial sums $S_n(t_0) := \sum_{i=0}^{n} c_i t_0^i$, some Padé approximation to the remainder $R_n(t_0)$ for the variable n.

But it is not sure that we will get a Diophantine approximation for two reasons.

(1) the Padé approximation to $R_n(t_0)$ may not converge to $R_n(t_0)$,
(2) the denominator of the approximant computed at t_0, can converge to infinity more rapidly than the approximation error does converge to zero.

So, this method works only for few cases.

3.1. *Irrationality of* $\zeta(2), \zeta(3)$, $\ln(1+\lambda)$ *and* $\sum_n 1/(q^n + r)$

3.1.1. *Zeta function*

The Zeta function of Riemann is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{7}$$

where the Dirichlet series on the right-hand side of (7) is convergent for $\mathrm{Re}(s) > 1$ and uniformly convergent in any finite region where $\mathrm{Re}(s) \geqslant 1 + \delta$, with $\delta > 0$. It defines an analytic function for $\mathrm{Re}(s) > 1$.

Riemann's formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx, \qquad \mathrm{Re}(s) > 1,$$

where

$$\Gamma(s) = \int_0^\infty y^{s-1} e^{-y}\,dy \text{ is the gamma function} \tag{8}$$

and

$$\zeta(s) = \frac{e^{-i\pi s}\Gamma(1-s)}{2i\pi} \int_{\mathscr{C}} \frac{z^{s-1}}{e^z - 1}\,dz \tag{9}$$

where $\mathscr{C}$ is some path in $\mathbb{C}$, provides the analytic continuation of $\zeta(s)$ over the whole s-plane.

If we write formula (7) as

$$\zeta(s) = \sum_{k=1}^{n} \frac{1}{k^s} + \sum_{k=1}^{\infty} \frac{1}{(n+k)^s}$$

and set $\Psi_s(x) := \Gamma(s) \sum_{k=1}^{\infty} (x/(1+kx))^s$ then

$$\zeta(s) = \sum_{k=1}^{n} \frac{1}{k^s} + \frac{1}{\Gamma(s)} \Psi_s(1/n). \tag{10}$$

The function $\sum_{k=1}^{\infty} (x/(1+kx))^s)$ is known as the generalized zeta-function $\zeta(s, 1+1/x)$ [32, Chapter XIII] and so we get another expression of $\Psi_s(x)$:

$$\Psi_s(x) = \int_0^\infty u^{s-1} \frac{e^{-u/x}}{e^u - 1}\,du, \quad x > 0,$$

whose asymptotic expansion is

$$\Psi_s(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \Gamma(k+s-1) x^{k+s-1},$$

where B_k are the Bernoulli numbers.

Outline of the method: In (10), we replace the unknown value $\Psi_s(1/n)$ by some Padé-approximant to $\Psi_s(x)$, at the point $x = 1/n$. We get the following approximation:

$$\zeta(s) \approx \sum_{k=1}^{n} \frac{1}{k^s} + \frac{1}{\Gamma(s)} [p/q]_{\Psi_s}(x = 1/n). \tag{11}$$

We only consider the particular case $p = q$.

Case $\zeta(2)$: If $s=2$ then (10) becomes

$$\zeta(2)=\sum_{k=1}^{n}\frac{1}{k^2}+\Psi_2(1/n),$$

and its approximation (11):

$$\zeta(2)\approx\sum_{k=1}^{n}\frac{1}{k^2}+[p/p]_{\Psi_2}(x=1/n), \tag{12}$$

where

$$\Psi_2(x)=\sum_{k=0}^{\infty}B_kx^{k+1}=B_0x+B_1x^2+B_2x^3+\cdots \quad \text{(asymptotic expansion)}. \tag{13}$$

The asymptotic expansion (13) is Borel-summable and its sum is

$$\Psi_2(x)=\int_0^{\infty}u\frac{\mathrm{e}^{-u/x}}{\mathrm{e}^u-1}\,\mathrm{d}u.$$

Computation of $[p/p]_{\Psi_2(x)/x}$: We apply Section 1.1, where function $f(x)=\Psi_2(x)/x$. The Padé approximants $[p/p]_f$ are linked with the orthogonal polynomial with respect to the sequence $B_0,B_1,B_2\ldots$.

As in Section 1, we define the linear functional B acting on the space of polynomials by

$$B:\mathscr{P}\to\mathbb{R}$$

$$x^i\to\langle B,x^i\rangle=B_i,\quad i=0,1,2,\ldots.$$

The orthogonal polynomials Ω_p satisfy

$$\langle B,x^i\Omega_p(x)\rangle=0,\quad i=0,1,\ldots,p-1. \tag{14}$$

These polynomials have been studied by Touchard ([31,9,28,29]) and generalized by Carlitz ([12,13]).

The following expressions

$$\Omega_p(x)=\sum_{2r\leqslant p}\binom{2x+p-2r}{p-2r}\binom{x}{r}^2$$

$$=(-1)^p\sum_{k=0}^{p}(-1)^k\binom{p}{k}\binom{p+k}{k}\binom{x+k}{k}=\sum_{k=0}^{p}\binom{p}{k}\binom{p+k}{k}\binom{x}{k} \tag{15}$$

hold (see [34,12]).

Note that the Ω_p's are orthogonal polynomials and thus satisfy a three-term recurrence relation.

The associated polynomials Λ_p of degree $p-1$ are defined as

$$\Lambda_p(t)=\left\langle B,\ \frac{\Omega_p(x)-\Omega_p(t)}{x-t}\right\rangle,$$

where B acts on x.

From expression (15) for Ω_p, we get the following formula for Λ_p:

$$\Lambda_p(t)=\sum_{k=0}^{p}\binom{p}{k}\binom{p+k}{k}\left\langle B,\frac{\binom{x}{k}-\binom{t}{k}}{x-t}\right\rangle.$$

The recurrence relation between the Bernoulli numbers B_i implies that

$$\left\langle B, \binom{x}{k} \right\rangle = \frac{(-1)^k}{k+1}.$$

Using the expression of the polynomial $(\binom{x}{k} - \binom{t}{k})/(x-t)$ on the Newton basis on $0, 1, \ldots, k-1$,

$$\frac{\binom{x}{k} - \binom{t}{k}}{x-t} = \binom{t}{k} \sum_{i=1}^{k} \frac{\binom{x}{i-1}}{i\binom{t}{i}},$$

we can write a compact formula for Λ_p:

$$\Lambda_p(t) = \sum_{k=1}^{p} \binom{p}{k} \binom{p+k}{k} \binom{t}{k} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{i^2 \binom{t}{i}} \in \mathscr{P}_{p-1}\ .$$

Approximation (12) for $\zeta(2)$ becomes

$$\zeta(2) \approx \sum_{k=1}^{n} \frac{1}{k^2} + t \left. \frac{\tilde{\Lambda}_p(t)}{\tilde{\Omega}_p(t)} \right|_{t=1/n} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{\Lambda_p(n)}{\Omega_p(n)}.$$

Using partial decomposition of $1/\binom{n}{i}$ with respect to the variable n, it is easy to prove that

$$\frac{d_n}{i\binom{n}{i}} \in \mathbb{N}, \quad \forall i \in \{1, 2, \ldots, n\} \tag{16}$$

with d_n:=LCM$(1, 2, \ldots, n)$.

A consequence of the above result is

$$d_n^2 \Lambda_p(n) \in \mathbb{N}, \quad \forall p \in \mathbb{N}$$

and

$$d_n^2 \Omega_p(n) \zeta(2) - d_n^2 (S_n \Omega_p(n) + \Lambda_p(n)) \tag{17}$$

is a Diophantine approximation of $\zeta(2)$, for all values of integer p, where S_n denotes the partial sums $S_n = \sum_{k=1}^{n} 1/k^2$. It remains to estimate the error for the Padé approximation:

$$\Psi_2(t) - [p/p]_{\Psi_2}(t) = \Psi_2(t) - [p-1/p]_{\Psi_2/t}(t).$$

Touchard found the integral representation for the linear functional B:

$$\langle B, x^k \rangle := B_k = -\mathrm{i}\frac{\pi}{2} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} x^k \frac{\mathrm{d}x}{\sin^2(\pi x)}, \quad -1 < \alpha < 0.$$

Thus, formula (1) becomes

$$t^{-1}\Psi_2(t)-[p-1/p]_{\Psi_{2/t}}(t)=-\mathrm{i}\frac{\pi}{2}\frac{t^{2p}}{\tilde{\Omega}_p^2(t)}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\frac{\Omega_p^2(x)}{1-xt}\frac{\mathrm{d}x}{\sin^2(\pi x)},$$

and we obtain the error for the Padé approximant to Ψ_2:

$$\Psi_2(t)-[p/p]_{\Psi_2}(t)=-\mathrm{i}\frac{\pi}{2}\frac{t}{\Omega_p^2(t^{-1})}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\frac{\Omega_p^2(x)}{1-xt}\frac{\mathrm{d}x}{\sin^2(\pi x)}$$

and the error for formula (17):

$$d_n^2\Omega_p(n)\zeta(2)-d_n^2(S_n\Omega_p(n)+\Lambda_p(n))=-d_n^2\mathrm{i}\frac{\pi}{2n}\frac{1}{\Omega_p(n)}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\frac{\Omega_p^2(x)}{1-x/n}\frac{\mathrm{d}x}{\sin^2(\pi x)}. \tag{18}$$

If $p=n$, we get Apéry's numbers [4]:

$$b'_n=\Omega_n(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$$

and

$$a'_n=S_n\Omega_n(n)+\Lambda_n(n)=\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)\quad b'_n+\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{k}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{i^2\binom{n}{i}}.$$

The error in formula (18) becomes

$$d_n^2b'_n\zeta(2)-d_n^2a'_n=-d_n^2\mathrm{i}\frac{\pi}{2n}\frac{1}{b'_n}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty}\frac{\Omega_n^2(x)}{1-x/n}\frac{\mathrm{d}x}{\sin^2\pi x} \tag{19}$$

In order to prove the irrationality of $\zeta(2)$, we have to show that the right-hand side of (19) tends to 0 when n tends to infinity, and is different from 0, for each integer n.

We have

$$\left|\int_{-1/2-\mathrm{i}\infty}^{-1/2+\mathrm{i}\infty}\frac{\Omega_n^2(x)}{1-x/n}\frac{\mathrm{d}x}{\sin^2\pi x}\right|\leqslant\left|\int_{-\infty}^{+\infty}\frac{\Omega_n^2(-\frac{1}{2}+\mathrm{i}u)}{1+1/2n}\frac{\mathrm{d}u}{\cosh^2\pi u}\right|\leqslant\frac{1}{1+1/2n}\left|\langle B,\Omega_n^2(x)\rangle\right|$$

since $\cosh^2\pi u$ is positive for $u\in\mathbb{R}$ and $\Omega_n^2(-\frac{1}{2}+\mathrm{i}u)$ real positive for u real (Ω_n has all its roots on the line $-\frac{1}{2}+\mathrm{i}\mathbb{R}$, because $\Omega_n(-\frac{1}{2}+\mathrm{i}u)$ is orthogonal with respect to the positive weight $1/\cosh^2\pi u$ on $\mathbb{R}$). The quantity $\langle B,\Omega_n^2(x)\rangle$ can be computed from the three term recurrence relation between the Ω'_ns [31]:

$$\langle B,\Omega_n^2(x)\rangle=\frac{(-1)^n}{2n+1}.$$

The Diophantine approximation (19) satisfies

$$|d_n^2b'_n\zeta(2)-d_n^2a'_n|\leqslant d_n^2\frac{\pi}{(2n+1)^2}\times\frac{1}{b'_n}.$$

In [14], it is proved that $b'_n \sim A'((1+\sqrt{5})/2)^{5n} n^{-1}$ when $n \to \infty$, for some constant A'. From a result concerning $d_n = \mathrm{LCM}(1,2,\dots,n)$: $(d_n = \mathrm{e}^{(n(1+\mathrm{o}(1))})$, we get

$$\lim_{n\to\infty} |d_n^2 b'_n \zeta(2) - d_n^2 a'_n| = 0, \tag{20}$$

where $d_n^2 b'_n$ and $d_n^2 a'_n$ are integers.

Relation (20) proves that $\zeta(2)$ is not rational.

Case $\zeta(3)$: If $s=3$ then equality (10) becomes

$$\zeta(3) = \sum_{k=1}^{n} \frac{1}{k^3} + \frac{1}{2}\Psi_3(1/n), \tag{21}$$

where

$$\Psi_3(x) = \int_0^\infty u^2 \frac{\mathrm{e}^{-u/x}}{\mathrm{e}^u - 1}\,\mathrm{d}u$$

whose asymptotic expansion is

$$\Psi_3(x) = \sum_{k=0}^{\infty} B_k (k+1) x^{k+2}.$$

Computation of $[p/p]_{\Psi_3(x)/x^2}$: Let us define the derivative of B by

$$\langle -B', x^k \rangle := \langle B, kx^{k-1} \rangle = kB_{k-1}, \quad k \geqslant 1,$$

$$\langle -B', 1 \rangle := 0.$$

So, the functional B' admits an integral representation:

$$\langle B', x^k \rangle = \mathrm{i}\pi^2 \int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} x^k \frac{\cos(\pi x)}{\sin^3(\pi x)}\,\mathrm{d}x, \quad -1 < \alpha < 0.$$

Let $(\Pi_n)_n$ be the sequence of orthogonal polynomial with respect to the sequence

$$-B'_0 := 0, \qquad -B'_1 = B_0, \qquad -B'_2 = 2B_1, \qquad -B'_3 = 3B_2, \dots .$$

The linear form B' is not definite and so the polynomials Π_n are not of exact degree n.

More precisely, Π_{2n} has degree $2n$ and $\Pi_{2n+1} = \Pi_{2n}$. For the general theory of orthogonal polynomials with respect to a nondefinite functional, the reader is referred to Draux [15]. If we take $\alpha = -\frac{1}{2}$, the weight $\cos \pi x / \sin^3(\pi x)\,\mathrm{d}x$ on the line $-\frac{1}{2} + \mathrm{i}\mathbb{R}$ becomes $\sinh \pi t / \cosh^3 \pi t\,\mathrm{d}t$ on $\mathbb{R}$, which is symmetrical around 0. So, $\Pi_{2n}(\mathrm{i}t - \frac{1}{2})$ only contains even power of t and we can write

$\Pi_{2n}(\mathrm{i}t - \frac{1}{2}) = W_n(t^2)$, W_n of exact degree n. Thus W_n satisfies

$$\int_{\mathbb{R}} W_n(t^2) W_m(t^2) \frac{t \sinh \pi t}{\cosh^3 \pi t}\,\mathrm{d}t = 0, \quad n \neq m.$$

The weight $t \sinh \pi t / \cosh^3 \pi t$ equals $(1/4\pi^3)|\Gamma(\frac{1}{2} + \mathrm{i}t)|^8 |\Gamma(2\mathrm{i}t)|^2$ and has been studied by Wilson [33,3]:

$$n \geqslant 0, \quad \Pi_{2n}(y) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{y+k}{k} \binom{y}{k}. \tag{22}$$

Let Θ_{2n} the polynomial associated to Π_{2n}:

$$\Theta_{2n}(t)=\left\langle -B',\frac{\Pi_{2n}(x)-\Pi_{2n}(t)}{x-t}\right\rangle,\quad B' \text{ acts on } x.$$

For the computation of Θ_{2n}, we need to expand the polynomial

$$\frac{\binom{x+k}{k}\binom{x}{k}-\binom{t+k}{k}\binom{t}{k}}{x-t}.$$

On the Newton basis with the abscissa $\{0,1,-1,\dots,n,-n\}$

$$\frac{\binom{x+k}{k}\binom{x}{k}-\binom{t+k}{k}\binom{t}{k}}{x-t}=\sum_{i=1}^{2k}\frac{N_{2k}(t)}{N_i(t)}\frac{N_{i-1}(x)}{[(i+1)/2]},$$

where $N_0(x){:=}1$, $N_1(x)=\binom{x}{1}$, $N_2(x)=\binom{x}{1}\binom{x+1}{1},\dots,N_{2i}(x)=\binom{x}{i}\binom{x+i}{i}$ $N_{2i+1}(x)=\binom{x}{i+1}\binom{x+i}{i}$.

By recurrence, the values $\langle -B',N_i(x)\rangle$ can be found in

$$i\in\mathbb{N},\quad \langle -B',N_{2i}(x)\rangle=0,\quad \langle -B',N_{2i+1}(x)\rangle=\frac{(-1)^i}{(i+1)^2}.$$

Using the linearity of B', we get the expression of Θ_{2n}:

$$\Theta_{2n}(t)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{i=1}^{k}\frac{(-1)^{i+1}}{i^3}\frac{\binom{t+k}{k-i}\binom{t-i}{k-i}}{\binom{k}{i}^2}\in\mathscr{P}_{2n-2}. \tag{23}$$

Eq. (16) implies that

$$d_n^3\Theta_{2n}(t)\in\mathbb{N},\quad \forall t\in\mathbb{N}.$$

The link between Π_{2n}, Θ_{2n} and the Apéry's numbers a_n, b_n is given by taking $y=n$ in (22) and $t=n$ in (23):

$$\Pi_{2n}(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=b_n,$$

$$\left(\sum_{k=1}^{n}\frac{1}{k^3}\right)\Pi_{2n}(n)+\frac{1}{2}\Theta_{2n}(n)=a_n.$$

Apéry was the first to prove irrationality of $\zeta(3)$. He only used recurrence relation between the a_n and b_n. We end the proof of irrationality of $\zeta(3)$ with the error term for the Padé approximation.

Let us recall equality (21),

$$\zeta(3)=\sum_{k=1}^{n}\frac{1}{k^3}+\frac{1}{2}\Psi_3\left(\frac{1}{n}\right)$$

in which we replace the unknown term $\Psi_3(1/n)$ by its Padé approximant $[2n/2n]_{\Psi_3}(x=1/n)$. It arises the following approximation for $\zeta(3)$:

$$\zeta(3) \approx \sum_{k=1}^{n} \frac{1}{k^3} + \frac{1}{2}\frac{\Theta_{2n}(n)}{\Pi_{2n}(n)}$$

and the expression

$$e_n = 2d_n^3 \Pi_{2n}(n)\zeta(3) - \left[\left(\sum_{k=1}^{n}\frac{1}{k^3}\right) 2\Pi_{2n}(n) + \Theta_{2n}(n)\right] d_n^3$$

will be a Diophantine approximation, if we prove that $\lim_n e_n = 0$ (since $\Pi_{2n}(n)$ and $d_n^3\Theta_{2n}(n)$ are integer).

Let us estimate the error e_n. The method is the same as for $\zeta(2)$:

$$\Psi_3(t) - [2n/2n]_{\Psi_3}(t) = \Psi_3(t) - t^2[2n-2/2n]_{\Psi_3/t^2}(t) = \Psi_3(t) - \frac{\Theta_{2n}(t^{-1})}{\Pi_{2n}(t^{-1})}.$$

The integral representation of B' gives

$$\Psi_3(t) - [2n/2n]_{\Psi_3}(t) = -\frac{t\pi^2 \mathrm{i}}{\Pi_{2n}^2(t^{-1})}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \frac{\Pi_{2n}^2(x)}{1-xt}\frac{\cos\pi x}{\sin^3\pi x}\,\mathrm{d}x.$$

The previous expression implies that the error $\Psi_3(t) - [2n/2n]_{\Psi_3}(t)$ is nonzero, and also that

$$|\Psi_3(t) - [2n/2n]_{\Psi_3}(t)| \leqslant \frac{\pi^2 t}{\Pi_{2n}^2(t^{-1})}\cdot\frac{1}{1+t/2}\cdot\int_{\mathbb{R}} W_n^2(u^2)\frac{u\sinh\pi u}{\cosh^3\pi u}\,\mathrm{d}u, \quad t\in\mathbb{R}^+.$$

From the expression of the integral (see [33]) we get

$$|\Psi_3(1/n) - [2n/2n]_{\Psi_3}(1/n)| \leqslant \frac{4\pi^2}{(2n+1)^2\Pi_{2n}^2(n)}.$$

The error term in the Padé approximation satisfies

$$\left|2\zeta(3) - 2\sum_{k=1}^{n}\frac{1}{k^3} - [2n/2n]_{\Psi_3}(1/n)\right| \leqslant \frac{4\pi^2}{(2n+1)^2\Pi_{2n}^2(n)}$$

and the error term e_n satisfies

$$|e_n| = \left|2d_n^3\Pi_{2n}(n)\zeta(3) - \left[2\left(\sum_{k=1}^{n}\frac{1}{k^3}\right)\Pi_{2n}(n) + \Theta_{2n}(n)\right] d_n^3\right| \leqslant \frac{8\pi^2}{(2n+1)^2}\frac{d_n^3}{\Pi_{2n}(n)}.$$

$\Pi_{2n}(n) = b_n$ implies that $\Pi_{2n}(n) = A(1+\sqrt{2})^{4n}n^{-3/2}$ [14], and so we get, since $d_n = \mathrm{e}^{n(1+\mathrm{o}(1))}$,

$$|2d_n^3 b_n\zeta(3) - 2d_n^3 a_n| \to 0, \quad n\to\infty, \tag{24}$$

where $2d_n^3 b_n$ and $2d_n^3 a_n$ are integers.

The above relation (24) shows that $\zeta(3)$ is irrational.

Of course, using the connection between Padé approximation and ε-algorithm, the Diophantine approximation of $\zeta(3)$ can be constructed by means of the following ε-array: $a_n/b_n = \sum_{k=1}^{n} 1/k^3 + \varepsilon_{4n}^{(0)}(T_m) = \varepsilon_{4n}^{(0)}(\sum_{k=1}^{n} 1/k^3 + T_m)$, where T_m is the partial sum of the asymptotic series (nonconvergent) $T_m = \frac{1}{2}\sum_{k=1}^{m} B_k(k+1)1/n^k$.

We get the following ε-arrays for $n=1$,

$$\begin{bmatrix} 0 & & \\ 0 & 0 & \\ 1 & 1/2 & 2/5 = \varepsilon_4^{(0)} \\ 0 & 1/3 & \\ 1/2 & & \end{bmatrix}, \qquad 1 + \tfrac{1}{2} * \varepsilon_4^{(0)} = \tfrac{6}{5} = a_1/b_1 \quad \text{(Apery's numbers)},$$

and for $n=2$,

$$\begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ 1/4 & 1/6 & 2/13 & & \\ 1/8 & 3/20 & 2/13 & 2/13 & \\ 5/32 & 5/32 & 21/136 & 37/240 & 45/292 = \varepsilon_8^{(0)} \\ 5/32 & 5/32 & 2/13 & 53/344 & \\ 59/384 & 59/384 & 37/240 & & \\ 59/384 & 59/384 & & & \\ 79/512 & & & & \end{bmatrix}$$

(we have only displayed the odd columns), $1 + 1/2^3 + 1/2 * \varepsilon_8^{(0)} = 351/292 = a_2/b_2$. ε-algorithm is a particular extrapolation algorithm as Padé approximation is particular case of Padé-type approximation. Generalization has been achieved by Brezinski and Hävie, the so-called E-algorithm. Diophantine approximation using E-algorithm and Padé-type approximation are under consideration.

3.1.2. *Irrationality of* $\ln(1+\lambda)$

In this part, we use the same method as in the preceding section:

$$\text{We set } \ln(1+\lambda) = \sum_{k=1}^{n} (-1)^{k+1}\frac{\lambda^k}{k} + \sum_{k=1}^{\infty} \frac{(-1)^{k+n+1}}{k+n}\lambda^{k+n}. \tag{25}$$

From the formula $1/(k+n) = \int_0^\infty e^{-(k+n)v}\,dv$, we get an integral representation for the remainder term in (25):

$$\sum_{k=1}^{\infty} (-1)^{k+n+1}\frac{\lambda^{k+n}}{k+n} = (-1)^n \int_0^\infty \lambda^{n+1} \frac{e^{-nv}}{e^v + \lambda}\,dv.$$

If we expand the function

$$\frac{1+\lambda}{e^v+\lambda}=\sum_{k=0}^{\infty}R_k(-\lambda)\frac{v^k}{k!},$$

where the $R_k(-\lambda)'$s are the Eulerian numbers [12], we get the following asymptotic expansion:

$$\sum_{k=1}^{\infty}(-1)^{k+n+1}\frac{\lambda^{k+n}}{k+n}=\frac{(-1)^n\lambda^{n+1}}{n(1+\lambda)}\left(\sum_{k=0}^{\infty}R_k(-\lambda)x^k\right)_{x=1/n}.$$

Let us set

$$\Phi_1(x)=\sum_{k=0}^{\infty}R_k(-\lambda)x^k.$$

Carlitz has studied the orthogonal polynomials with respect to $R_0(-\lambda)$, $R_1(-\lambda),\dots$.

If we define the linear functional R by

$$\langle R,x^k\rangle:=R_k(-\lambda),$$

then the orthogonal polynomials P_n with respect to R, i.e.,

$$\langle R,x^kP_n(x)\rangle=0,\quad k=0,1,\dots,n-1,$$

satisfy $P_n(x)=\sum_{k=0}^{n}(1+\lambda)^k\binom{n}{k}\binom{x}{k}$ [12].

The associated polynomials are

$$Q_n(t)=\sum_{k=0}^{n}(1+\lambda)^k\binom{n}{k}\left\langle R,\frac{\binom{x}{k}-\binom{t}{k}}{x-t}\right\rangle. \tag{26}$$

Carlitz proved that $\langle R,\binom{x}{k}\rangle=(-\lambda-1)^{-k}$ and thus, using (26),

$$Q_n(t)=\sum_{k=0}^{n}(1+\lambda)^k\binom{n}{k}\binom{t}{k}\sum_{i=1}^{k}\frac{1}{i\binom{t}{i}}\left(\frac{-1}{\lambda+1}\right)^{i-1}.$$

If we set $\lambda=p/q$, p and $q\in\mathbb{Z}$ and $t=n$, then

$$q^nd_nQ_n(n)\in\mathbb{Z}.$$

An integral representation for $R_k(-\lambda)$ is given by Carlitz:

$$R_k(-\lambda)=-\frac{1+\lambda}{2i\lambda}\int_{\alpha-i\infty}^{\alpha+i\infty}z^k\frac{\lambda^{-z}}{\sin\pi z}\,dz,\quad -1<\alpha<0, \tag{27}$$

and thus

$$\Phi_1(x)=-\frac{1+\lambda}{2i\lambda}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{1-xz}\frac{\lambda^{-z}}{\sin\pi z}\,dz.$$

The orthogonal polynomial P_n satisfies [12]

$$\int_{\alpha-i\infty}^{\alpha+i\infty}P_n^2(z)\frac{\lambda^{-z}}{\sin\pi z}dz=\frac{+2i}{i+\lambda}(-\lambda)^{n+1},$$

and since $\mathrm{Re}(\lambda^{-z}\sin\pi z) > 0$ for $z \in -\frac{1}{2} + \mathrm{i}\mathbb{R}$, we obtain a majoration of the error for the Padé approximation to Φ_1:

$$x > 0, \left|\Phi_1(x) - [n-1/n]_{\Phi_1}(x)\right| \leqslant \frac{\lambda^n}{|1+x/2|}$$

and if $x = 1/n$, we get

$$\left|\Phi_1\left(\frac{1}{n}\right) - [n-1/n]_{\Phi_1}(1/n)\right| \leqslant \frac{|\lambda|^n}{1+1/2n}.$$

Let us replace in (25) the remainder term by its Padé approximant:

$$\ln(1+\lambda) \approx \sum_{k=1}^{n} (-1)^{k+1}\frac{\lambda^k}{k} + \frac{(-1)^n\lambda^{n+1}}{(1+\lambda)n}[n-1/n]_{\Phi_1}(1/n),$$

we obtain a Diophantine approximation for $\ln(1+p/q)$:

$$\left|\ln\left(1+\frac{p}{q}\right) d_n q^{2n} P_n(n) - d_n q^{2n} T_n(n)\right| \leqslant \frac{\lambda^{2n} d_n q^{2n}}{(n+2)P_n(n)}, \tag{28}$$

where $T_n(n) = P_n(n)\sum_{k=1}^{n}(-1)^{k+1}p^k/kq^k + (-1)^{n+1}Q_n(n)q^n$.

From the expression of $P_n(x)$ we can conclude that

$$P_n(n) = \sum_{k=0}^{n}(1+\lambda)^k \binom{n}{k}^2 = \text{Legendre}\left(n, \frac{2}{\lambda}+1\right)\lambda^n,$$

where Legendre (n,x) is the nth Legendre polynomial and thus

$$\frac{T_n(n)}{P_n(n)} = [n/n]_{\ln(1+x)}\ (x=1).$$

So, the classical proof for irrationality of $\ln(1+p/q)$ based on Padé approximants to the function $\ln(1+x)$ is recovered by formula (28).

Proof of irrationality of $\zeta(2)$ with alternated series: Another expression for $\zeta(2)$ is

$$\zeta(2) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}.$$

Let us write it as a sum

$$\zeta(2) = 2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2} + 2\sum_{k=1}^{\infty}\frac{(-1)^{k+n+1}}{(k+n)^2}.$$

Let Φ_2 be defined by $\Phi_2(x) = \sum_{k=0}^{\infty} R_k(-1)(k+1)x^k$. So

$$\zeta(2) = 2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2} + \frac{(-1)^n}{n^2}\Phi_2(1/n).$$

With the same method, we can prove that the Padé approximant $[2n/2n]_{\Phi_2}(x)$ computed at $x = 1/n$ leads to Apéry's numbers a_n' and b_n' and so proves the irrationality of $\zeta(2)$ with the integral representation for the sequence $(kR_{k-1}(-1))_k$:

$$kR_{k-1}(-1) = -\frac{\pi(1+\lambda)}{2\mathrm{i}\lambda}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} z^k \frac{\cos\pi z}{\sin^2\pi z}\,\mathrm{d}z, \quad k \geqslant 1.$$

obtained with an integration by parts applied to (27).

3.1.3. Irrationality of $\sum 1/(q^n + r)$

In [7], Borwein proves the irrationality of $L(r) = \sum 1/(q^n - r)$, for q an integer greater than 2, and r a non zero rational (different from q^n, for any $n \geqslant 1$), by using similar method. It is as follows:

Set

$$L_q(x) := \sum_{n=1}^{\infty} \frac{x}{q^n - x} = \sum_{n=1}^{\infty} \frac{x^n}{q^n - 1}, \quad |q| > 1.$$

Fix N a positive integer and write $L_q(r) = \sum_{n=1}^{N} r/(q^n - r) + L_q(r/q^N)$.

Then, it remains to replace $L_q(r/q^N)$ by its Padé approximant $[N/N]_{L_q}(r/q^N)$.

The convergence of $[N/N]_{L_q}$ to L_q is a consequence of the following formula:

$$\forall t \in \mathbb{C} \setminus \{q^j, j \in \mathbb{N}\}, \quad \forall n \in \mathbb{N}, \quad \limsup_N |L_q(t) - [N/N]_{L_q}(t)|^{1/3N^2} \leqslant 1/q.$$

p_n/q_n defined by $p_n/q_n := \sum_{n=1}^{N} r/(q^n - r) + [N/N]_{L_q}(r/q^N)$ leads to Diophantine approximation of $L_q(r)$ and so proves the irrationality of $L_q(r)$.

For further results concerning the function L_q, see [17–19].

Different authors used Padé or Padé Hermite approximants to get Diophantine approximation, see for example [8,20–23,27].

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