On the Hankel transform of generalized central trinomial coeffcients

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Outline of the talk

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1. Introduction
   - Integer sequences
   - Hankel transform
   - Computing the Hankel transform via orthogonal polynomials
2. Generalized central trinomial coefficients
   - Definition and generating function
   - Hankel transform of generalized central trinomial coefficients
   - Coefficient array
3. Hankel transform and k-binomial transforms
   - Binomial transform and generalizations
   - Generalized binomial transforms and Hankel transform
   - Connection with generalized central trinomial coefficients

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Integer sequences

Let $a : \mathbb{N}_0 \longrightarrow \mathbb{Z}$ represent an integer sequence with $a_n = a(n)$.
The ordinary generating function of a_n is $f(x) = \sum_{k=0}^{\infty} a_n x^n$.

Example . Central binomial coefficients

$$a_n = \binom{2n}{n} \qquad f(x) = \frac{1}{\sqrt{1-4x}} \qquad 1, 2, 6, 20, 70, \ldots$$

Example . Central trinomial coefficients

$$t_n = [x^n](1+x+x^2)^n \qquad f(x) = \frac{1}{\sqrt{1-2x-3x^2}} \qquad 1, 1, 3, 7, 19, \ldots$$

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Hankel transform

- J.W. Layman, *The Hankel Transform and Some of its Properties*, Journal of Integer Sequences, Article 01.1.5, Volume 4, 2001..

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The Hankel transform of a given sequence $a = \{a_n\}_{n\in\mathbb{N}_0}$ is the sequence of Hankel determinants $h = \mathbf{H}(a) = \{h_n\}_{n\in\mathbb{N}_0}$ where $h_n = \det[a_{i+j-2}]_{i,j=1}^{n}$, i.e

$$a = \{a_n\}_{n\in\mathbb{N}_0} \quad \stackrel{\mathbf{H}}{\Longrightarrow} \quad h = \{h_n\}_{n\in\mathbb{N}_0} : \quad h_n = \det\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{bmatrix}$$

Example

The Hankel transform of the central binomial coefficients $\left\{\binom{2n}{n}\right\}_{n\in\mathbb{N}_0}$ is the sequence $\{2^n\}_{n\in\mathbb{N}_0}$. That is,

$$|1| = 1, \quad \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 2, \quad \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 20 \\ 6 & 20 & 70 \end{vmatrix} = 4, \quad \ldots$$

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Example . (Cvetković, Rajković, Ivković, 2002)

Let $a_n = C_n + C_{n+1}$ where C_n is n-th Catalan number. Then $\mathbf{H}(a) = \{F_{2n+1}\}_{n\in\mathbb{N}_0}$, where F_n is n-th Fibonacci number.

Generalization:

- P.M. Rajković, M.D. Petković, P. Barry, *The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers*, Integral Transforms and Special Functions, Vol **18/4** (January 2007), 285 – 296..

Some applications:

1. Aztec diamond counting.

- R. Brualdi, S. Kirkland, *Aztec diamonds and digraphs, and Hankel determinants of Schroder numbers*, Journal of Combinatorial Theory, Series B **94** (2005) 334 - 351..

2. Solving the Toda equation.

- K. Kajiwara, M. Mazzocco, Y. Ohta, *A Remark on the Hankel Determinant Formula for Solutions of the Toda Equation*, Journal of Physics A: Mathematical and Theoretical, Vol **40** (2007), Issue 42, 12661–12675..

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Computing the Hankel transform via orthogonal polynomials

Let $d\mu$ be the positive measure on $\mathbb{R}$, $\{\pi_n(x)\}_{n\in\mathbb{N}_0}$ corresponding MOPS and $a_n = \int_{\mathbb{R}} x^n d\mu$. Consider the three-term recurrence relation

$$\pi_{n+1}(x) = (x - \alpha_n)\pi_n(x) - \beta_n \pi_{n-1}(x), \quad n \in N_0$$

Theorem . (Heilermann)

The Hankel determinant $\det_{0\le i,j\le n-1}(a_{i+j})$ *is given by*

$$\det_{0\le i,j\le n-1}[a_{i+j}] = a_0^n \beta_1^{n-1} \beta_2^{n-2} \ldots \beta_{n-2}^2 \beta_{n-1}.$$

Applicable to any sequence whose Hankel transform is positive (Hamburger moment problem).

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Let $\{a_n\}_{n\in\mathbb{N}_0}$ is the sequence whose Hankel transform is positive. Let

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n, \qquad g(z) = \tfrac{1}{z} f(\tfrac{1}{z}).$$

Theorem . (Stieltjes-Perron inversion formula)

Then $a_n = \int_{\mathbb{R}} x^n d\lambda$ where

$$\lambda(t) - \lambda(0) = -\frac{1}{2\pi i} \lim_{y\to 0^+} \int_0^t \Big[g(x+iy) - g(x-iy)\Big] dx.$$

Moreover if $d\lambda = w(t)dt$ and $g(\overline{z}) = \overline{g(z)}$ then holds

$$w(t) = \frac{1}{\pi} \lim_{y\to 0^+} \Im g(t+i, y)$$

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Lemma . (Transformation lemma)

(1) *If $\tilde{w}(x) = Cw(x)$ where $C > 0$ then holds $\tilde{\alpha}_n = \alpha_n$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = C\beta_0$, $\tilde{\beta}_n = \beta_n$ for $n \in \mathbb{N}$. Additionally holds $\tilde{\pi}_n(x) = \pi_n(x)$ for all $n \in \mathbb{N}_0$.*

(2) *If $\tilde{w}(x) = w(ax+b)$ where $a, b \in \mathbb{R}$ and $a \neq 0$ there holds $\tilde{\alpha}_n = \frac{\alpha_n - b}{a}$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = \frac{\beta_0}{|a|}$ and $\tilde{\beta}_n = \frac{\beta_n}{a^2}$ for $n \in \mathbb{N}$. Additionally holds $\tilde{\pi}_n(x) = \frac{1}{a^n}\pi_n(ax+b)$.*

(3) *If $\tilde{w}(x) = (x-c)w(x)$ where $c < \inf \operatorname{supp}(w)$, there holds*

$$\tilde{\beta}_0 = \int_{\mathbb{R}} \tilde{w}(x)dx, \quad \tilde{\beta}_n = \beta_n \frac{r_n}{r_{n-1}}, (n \in \mathbb{N}), \quad \tilde{\alpha}_n = \alpha_{n+1} + r_{n+1} - r_n, (n \in \mathbb{N}_0)$$

where temporary sequence $\{r_n\}_{n \in \mathbb{N}_0}$ is defined by

$$r_0 = c - \alpha_0, \qquad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \qquad (n \in \mathbb{N}).$$

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Generalized central trinomial coefficients

- T. D. Noe, *On the Divisibility of Generalized Central Trinomial Coefficients*, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7..

$$\tau_n(a,b,c) = [t^n](a+bt+ct^2)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \binom{n}{2k} b^{n-2k}(ac)^k$$

First few members are $1, b, b^2+2ac, b^3+6abc, b^4+12ab^2c+6a^2c^2, \ldots$
Holds $t_n = \tau_n(1,1,1)$.

Theorem . (Noe 2006)

Generating function of the sequence $\{\tau_n(a,b,c)\}_{n\in\mathbb{N}_0}$ is given by

$$f(x) = \frac{1}{\sqrt{1-2bx+dx^2}} = \sum_{n=0}^{\infty} \tau_n(a,b,c)x^n.$$

where $d = b^2-4ac$ is the discriminant.

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Hankel transform of generalized trinomial coefficients

Since

$$g(x) = \tfrac{1}{x} f(\tfrac{1}{x}) = \frac{1}{\sqrt{(x-b)^2 - 4ac}},$$

Applying the Stieltjes-Perron inversion formula and taking into account the regular branches of square root in $g(x)$ we obtain:

Theorem .

There holds

$$\tau_n(a,b,c) = \frac{1}{\pi} \int_{b-2\sqrt{c}}^{b+2\sqrt{c}} \frac{y^n}{\sqrt{4ac - (y-b)^2}} dy$$

Note that

$$w_t(x) = \frac{1}{\sqrt{4ac - (x-b)^2}} = \frac{1}{2\sqrt{ac}} w\left(\frac{x}{2\sqrt{ac}} - \frac{b}{2\sqrt{ac}}\right)$$

where $w(x) = \frac{1}{\sqrt{1-x^2}}$ is Chebyshev weight.

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Note that

$$w_t(x) = \frac{1}{\sqrt{4ac-(x-b)^2}} = \frac{1}{2\sqrt{ac}} w\left(\frac{x}{2\sqrt{ac}} - \frac{b}{2\sqrt{ac}}\right)$$

where $w(x) = \frac{1}{\sqrt{1-x^2}}$ is Chebyshev weight.

For $w(t) = \frac{1}{\sqrt{1-x^2}}$ we know that

$$\alpha_n = 0, \qquad \beta_0 = \pi, \qquad \beta_1 = \frac{1}{2}, \qquad \beta_n = \frac{1}{4}, \quad (n \in \mathbb{N}).$$

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$$w_t(x) = \frac{1}{2\sqrt{ac}} w\left(\frac{x}{2\sqrt{ac}} - \frac{b}{2\sqrt{ac}}\right)$$

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$$\alpha_{t,n} = b, \qquad \beta_{t,0} = 1, \qquad \beta_{t,1} = 2ac, \qquad \beta_{t,n} = ac.$$

Moreover, MOPS corresponding to weight $w_t(x)$ are given by

$$P_{t,n}(x) = 2(ac)^n T_n\left(\frac{x-b}{ac}\right).$$

By direct application of the Heilermann formula we obtain:

Theorem . (Main)

The Hankel transform of the sequence $\{\tau_n(a,b,c)\}_{n\in\mathbb{N}_0}$ is equal to $\left\{2^n (ac)^{\binom{n+1}{2}}\right\}_{n\in\mathbb{N}_0}$

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Coefficient array

We now wish to look at the coefficient array of the $P_{t,n}(x)$.

Definition . Riordan Array

Consider a pair $f(x)$, $g(x)$ such that $g(x) = 1 + g_1 x + g_2 x^2 + \ldots$, $(g(0) = 1)$, and $f(x) = f_1 x + f_2 x^2 + \ldots$ with $f_1 \neq 0$ (so $f(0) = 0$), both with integer coefficients. We let (g, f) denote the infinite lower triangular matrix whose k-th column has g.f. $g(x)f(x)^k$.

The set of such matrices forms a group, with multiplication law

$$(g, f) * (h, l) = (g(h \circ f), l \circ f).$$

The identity is $(1, x)$, and

$$(g, f)^{-1} = \left(\frac{1}{g \circ \bar{f}}, \bar{f}\right)$$

where $\bar{f}$ is the series reversion (compositional inverse) of f:

$$f(\bar{f}(x)) = x.$$

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Example .

The Riordan array $(\frac{1}{1-x}, \frac{x}{1-x})$ is given by

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \dots \\ 1 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with general term $\binom{n}{k}$. This therefore represents the binomial transform. We have

$$(\frac{1}{1-x}, \frac{x}{1-x})^{-1} = (\frac{1}{1+x}, \frac{x}{1+x}).$$

The coefficient array $\{c_{n,k}\}_{n,k\in\mathbb{N}_0}$ where $P_{t,n}(x)=\sum_{k=0}^{n} c_{n,k}x^k$ is given by

$$\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
-b & 1 & 0 & 0 & 0 \\
b^2-2(ac) & -2b & 1 & 0 & 0 \\
b(3(ac)-b^2) & 3(b^2-(ac)) & -3b & 1 & 0 \\
b^4-4b^2(ac)+2(ac)^2 & 4b(2(ac)-b^2) & 2(3b^2-2(ac)) & -4b & 1 \\
-b(b^4-5b^2ac+5(ac)^2) & 5(b^4-3b^2ac+(ac)^2) & 5b(3ac-2b^2) & 5(2b^2-ac) & -5b \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right.$$

which is the Riordan array

$$\begin{aligned}
\{c_{n,k}\}_{n,k\in\mathbb{N}_0} &= \left(\frac{1-acx^2}{1+bx+acx^2}, \frac{x}{1+bx+acx^2}\right) \\
&= \left(\frac{1-acx^2}{1+acx^2}, \frac{x}{1+acx^2}\right) * \left(\frac{1}{1+bx}, \frac{x}{1+bx}\right)
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Outline of the talk

Binomial transform and generalizations

Definition .

The binomial transform of a given sequence $a = \{a_n\}_{n\in\mathbb{N}_0}$ the sequence $\{b_n\}_{n\in\mathbb{N}_0} = \mathbf{B}(a)$ given by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

Definition . (Spivey, Stail 2006)

Rising and falling k-binomial transforms of sequence $a = \{a_n\}_{n\in\mathbb{N}_0}$ are sequences $\{r_n\}_{n\in\mathbb{N}_0} = \mathbf{Br}(a;k)$ and $\{f_n\}_{n\in\mathbb{N}_0} = \mathbf{Bf}(a;k)$ given by

$$r_n = \sum_{i=0}^{n} \binom{n}{i} k^i a_i; \qquad f_n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} a_i.$$

$$B(\cdot) = Bf(\cdot;1) = Br(\cdot;1)$$

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Generalized binomial transforms and Hankel transform

Theorem . (Spivey, Stail 2006)

Given an integer sequence $\{a_n\}_{n\in\mathbb{N}_0}$, *let* $\{h_n\}_{n\in\mathbb{N}_0} = \mathbf{H}(a)$. *Then holds:*

a) $\mathbf{H}(a) = \mathbf{H}(\mathbf{Bf}(a;k)) = \{h_n\}_{n\in\mathbb{N}_0}$

b) $\mathbf{H}(\mathbf{Br}(a;k)) = \left\{k^{n(n+1)}h_n\right\}_{n\in\mathbb{N}_0}$.

Corollary . (Layman 2001)

Hankel transform is invariant under the binomial transform, i.e. $\mathbf{H}(\mathbf{B}(a)) = \mathbf{H}(a)$ *for any sequence* a.

We will give an alternative proof for the moment sequences, i.e. sequences $a_n = \int_{\mathbb{R}} x^n d\mu$ where μ is any positive measure.

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Proof:
a) Let $\{f_n\}_{n\in\mathbb{N}_0} = \mathbf{Bf}(a;k)$. Sequence $\{f_n\}_{n\in\mathbb{N}_0}$ is the n-th order moment sequence of the weight $w_f(x) = w(x-k)$.

$$\begin{aligned} f_n &= \sum_{i=0}^{n} \binom{n}{i} k^{n-i} \int_{\mathbb{R}} x^i w(x)dx \\ &= \int_{\mathbb{R}} \left(\sum_{i=0}^{n} \binom{n}{i} k^{n-i} x^i \right) dx = \int_{\mathbb{R}} (x+k)^n w(x)dx \end{aligned}$$

By applying the transformation lemma we obtain $\beta_{f,n} = \beta_n$, for every $n \in \mathbb{N}$. Hence $\mathbf{H}(a) = \mathbf{H}(f)$.

b) Let $\{r_n\}_{n\in\mathbb{N}_0} = \mathbf{Br}(a;k)$. We can prove that sequence $\{r_n\}_{n\in\mathbb{N}_0}$ is the moment sequence of the weight $w_r(x) = w\left(\frac{x-1}{k}\right)$.

$$\begin{aligned} r_n &= \sum_{i=0}^{n}\binom{n}{i}k^i\int_{\mathbb{R}} x^i w(x)dx = \int_{\mathbb{R}}\left(\sum_{i=0}^{n}\binom{n}{i}k^i x^i\right)dx \\ &= \int_{\mathbb{R}}(1+kx)^n w(x)dx = \int_{\mathbb{R}} x^n w\left(\frac{x-1}{k}\right)dx \end{aligned} \tag{1}$$

Applying the transformation lemma yields to $\beta_{r,n} = k^2\beta_n$ and therefore $\mathbf{H}(\mathbf{Br}(a;k)) = \left\{k^{n(n+1)}h_n\right\}_{n\in\mathbb{N}_0}$.

Connection with generalized central trinomial coefficients

Let $t_n = \tau_n(1,1,1)$ be central trinomial coefficients.

Lemma .

Sequence $\{\tau_n(a,b,c)\}_{n\in\mathbb{N}_0}$ is the falling α-th binomial transform ($\alpha = b - \sqrt{ac}$) of the sequence $\left\{(ac)^{n/2}t_n\right\}_{n\in\mathbb{N}_0}$, i.e. holds

$$\mathbf{Bf}\left(\left\{(ac)^{n/2}t_n\right\}_{n\in\mathbb{N}_0};\alpha\right) = \{\tau_n(a,b,c)\}_{n\in\mathbb{N}_0}.$$

Lemma .

The Hankel transform of $\{t_n\}_{n\in\mathbb{N}_0}$ is $\{2^n\}_{n\in\mathbb{N}_0}$.

Last two lemmas and previous results directly yields to

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Thanks for attention!