Riemann Zeta Function

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Introduction

The notes on the Riemann zeta function reproduced below are informal lecture notes from two lectures from the graduate complex variable course that I taught 20 years ago. Much of the material is cribbed from the books of Edwards and Conway (see bibliography) and, of course, from Riemann's 1859 paper on the distribution of primes. The only original mathematics that I can claim is any errors that I may have added. I have not updated the notes except to correct errors. [1]

These notes were prepared using LaTeX 2_ε. The original notes, distributed in February of 1976, were duplicated using hand–written ditto masters. We've come a long way in desktop mathematical document preparation!

The purpose of these lectures on the zeta function was to illustrate some interesting contour integral arguments in a nontrivial context and to make sure that the students learned about the Riemann hypothesis – an important part of our mathematical heritage and culture.

[1] Thanks to Mary Flahive for pointing out numerous errors.

The Zeta Function

If $\mathfrak{Re}\, z \geq 1 + \epsilon$ where $\epsilon > 0$ then

$$\sum_{k=m}^{n} \left| k^{-z} \right| = \sum_{k=m}^{n} \left| k^{-\mathfrak{Re}\, z} \right| \leq \sum_{k=m}^{n} k^{-1-\epsilon} \tag{1}$$

implies $\sum_{n=1}^{\infty} \left| n^{-z} \right|$ converges uniformly on $\{ z \in \mathbb{C} \mid \mathfrak{Re}\, z \geq 1 + \epsilon \}$. Thus the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \tag{2}$$

converges normally in the half plane $H = \{ z \in \mathbb{C} \mid \mathfrak{Re}\, z > 1 \}$ and so defines an analytic function ζ in H. The function ζ is called the *Riemann zeta function*.

Note substituting nt for t yields

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-t}\, t^{z-1}\, dt = n^z \int_0^{\infty} \mathrm{e}^{-nt}\, t^{z-1}\, dt. \tag{3}$$

Therefore

$$\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty} \int_0^{\infty} \mathrm{e}^{-nt}\, t^{z-1}\, dt \tag{4}$$

for $\mathfrak{Re}\, z > 1$. Now

$$\sum_{n=1}^{\infty} \mathrm{e}^{-nt} = \frac{\mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} = \left(\mathrm{e}^{t} - 1\right)^{-1} \tag{5}$$

if $t > 0$. If $z = x + \mathrm{i}y$ then

$$\int_0^{\infty} \sum_{n=1}^{\infty} \left| \mathrm{e}^{-nt} t^{z-1} \right| dt = \int_0^{\infty} \left(\mathrm{e}^{t} - 1\right)^{-1} t^{x-1}\, dt. \tag{6}$$

For large t we have $\left(\mathrm{e}^{t} - 1\right)^{-1} \approx \mathrm{e}^{-t}$ and for small t we have $\left(\mathrm{e}^{t} - 1\right)^{-1} \approx t^{-1}$. It follows the integral in equation (6) converges if $x > 1$. Then by the Fubini–Tonelli theorem we may interchange the order of integration in equation (4) (where we think of the summation as an integral relative to the appropriate measure). Thus

$$\zeta(z)\Gamma(z) = \int_0^{\infty} \left(\mathrm{e}^{t} - 1\right)^{-1} t^{z-1}\, dt \tag{7}$$

for $\Re\mathfrak{e}\, z > 1$. Equation (7) is the very first result derived in Riemann's famous 8–page paper *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, 1859.

The integral in equation (7) is badly behaved for $\Re\mathfrak{e}\, z$ near 1 since then the integrand behaves roughly as t^{-1} for small t. Riemann therefore considers a related contour integral where we avoid the origin

$$I(z) = \int_\gamma (\mathrm{e}^w - 1)^{-1}\ (-w)^z\ \frac{dw}{w}. \tag{8}$$

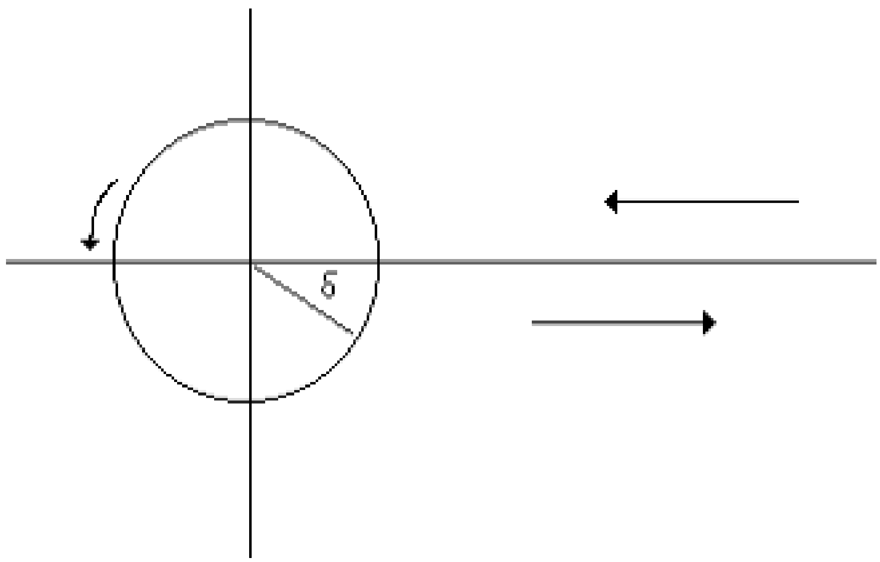

Figure 1: The contour for equation (8).

Here γ is the contour along the real axis from ∞ to $\delta > 0$, counterclockwise around the circle of radius δ with center at the origin, and then along the real axis from δ to ∞. We take $-w$ to have argument $-\pi$ when we are going towards the origin and argument π when we are going towards ∞. (Strictly speaking we should open this contour up a little and then pass to a limit, or else view it as lying in the appropriate Riemann surface.) The integral (8) converges for all z and defines an entire function. Moreover, by Cauchy's theorem it is independent of the choice of $\delta > 0$. Note moreover that $w(\mathrm{e}^w - 1)^{-1}$ has a removeable singularity at the origin and so by Cauchy's theorem

$$I(k) = 0 \quad \text{for} \quad k = 2, 3, 4, \cdots \tag{9}$$

(since when z is an integer, the integrals along the real axis in (8) cancel and so we may regard γ as just the circle of radius δ).

Now

$$
\begin{aligned}
I(z) = & \int_{\infty}^{\delta} \left(\mathrm{e}^{t}-1\right)^{-1} \mathrm{e}^{z(\log(t)-\mathrm{i}\pi)} \frac{dt}{t} \\
& + \int_{|w|=\delta} \left(\mathrm{e}^{w}-1\right)^{-1} (-w)^{z} \frac{dw}{w} \\
& + \int_{\delta}^{\infty} \left(\mathrm{e}^{t}-1\right)^{-1} \mathrm{e}^{z(\log(t)+\mathrm{i}\pi)} \frac{dt}{t}.
\end{aligned} \tag{10}
$$

We cannot use the Cauchy formula to evaluate the middle integral in (10), but with $w = \delta \mathrm{e}^{\mathrm{i}\theta}$ we have $\frac{dw}{w} = \mathfrak{i}\, d\theta$ and so since $w(\mathrm{e}^{w}-1)^{-1}$ has a removeable singularity at the origin we see the integral is bounded by $C\,\delta^{\mathfrak{Re}\, z - 1}$. In particular the integral goes to 0 as $\delta \to 0$ provided that $\mathfrak{Re}\, z > 1$. Thus letting $\delta \to 0$ we obtain

$$
\begin{aligned}
I(z) &= \left(\mathrm{e}^{\pi \mathfrak{i} z} - \mathrm{e}^{-\pi \mathfrak{i} z}\right) \int_{0}^{\infty} \left(\mathrm{e}^{t}-1\right)^{-1} t^{z-1}\, dt \\
&= 2\mathfrak{i}\, \sin(\pi z)\, \zeta(z)\, \Gamma(z) \quad \text{if } \mathfrak{Re}\, z > 1.
\end{aligned} \tag{11}
$$

Recalling the functional equation for the gamma function

$$
\Gamma(1-z)\, \Gamma(z) = \frac{\pi}{\sin(\pi z)}, \tag{12}
$$

we obtain

$$
\zeta(z) = \frac{\Gamma(1-z)}{2\pi \mathfrak{i}}\, I(z). \tag{13}
$$

Now equation (13) has been proved for $\mathfrak{Re}\, z > 1$, but the right side is analytic in the whole plane, except that $\Gamma(1-z)$ has simple poles at $z = 1, 2, 3, \cdots$. On the other hand $I(z)$ has zeros at $z = 2, 3, \cdots$. Thus $\zeta(z)$ is actually analytic in $\mathbb{C} \sim \{1\}$. At $z = 1$ there is at worst a simple pole. We see the pole is actually there by computing the residue

$$
\begin{aligned}
\lim_{z \to 1} \frac{(z-1)\Gamma(1-z)}{2\pi \mathfrak{i}}\, I(z) &= -\frac{1}{2\pi \mathfrak{i}}\, I(1) \\
&= -\frac{1}{2\pi \mathfrak{i}} \int_{|w|=\delta} \left(\mathrm{e}^{w}-1\right)^{-1} \frac{dw}{w} \\
&= -\lim_{w \to 0} \left(\mathrm{e}^{w}-1\right)^{-1} (-w) \\
&= 1.
\end{aligned} \tag{14}
$$

We have shown

Theorem 1. *The zeta function ζ continues analytically to a meromorphic function in $\mathbb{C}$ with a simple pole at $z = 1$. The residue at the pole is* 1.

Riemann now goes on to deduce the functional equation for the zeta function, but first he remarks that ζ vanishes at the negative even integers. This fact may be seen as follows: if $n \geq 0$ is an integer then

$$\zeta(-n) = \frac{n!}{2\pi\mathrm{i}}\, I(-n) \tag{15}$$

where (by the parenthetical remark following equation (9))

$$\begin{aligned} I(-n) &= \int_\gamma \left(\mathrm{e}^w - 1\right)^{-1} \,(-w)^{-n}\, \frac{dw}{w} \\ &= (-1)^n \int_{|\,w\,|=\delta} \frac{w}{\mathrm{e}^w - 1}\, \frac{dw}{w^{n+2}} \end{aligned} \tag{16}$$

Now $w\left(\mathrm{e}^w - 1\right)^{-1}$ is analytic in a neighborhood of the origin. Thus

$$\frac{w}{\mathrm{e}^w - 1} = \sum_{n=0}^{\infty} \frac{1}{n!}\, B_n\, w^n \tag{17}$$

for $|\,w\,| < 2\pi$. The numbers B_n defined by equation (17) are called the *Bernoulli numbers* (though there is some disagreement on how these numbers should be defined). Since

$$\frac{w}{\mathrm{e}^w - 1} + \frac{w}{2} \tag{18}$$

is an even function we see that $B_1 = -1/2$ and the other odd Bernoulli numbers all vanish. We can easily compute the even ones: for example

$$\begin{aligned} B_2 &= \frac{1}{6} \\ B_4 &= \frac{-1}{30} \\ B_6 &= \frac{1}{42} \\ B_8 &= \frac{-1}{30}. \end{aligned} \tag{19}$$

Now by Cauchy's integral formula we have

$$\frac{(n+1)!}{2\pi\mathrm{i}} \int_{|\,w\,|=\delta} \frac{w}{\mathrm{e}^w - 1}\, \frac{dw}{w^{n+2}} = B_{n+1} \tag{20}$$

and therefore

$$\zeta(-n) = (-1)^n\, \frac{B_{n+1}}{n+1} \tag{21}$$

for each integer $n \geq 0$. It follows

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0.$$

These roots are called the *trivial zeros* of the zeta function. The remaining roots are called the *nontrivial zeros* or *critical roots* of the zeta function.

The Functional Equation

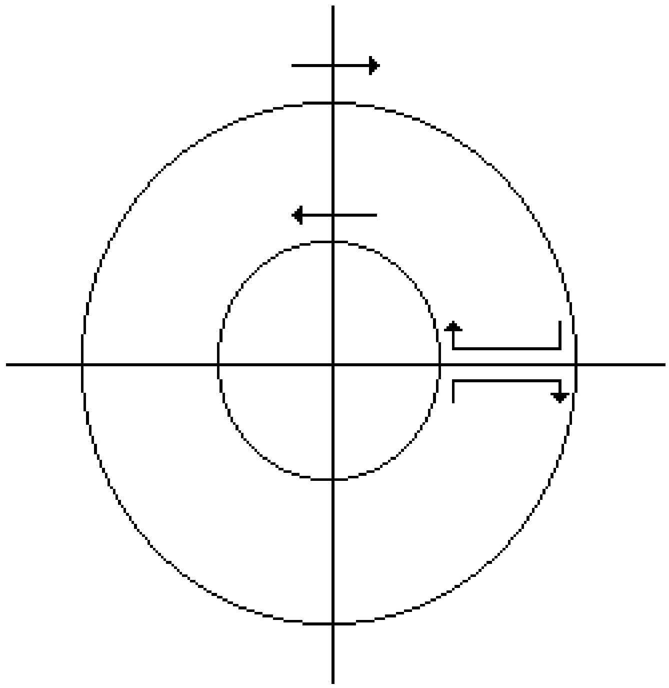

Figure 2: The contour for equation (22).

Let γ_n be the contour consisting of two circles centered at the origin and a radius segment along the positive reals. The outer circle has radius $(2n+1)\pi$ and the innner circle has radius $\delta < \pi$. The outer circle is traversed clockwise and the inner one counterclockwise. The radial segment is traversed in both directions. We make the same conventions concerning the argument of $-w$ as above. If we open the contour a little bit along the real axis we can employ the residue theorem, and then pass to a limit, to obtain

$$\begin{aligned} -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_n} (\mathrm{e}^w - 1)^{-1} \, \frac{(-w)^z}{w} \, dw & \\ &= \sum_{k=-n\cdots n, k\neq 0} \mathfrak{Res}\left((\mathrm{e}^w - 1)^{-1} \, \frac{(-w)^z}{w}, \, w = 2\pi \mathrm{i} k \right) \\ &= -\sum_{k=1}^{n} \left((2\pi \mathrm{i} k)^{z-1} + (-2\pi \mathrm{i} k)^{z-1} \right). \end{aligned} \tag{22}$$

Since

$$\begin{aligned} \mathfrak{i}^{z-1} + (-\mathfrak{i})^{z-1} &= \frac{1}{\mathfrak{i}}\left(e^{z\log(\mathrm{i})} - e^{z\log(-\mathrm{i})}\right) \\ &= \frac{1}{\mathfrak{i}}\left(e^{\frac{z\pi\mathrm{i}}{2}} - e^{-\frac{z\pi\mathrm{i}}{2}}\right) \\ &= 2\,\sin\left(\frac{\pi z}{2}\right), \end{aligned} \tag{23}$$

we obtain

$$\frac{1}{2\pi\mathfrak{i}}\int_{\gamma_n} (e^w - 1)^{-1}\,\frac{(-w)^z}{w}\,dw = 2(2\pi)^{z-1}\,\sin\left(\frac{\pi z}{2}\right)\sum_{k=1}^{n} k^{z-1}. \tag{24}$$

Now on the circle $|\,w\,| = (2n+1)\pi$ we have $|\,(e^z - 1)\,|$ is bounded independently of n and we have $|\,(-w)^z/w\,| \le |\,w\,|^{\mathfrak{Re}\,z-1}$. Thus if $\mathfrak{Re}\,z < 0$ the integral over the large circle tends to 0 as $n \to \infty$. It follows that

$$\frac{1}{2\pi\mathfrak{i}}\int_{\gamma} (e^w - 1)^{-1}\,\frac{(-w)^z}{w}\,dw = 2(2\pi)^{z-1}\,\sin\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty} n^{z-1}. \tag{25}$$

By equation (13) it now follows that

$$\boxed{\zeta(z) = 2(2\pi)^{z-1}\,\sin\left(\frac{\pi z}{2}\right)\,\Gamma(1-z)\,\zeta(1-z)} \tag{26}$$

for $\mathfrak{Re}\,z < 0$. By uniqueness of analytic continuation equation (26) is valid for all $z \ne 1$. Note $\zeta(1-z)$ has a simple pole at $z = 0$ and roots at the positive odd integers greater than 1, $\Gamma(1-z)$ has simple poles at the positive integers, and $\sin(\pi z/2)$ has roots at the even integers. Thus we see explicitly that all the singularities on the right side of equation (26), except $z = 1$, are removeable.

Equation (26) is Riemann's *functional equation for the zeta function.* Riemann gives a second proof of the functional equation. Since he is otherwise economical in the extreme, this duplication is something of a mystery. Edwards comments on this mystery in a footnote in section 1.6 in his book (see bibliography).

The Zeros of the Zeta Function

Riemann's paper starts by quoting the following result of Euler:

$$\zeta(z) = \prod_{n=1}^{\infty}\left(1 - p_n^{-z}\right)^{-1} \tag{27}$$

if $\mathfrak{Re}\, z > 1$. Here $p_1, p_2, p_3, \cdots$ is the sequence of prime numbers. To prove this result note that

$$\left(1 - p_n^{-z}\right)^{-1} = \sum_{m=0}^{\infty} p_n^{-mz} \tag{28}$$

(geometric series) and therefore

$$\prod_{k=1}^{N} \left(1 - p_k^{-z}\right)^{-1} = \sum_{j=1}^{\infty} n_{N,j}^{-z} \tag{29}$$

where the integers $n_{N,1}, n_{N,2}, n_{N,3}, \cdots$ are all the integers that can be factored as a product of powers of the primes $p_1, p_2, \cdots, p_N$. Letting $N \to \infty$ we obtain equation (27).

Since the product (27) contains no zero factors we see

$$\zeta(z) \neq 0 \quad \text{if} \quad \mathfrak{Re}\, z > 1. \tag{30}$$

Suppose now that $\zeta(z) = 0$ and $\mathfrak{Re}\, z < 0$. Since $\zeta(1-z) \neq 0$ the functional equation (26) implies

$$\Gamma(1-z) \sin\left(\frac{\pi z}{2}\right) = 0. \tag{31}$$

Since Γ has no roots we have $z = 2k$ where $k < 0$ is an integer, that is, z is a trivial zero. The strip $\{\, z \in \mathbb{C} \mid 0 \leq \mathfrak{Re}\, z \leq 1 \,\}$ is called the *critical strip*. We have seen that all the nontrivial roots of ζ lie in the critical strip.

Suppose now z is in the critical strip and $\zeta(z) = 0$. Since $\sin(\pi z/2) \neq 0$ we see that $\zeta(1-z) = 0$. That is, the nontrivial roots (or critical roots) are symmetric with respect to the *critical line* $\mathfrak{Re}\, z = \frac{1}{2}$. Note also, since $\zeta(z)$ is real for real z, it is trivial that the roots of the zeta function are symmetric with respect to the real axis.

For his study of the distribution of primes Riemann needs to estimate the numbers of roots of the zeta function in a box in the critical strip symmetric about the critical line. He obtains such an estimate (finally proved in 1905 by van Mangoldt) and then remarks that the number of zeros *on* the critical line in the box is about the same (no one has ever proved this statement). He then goes on to say that it is "very likely" that all the roots in the critical strip lie on the critical line. This statement is the *Riemann hypothesis*. Riemann goes on to say:

> "One would of course like to have a rigorous proof of this, but I have put aside the search for a such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation." *(This translation is from Edward's book.)*

Now, 117 years later, after a great deal of effort by many mathematicians, there is still no compelling reason to believe or to disbelieve the Riemann hypothesis.

Hardy in 1914 proved that ζ has an infinite number of roots on the critical line. Also in 1914 Bohr and Landau proved that in a certain sense "most" of the critical roots lie on the critical line. That is, if $\delta > 0$ then

$$\lim_{T\to\infty}\left(\frac{\text{number of roots } z \text{ with } 0 \le \mathfrak{Im}\, z \le T, \quad \delta + 1/2 \le \mathfrak{Re}\, z \le 1}{\text{number of roots } z \text{ with } 0 \le \mathfrak{Im}\, z \le T, \quad 0 \le \mathfrak{Re}\, z \le 1}\right) = 0. \tag{32}$$

In 1921 Hardy and Littlewood showed that the number of roots on the imaginary segment $[\,1/2\,,\,1/2 + \mathrm{i}T\,]$ is at least KT for all large T, for some constant K. In 1942 Selberg showed the number of roots on this segment is a least $KT\log(T)$ for all large T. Selberg's work implies that a positive fraction (in an appropriate sense) of the critical roots lie on the critical line.

The MIT Mathematics Department Alumni Newsletter in February 1975 announced that N. Levinson had proved at least 1/3 of of the critical roots lie on the critical line. During 1975 I heard a rumor that Levinson had improved his estimate to at least 98 per cent. On October 10, 1975, Levinson died.

Rosser, Yohe and Schoenfeld in 1968 showed that for a certain T_0 the zeta function has 3,500,000 roots in the box $\{\,z \in \mathbb{C} \;\mid\; 0 \le \mathfrak{Im}\, z \le T_0, \quad 0 \le \mathfrak{Re}\, z \le 1\,\}$ and that all these roots are simple and lie on the critical line. Lehman in 1966 had obtained the same result, but just for the first 250,000 roots. It turns out that certain behavior observed during the calculations may imply that eventually a few critical roots, not on the critical line, will be found. No one really knows.

Stieltjes and Hadamard

From the Euler product formula we have

$$\frac{1}{\zeta(z)} = \prod_{n=1}^{\infty}\left(1 - p_n^{-z}\right) \tag{33}$$

for $\Re z > 1$. It follows that

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} \tag{34}$$

where

$$\mu(n) = \begin{cases} +1 & \text{if } n = 1 \\ +1 & \text{if } n \text{ is the product of an } \textit{even} \\ & \qquad \text{number of } \textit{distinct} \text{ primes} \\ -1 & \text{if } n \text{ is the product of an } \textit{odd} \\ & \qquad \text{number of } \textit{distinct} \text{ primes} \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

Since ζ has a pole at 1, $1/\zeta$ has a root at 1. It turns out that the series (34) actually converges for $z = 1$ (van Mangoldt 1897, de la Vallée Poussin 1899). Thus we obtain Euler's curious formula (1748)

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} - \frac{1}{17} \cdots \tag{36}$$

Let M be the step function defined by $M(0) = 0$, M piecewise constant, M has a jump $\mu(n)$ at n, and M is equal to the average of its left and right limits at each jump. Then

$$\frac{1}{\zeta(z)} = \int_0^\infty x^{-z}\, dM(x) = z \int_0^\infty M(x)\, x^{-z-1}\, dx \tag{37}$$

for $\Re z > 1$.

If M grows less rapidly than x^α for some $\alpha > 0$ then the second integral above will converge absolutely for $\Re z > \alpha$. Then, by uniqueness of analytic continuation, we can conclude that ζ has no roots in $\{\, z \in \mathbb{C} \mid \Re z > \alpha \,\}$. In 1912 Littlewood proved the converse and therefore:

Theorem 2. *The Riemann hypothesis is true if and only if for each $\epsilon > 0$ we have*

$$\lim_{x\to\infty} M(x)\, x^{-1/2-\epsilon} = 0. \tag{38}$$

In 1885 Stieltjes wrote to Hermite that he had proved that $M(x)x^{-1/2}$ is bounded for large x and that therefore the Riemann hypothesis is true. Stieltjes, however, was unable to recall his proof in later years. Hadamard in 1896 published a paper on the zeta function in which he shows that ζ has no roots on the line $\Re z = 1$. He says that he is publishing his proof only because Stieltjes' proof that there are no zeros in $\Re z > 1/2$ has not yet been published.

> Stieltjes avait démontré, en conformément aux prévisions de Riemann, que ces zéros sont tous de la forme $\frac{1}{2} + t\mathfrak{i}$ (le nombre t étant réel); mais sa démonstration n'a jamais été publiée, et il n'a même pas été établi que la fonction ζ n'ait pas de zéros sur la droite $\mathfrak{Re}\,(s) = 1$.
>
> C'est cette dernière conclusion que je me propose de démontrer.

It now seems likely that Stieltjes had in fact made an error.

Odds and Ends. Euler Relation

If we form the Dirichlet product we have

$$\zeta(z)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^z} \tag{39}$$

where $d(n)$ is the number of divisors of n.

We have similar series involving sums of divisors of n or the number of positive integers less than n relatively prime to n. See for example Conway's book.

Note

$$2 \sum_{n=1}^{\infty} \frac{1}{(2n)^z} = 2^{1-z}\,\zeta(z) \tag{40}$$

and therefore

$$\left(1 - 2^{1-z}\right)\,\zeta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}. \tag{41}$$

Thus we have

$$\zeta(z) = \left(1 - 2^{1-z}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \tag{42}$$

where the series converges uniformly, but not absolutely, for $\mathfrak{Re}\, z \geq \delta$ for any $\delta > 0$. Equation (42) therefore yields a representation of $\zeta(z)$ valid for $\mathfrak{Re}\, z > 0$.

The equation (4) can be analytically continued explicitly to obtain

$$\begin{aligned}\zeta(z) &= \frac{1}{\Gamma(z)} \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt \quad \text{if } 0 < \Re\mathfrak{e}\, z < 1 \\ &= \frac{1}{\Gamma(z)} \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt \quad \text{if } -1 < \Re\mathfrak{e}\, z < 0\end{aligned} \tag{43}$$

These representations are useful for dealing with $\zeta(z)$ and $\zeta(1-z)$ for z in the critical strip.

If $n \geq 0$ is an integer then

$$\zeta(-(2n-1)) = (-1)^{2n-1} \frac{B_{2n}}{2n} \tag{44}$$

by equation (15). By the functional equation (26)

$$\zeta(-(2n-1)) = 2(2\pi)^{-2n}\, \Gamma(2n)\, (-1)^n \zeta(2n). \tag{45}$$

Thus we obtain the Euler relation

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1} B_{2n}}{2(2n)!}. \tag{46}$$

Taking $n = 1, 2, \cdots$ we obtain familiar expressions

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

and so on.

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