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Unified representation of the family of L-functions

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Abstract
The aim of this paper is to unify the family of L-functions. By using the generating functions of the Bernoulli, Euler and Genocchi polynomials, we construct unification of the L-functions. We also derive new identities related to these functions. We also investigate fundamental properties of these functions.

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1 Introduction

The theory of the family of L-functions has become a very important part in the analytic number theory. In this paper, using a new type generating function of the family of special numbers and polynomials, we construct unification of the L-functions.

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $\mathbb{Z}^- = \{-1, -2, \ldots\}$. Also, as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real number and $\mathbb{C}$ denotes the set of complex numbers. We assume that $\ln(z)$ denotes the principal branch of the multi-valued function $\ln(z)$ with the imaginary part $\Im(\ln(z))$ constrained by $-\pi < \Im(\ln(z)) \leq \pi$.

Recently, the first author [1] introduced and investigated the following generating functions which give a unification of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials:

$$g_{a,b}(x; t, k, \beta) := \frac{2^{1-k} t^k e^{tx}}{\beta^b e^t - a^b} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x; k, a, b) \frac{t^n}{n!}, \tag{1}$$

where ($|t| < 2\pi$ when $\beta = a$; $|t| < |b \log(\frac{\beta}{a})|$ when $\beta \neq a$; $k \in \mathbb{N}_0$; $\beta \in \mathbb{C}$ ($|\beta| < 1$); $a, b \in \mathbb{C} \backslash \{0\}$).

For the special values of a, b, k, b and β, the polynomials $\mathcal{Y}_{n,\beta}(x; k, a, b)$ provide us with a generalization and unification of the classical Bernoulli polynomials, Euler polynomials and Genocchi polynomials and also of the Apostol-type (Apostol-Bernoulli, Apostol-Euler, Apostol-Genocchi) polynomials.

Remark 1.1 If we set $k = a = b = 1$ in (1), we get a special case of the generalized Bernoulli polynomials $\mathcal{Y}_{n,\beta}(x, k, 1, 1)$, that is, the so-called Apostol-Bernoulli polynomials $\mathcal{B}_n(x, \beta)$

generated by

$$\frac{t}{\beta e^t - 1}e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n(x,\beta)\frac{t^n}{n!}$$

(*cf.* [1–28]).

Remark 1.2 By substituting $k+1 = -a = b = 1$ in (1), we are led to Apostol-Euler polynomials $\mathcal{E}_n(x,\beta)$ which are defined by means of the following generating function:

$$\frac{2}{\beta e^t + 1}e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n(x,\beta)$$

(*cf.* [1–28]).

Remark 1.3 Setting $k = -a = b = 1$ into (1), we get the Apostol-Genocchi polynomials $\mathcal{G}_n(x,\beta)$ which are defined by means of the following generating function:

$$\frac{2t}{\beta e^t + 1}e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n(x,\beta)\frac{t^n}{n!}$$

(*cf.* [1–28]).

In terms of a Dirichlet character χ of conductor $f \in \mathbb{N}$, Ozden *et al.* [16] extended and investigated the generating functions of the generalized Bernoulli, Euler and Genocchi numbers and the generalized Bernoulli, Euler and Genocchi polynomials with parameters a, b, β and k. Such χ-extended polynomials and χ-extended numbers are useful in many areas of mathematics and mathematical physics.

Definition 1.4 (Ozden *et al.* [16, p.2783]) Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Then the aforementioned χ-extended generalized Bernoulli-Euler-Genocchi numbers $\mathcal{Y}_{n,\chi,\beta}(k,a,b)$ and the aforementioned χ-extended generalized Bernoulli-Euler-Genocchi polynomials $\mathcal{Y}_{n,\chi,\beta}(x;k,a,b)$ are given by the following generating functions:

$$F_{\chi,\beta}(t;k,a,b) = 2^{1-k}t^k \sum_{j=1}^{f} \frac{\chi(j)(\frac{\beta}{a})^{bj}e^{jt}}{\beta^{bf}e^{ft} - a^{bf}} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}(k,a,b)\frac{t^n}{n!}, \tag{2}$$

where ($|t| < 2\pi$ when $\beta = a$; $|t| < |b\log(\frac{\beta}{a})|$ when $\beta \neq a$; $k \in \mathbb{N}_0$; $\beta \in \mathbb{C}$ ($|\beta| < 1$); $a, b \in \mathbb{C}\backslash\{0\}$) and

$$\mathfrak{H}_{\chi,\beta}(x,t;k,a,b) = F_{\chi,\beta}(t,k;a,b)e^{tx} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}(x;k,a,b)\frac{t^n}{n!} \tag{3}$$

($|t| < 2\pi$ when $\beta = a$; $|t| < |b\log(\frac{\beta}{a})|$ when $\beta \neq a$; $k \in \mathbb{N}_0$; $\beta \in \mathbb{C}$ ($|\beta| < 1$); $a, b \in \mathbb{C}\backslash\{0\}$).

Remark 1.5 Substituting $k = a = b = \beta = 1$ into (2), we are led immediately to the generating function of the generalized Bernoulli numbers which are defined by means of the following generating function:

$$\sum_{j=1}^{f} \frac{\chi(j)te^{jt}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} \tag{4}$$

(*cf.* [1–26]).

2 Unification of the *L*-functions

Our aim in this section is to apply the Mellin transformation to the generating function (3) of the polynomials $\mathcal{Y}_{n,\chi,\beta}(x;k,a,b)$ in order to construct a unification of the various members of the family of the L-functions and to thereby interpolate $\mathcal{Y}_{n,\chi,\beta}(x;k,a,b)$ for negative integer values of n.

Throughout this section, we assume that $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $s \in \mathbb{C}$.

By substituting (1) into (2), we obtain the following functional equation:

$$F_{\chi,\beta}(t;k,a,b) = \frac{1}{f^k} \sum_{j=1}^{f} \chi(j) \left(\frac{\beta}{a}\right)^{bj} g_{a^f,b}\left(\frac{j}{f}, tf; k, \beta^f\right). \tag{5}$$

By using this functional equation, we arrive at the following theorem.

Theorem 2.1 *Let χ be a Dirichlet character of conductor f. Then we have*

$$\mathcal{Y}_{n,\chi,\beta}(k,a,b) = f^{n-k} \sum_{j=1}^{f} \chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathcal{Y}_{n,\beta^f}\left(\frac{j}{f}; k, a^f, b\right). \tag{6}$$

By using (5), we modify (3) as follows:

$$\mathfrak{H}_{\chi,\beta}(x,t;k,a,b) = \frac{1}{f^k} \sum_{j=1}^{f} \chi(j) \left(\frac{\beta}{a}\right)^{bj} g_{a^f,b}\left(\frac{j+x}{f}, tf; k, \beta^f\right). \tag{7}$$

By using (7), we derive the following result.

Corollary 2.2 *Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Then we have*

$$\mathcal{Y}_{n,\chi,\beta}(x;k,a,b) = f^{n-k} \sum_{j=1}^{f} \chi(j) \left(\frac{\beta}{a}\right)^{bj} \mathcal{Y}_{n,\beta^f}\left(\frac{j+x}{f}; k, a^f, b\right). \tag{8}$$

By applying the Mellin transformation to the generating function (1), Ozden *et al.* [16, p.2784 Equation (4.1)] gave an integral representation of the unified zeta function $\zeta_\beta(s,x;k,a,b)$:

$$\zeta_\beta(s,x;k,a,b) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-k-1} g_{a,b}(x;-t;k,\beta)\,dt \quad \left(\min\{\Re(s),\Re(x)\} > 0\right), \tag{9}$$

where the additional constraint $\Re(x) > 0$ is required for the convergence of the infinite integral, which is given in (9), at its upper terminal. By making use of the above integral representation, Ozden *et al.* [16, p.2784 Equation (4.1)] defined the unified zeta function $\zeta_\beta(s,x;k,a,b)$ as follows:

$$\zeta_\beta(s,x;k,a,b) = \left(-\frac{1}{2}\right)^{k-1}\sum_{m=0}^{\infty}\frac{\beta^{bm}}{a^{b(m+1)}(m+x)^s} \quad \bigl(\beta\in\mathbb{C}\bigl(|\beta|<1\bigr); s\in\mathbb{C}\bigl(\Re(s)>1\bigr)\bigr). \quad (10)$$

By applying the Mellin transformation to the generating function (7), we have the following integral representation of the unified two-variable L-functions $L_{\chi,\beta}(s,x;k,a,b)$:

$$L_{\chi,\beta}(s,x;k,a,b) = \sum_{j=1}^{f}\frac{\chi(j)(\frac{\beta}{a})^{bj}}{f^k\Gamma(s)}\int_0^\infty t^{s-k-1}g_{a^f,b}\left(\frac{j+x}{f},-tf;k,\beta^f\right)dt$$
$$\bigl(\min\bigl\{\Re(s),\Re(x)\bigr\}>0\bigr) \quad (11)$$

in terms of the generating function $\mathfrak{H}_{\chi,\beta}(x,t;k;a,b)$ defined in (7). By substituting (9) into (11), we obtain

$$L_{\chi,\beta}(s,x;k,a,b) = \frac{1}{f^{k+s}}\sum_{j=1}^{f}\chi(j)\left(\frac{\beta}{a}\right)^{bj}\zeta_{\beta^f}\left(s,\frac{j+x}{f};k,a^f,b\right) \quad (12)$$

where $(\beta\in\mathbb{C}\ (|\beta|<1); s\in\mathbb{C}\ (\Re(s)>1))$.

Consequently, by making use of (10) and (12), we are ready to define a two-variable unification of the Dirichlet-type L-functions $L_{\chi,\beta}(s,x;k,a,b)$ as follows.

Definition 2.3 Let χ be a Dirichlet character of conductor $f\in\mathbb{N}$. For $s,\beta\in\mathbb{C}\ (|\beta|<1)$, we define a two-variable unified L-function $L_{\chi,\beta}(s,x;k,a,b)$ by

$$L_{\chi,\beta}(s,x;k,a,b) = f^{-k}\left(-\frac{1}{2}\right)^{k-1}\sum_{m=0}^{\infty}\frac{\beta^{bm}\chi(m)}{a^{b(m+f)}(m+x)^s} \quad \bigl(\beta\in\mathbb{C}\ \bigl(|\beta|<1\bigr);\Re(s)>1\bigr). \quad (13)$$

Remark 2.4 If we substitute $x=1$ into (13), we get the unified L-function

$$L_{\chi,\beta}(s;k,a,b) := L_{\chi,\beta}(s,1;k,a,b)$$

by

$$L_{\chi,\beta}(s;k,a,b) = f^{-k}\left(-\frac{1}{2}\right)^{k-1}\sum_{m=1}^{\infty}\frac{\beta^{bm}\chi(m)}{a^{b(m+f)}m^s},$$

where $(\Re(s)>1,\ \beta\in\mathbb{C}\ (|\beta|<1))$.

Remark 2.5 Upon substituting $k=a=b=1$ and $\beta=\frac{\xi}{u}$ into (13), we arrive at the interpolation function for twisted generalized Eulerian numbers and polynomials, which is given as follows:

$$l_1\left(\frac{u}{\xi},s,\chi\right) = L_{\chi,\frac{\xi}{u}}(s,x;1,1,1),$$

where, for a positive integer r, ξ is the rth root of 1.

$$l_1\left(\frac{u}{\xi}, s; \chi\right) = \sum_{m=0}^{\infty}\left(\frac{\xi}{u}\right)^m \frac{\chi(m)}{(m+x)^s}$$

(*cf.* [18]).

Remark 2.6 Substituting $x = 1$ into (13), we get a unification of the L-functions

$$L_{\chi,\beta}(s,1;k,a,b) = L_{\chi,\beta}(s;k,a,b).$$

Substituting $\chi \equiv 1$ into (13), we get a unification $\zeta_\beta(s,x;k,a,b)$ of the Hurwitz-type zeta function which is given in (10). We also note that both the Hurwitz (or generalized) zeta function

$$\zeta(s,x) = \zeta_1(s,x;1,1,1) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

(*cf.* [27, 28]) and the Riemann zeta function

$$\zeta(s) = \zeta_1(s,1;1,1,1) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

are obvious special cases of the unified zeta function $\zeta_\beta(s,x;k,a,b)$ (*cf.* [16, 27, 28]). *The relationship between the unified zeta function and the Hurwitz-Lerch zeta function* $\Phi(z,s,a)$ *was given by Ozden et al.* [16]*:*

$$\zeta_\beta(s,x;k,a,b) := \left(-\frac{1}{2}\right)^{k-1} a^{-b}\Phi\left(\frac{\beta^b}{a^b}, s, x\right), \tag{14}$$

where *the Hurwitz-Lerch zeta function is defined by*

$$\Phi(z,s,x) = \sum_{n=0}^{\infty} \frac{z^n}{(n+x)^s},$$

which converges for ($x \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$), where as usual

$$\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$$

(*cf.* [27, 28]).

A relationship between the functions $L_{\chi,\beta}(s,x;k,a,b)$ and $\zeta_\beta(s,x;k,a,b)$ is provided by the next theorem.

Theorem 2.7 *Let* $s \in \mathbb{C}$. *Let* χ *be a Dirichlet character of conductor* $f \in \mathbb{N}$. *Then we have*

$$L_{\chi,\beta}(s,x;k,a,b) = f^{-s-k}\sum_{j=1}^{f}\left(\frac{\beta}{a}\right)^{jb}\chi(j)\zeta_{\beta^f}\left(s, \frac{j+x}{f}; k, a^f, b\right). \tag{15}$$

Proof Substituting $m = nf + j$, $j = 1, 2, \ldots, f$, $n = 0, \ldots, \infty$ into (13), we obtain

$$L_{\chi,\beta}(s,x;k,a,b) = \left(-\frac{1}{2}\right)^{k-1} f^{-s-k} \sum_{j=1}^{f} \left(\frac{\beta}{a}\right)^{jb} \chi(j) \sum_{n=0}^{\infty} \frac{\beta^{bnf}}{a^{bnf}(n + \frac{j+x}{f})^s}.$$

After some algebraic manipulations, we arrive at the desired result. □

Remark 2.8 Substituting $a = b = k = 1$ into (13), we have

$$L_{\chi,\beta}(s,x;1,1,1) = \sum_{m=0}^{\infty} \frac{\beta^m \chi(m)}{(m+x)^s} \quad \bigl(\Re(s) > 1, \beta \in \mathbb{C}\bigl(|\beta| < 1\bigr)\bigr)$$

which interpolates the Apostol-Bernoulli polynomials attached to the Dirichlet character, which are given by means of the following generating functions:

$$\sum_{j=1}^{f} \frac{\chi(j) t \beta^j e^{t(j+x)}}{\beta^f e^{tf} - 1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\chi}(x,\beta) \frac{t^n}{n!}.$$

Let f be an odd integer. If we set $a = -1$ and $k = 0$ into (13), then we have

$$L_{\chi,\beta}(s,x;1,-1,1) = 2 \sum_{m=1}^{\infty} (-1)^m \frac{\chi(m)\beta^m}{(m+x)^s} \quad \bigl(\Re(s) > 1, \beta \in \mathbb{C}\bigl(|\beta| < 1\bigr)\bigr),$$

which interpolate the Apostol-Euler polynomials attached to the Dirichlet character, which are defined by the following generating functions:

$$\sum_{j=1}^{f} \frac{2\chi(j) \beta^j e^{t(j+x)}}{\beta^f e^{tf} + 1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi}(x,\beta) \frac{t^n}{n!}$$

(*cf.* [1–29]).

By using (15) and (14), we arrive at the following result.

Corollary 2.9 *Let $s \in \mathbb{C}$. Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Then we have*

$$L_{\chi,\beta}(s,x;k,a,b) = \left(-\frac{1}{2}\right)^{k-1} a^{-fb} f^{-s-k} \sum_{j=1}^{f} \left(\frac{\beta}{a}\right)^{jb} \chi(j) \Phi\left(\frac{\beta^{fb}}{a^{fb}}, s, \frac{j+x}{f}\right).$$

Theorem 2.10 *Let χ be a Dirichlet character of conductor f. Let n be a positive integer. Then we have*

$$L_{\chi,\beta}(1-n,x;k,a,b) = \frac{(-1)^k}{f} \frac{(n-1)!}{(n+k-1)!} \mathcal{Y}_{n+k-1,\chi,\beta}(x;k,a,b). \tag{16}$$

Proof By substituting $s = 1 - n$ into (15), we get

$$L_{\chi,\beta}(1-n,x;k,a,b) = f^{n-1-k} \sum_{j=1}^{f} \left(\frac{\beta}{a}\right)^{jb} \chi(j) \zeta_{\beta^f}\left(1-n, \frac{j+x}{f}; k, a^f, b\right).$$

By using Theorem 7 in [16], we get

$$\begin{aligned}
&L_{\chi,\beta}(1-n,x;k,a,b) \\
&\quad = (-1)^k \frac{(n-1)!}{(n+k-1)!} f^{n-1-k} \sum_{j=1}^{f} \left(\frac{\beta}{a}\right)^{jb} \chi(j) \mathcal{Y}_{n+k-1,\beta}\left(\frac{j+x}{f};k,a,b\right).
\end{aligned}$$

By substituting (8) into the above, we arrive at the desired result. □

Remark 2.11 The two-variable Dirichlet L-function and the Dirichlet L-function are obvious special cases of the unified Dirichlet-type L-functions $L_{\chi,\beta}(s,x;k,a,b)$ defined by (13). We thus have (*cf.* [13])

$$L(s,x;\chi) = \sum_{m=0}^{\infty} \frac{\chi(m)}{(m+x)^s}$$

and

$$L(s;\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

where $\Re(s) > 1$. By analytic continuation, this function can be extended to a meromorphic function on the whole complex plane. We have

$$L(1-n;\chi) = -\frac{B_{n,\chi}}{n},$$

where $n \in \mathbb{Z}^+$ and $B_{n,\chi}$, the usual generalized Bernoulli number, is defined by (4). The Dirichlet L-function is used to prove the theorem on primes in arithmetic progressions. Dirichlet shows that $L(s;\chi)$ is non-zero at $s = 1$. Furthermore, if χ is a principal character, then the corresponding Dirichlet L-function has a simple pole at $s = 1$ (*cf.* [6, 7, 9, 18, 24, 27, 28, 30, 31]).

3 Applications

In this section, by using (16) and the following formula, which was proved by Ozden *et al.* [16, Theorem 5, Equation (3.10)]

$$\mathcal{Y}_{n,\chi,\beta}(x;k,a,b) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \mathcal{Y}_{j,\chi,\beta}(k,a,b), \tag{17}$$

we construct a meromorphic function involving a unified family of L-functions. Therefore, using (16) and (17),

$$L_{\chi,\beta}(1-n,x;k,a,b) = \frac{x^{n+k-1}}{f\prod_{l=0}^{k-1}(n+l)} \sum_{j=0}^{n+k-1} \binom{n+k-1}{j} \frac{1}{x^j} \mathcal{Y}_{j+k-1,\chi,\beta}(k,a,b).$$

From the above equation, we arrive at the following theorem.

Theorem 3.1 *Let $x \neq 0$. Let χ be a Dirichlet character of conductor f. Then we have*

$$L_{\chi,\beta}(s,x;k,a,b)=\frac{x^{k-s}}{f\prod_{l=0}^{k-1}(s-1-l)}\sum_{j=0}^{\infty}\binom{k-s}{j}\frac{1}{x^{j}}\mathcal{Y}_{j+k-1,\chi,\beta}(k,a,b).$$

The function $L_{\chi,\beta}(s,x;k,a,b)$ is an analytic function at $s=0$. We now compute the value of this function at this point as follows:

$$L_{\chi,\beta}(0,x;k,a,b)=\frac{x^{k}}{(-1)^{k}f\prod_{l=0}^{k-1}(1+l)}\sum_{j=0}^{k}\binom{k}{j}\frac{1}{x^{j}}\mathcal{Y}_{j+k-1,\chi,\beta}(k,a,b).$$

The function $L_{\chi,\beta}(s,x;k,a,b)$ is a meromorphic function. This function has simple poles which are

$$s=1,2,3,\ldots,k.$$

The residues of this function at the simple poles at $s=1$ and $s=k$ are given, respectively, as follows:

$$\operatorname{Res}_{s=1}\left\{L_{\chi,\beta}(s,x;k,a,b)\right\}=\frac{x^{k-1}}{f(-1)^{k}\prod_{l=0}^{k-1}(2+l)}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{1}{x^{j}}\mathcal{Y}_{j+k-1,\chi,\beta}(k,a,b)$$

and

$$\operatorname{Res}_{s=k}\left\{L_{\chi,\beta}(s,x;k,a,b)\right\}=\frac{\mathcal{Y}_{k-1,\chi,\beta}(k,a,b)}{f\prod_{l=0}^{k-2}(k-1-l)}.$$

Remark 3.2 Simsek (*cf.* [20, 21]) defined a twisted two-variable L-function $L_{\xi,q}^{(h)}(s,x;\chi)$ as follows:

$$L_{\xi,q}^{(h)}(s,x;\chi)=\sum_{m=0}^{\infty}\frac{\chi(m)\phi_{\xi}(m)q^{hm}}{(x+m)^{s}}-\frac{\log q^{h}}{s-1}\sum_{m=0}^{\infty}\frac{\chi(m)\phi_{\xi}(m)q^{hm}}{(x+m)^{s-1}},$$

where $q\in\mathbb{C}$ ($|q|<1$); $\xi^{r}=1$ ($r\in\mathbb{Z}$); $\xi\neq1$. Observe that if $\xi=1$, then $L_{\xi,q}^{(h)}(s,x;\chi)$ is reduced to the work of Kim [9].

Relationship between the function $L_{\chi,\beta}(s,x;k,a,b)$ and $L_{\xi,q}^{(h)}(s,x;\chi)$ is given as the following result.

Corollary 3.3 *Let χ be a Dirichlet character of conductor f. Then we have*

$$L_{1,\frac{\beta^{b}}{a^{b}}}^{(b)}(s,x;\chi)=(-2)^{k}a^{bf}f^{k}\left(L_{\chi,\beta}(s,x;k,a,b)-\frac{\log q^{h}}{s-1}L_{\chi,\beta}(s-1,x;k,a,b)\right).$$

We conclude our present investigation by remarking that the existing literature contains several interesting generalizations and extensions of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$, Hurwitz zeta function $\zeta(s,x)$ and L-function (*cf.* [1–30]); see also the references cited in each of these earlier works.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors completed the paper together. Both authors read and approved the final manuscript.

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