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Non-standard orthogonality for the little q-Laguerre polynomials

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ABSTRACT

Little q-Laguerre polynomials $\{p_n(\cdot; a|q)\}_{n=0}^{\infty}$ are classically defined for $0 < q < 1$ and $0 < aq < 1$. After extending this family to a new one in which arbitrary real values of the parameter a are allowed, we give an orthogonality condition for those cases for which Favard's Theorem fails to work.

1. Preliminaries

The orthogonality of the Laguerre polynomials $\{L_n^{(-N)}\}_{n=0}^{\infty}$, for positive integers N, was stated in [1]. A first extension of this result, in the discrete setting, can be found in [2], in which a Δ-Sobolev orthogonality for the Meixner polynomials $\{M_n^{(-N,\mu)}\}_{n=0}^{\infty}$, for nonnegative integers N, is established. Recently [3], we have given non-standard inner products with respect to which the Meixner–Pollaczek polynomials $\{P_n^{((1-N)/2)}(\cdot;\phi)\}_{n=0}^{\infty}$, for positive integers N (another discrete extension of the Laguerre polynomials), become orthogonal. The parameters considered in the previous cases are out of classical considerations and coincide with those values for which Favard's Theorem does not work. In this paper we state a q-Sobolev orthogonality for the family of the little q-Laguerre polynomials $\{p_n(\cdot; q^{-N}|q)\}_{n=0}^{\infty}$ (one q-analogue of the Laguerre family), for positive integers N, that is, for those values of the parameter for which Favard's Theorem fails to work.

Throughout this work, $\mathbb{R}$, $\mathbb{N}_0$ and $\mathbb{P}_k$ respectively denote the set of real numbers, the set of nonnegative integers and the set of real-valued polynomials in one real variable of degree not greater than k. For a fixed $q \in (0, 1)$, we use the (standard) definitions and notations below (see [4,5]):

$$\text{Shifted factorial:} \quad (x)_0 = 1, \quad (x)_{n+1} = x(x+1)\cdots(x+n), \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R},$$

$$q\text{-Shifted factorial:} \quad (x;q)_0 = 1, \quad (x;q)_{n+1} = \prod_{k=0}^{n}(1 - xq^k), \quad n \in \mathbb{N}_0 \cup \{\infty\}, \quad x \in \mathbb{R},$$

$$\text{Binomial coefficient:} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n, k \in \mathbb{N}_0, \quad k \le n,$$

$$q\text{-Binomial coefficient:} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad n, k \in \mathbb{N}_0, \quad k \le n.$$

We will use also the standard notations and definitions for the q-hypergeometric (also, basic hypergeometric) series ${}_m\phi_n$, for the function e_q (one of the q-analogues of the exponential function) and for the q-derivative operator acting on polynomials p:

$${}_m\phi_n\left(\left.\begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{matrix}\right| q; x\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_m;q)_k}{(b_1;q)_k \cdots (b_n;q)_k} \frac{(-1)^{(n-m+1)k}}{q^{(m-n-1)k(k-1)/2}} \frac{x^k}{(q;q)_k},$$

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$$e_q(x) = \frac{1}{(x;q)_\infty}, \quad |x| < 1,$$
$$D_q p(x) = (p(x) - p(qx))/((1-q)x) \quad \text{if } x \neq 0, \text{ and } D_q p(0) = p'(0).$$

As usual, the nth iteration of the q-derivative operator is recursively defined by means of $D_q^0 = I$ (I is the identity operator) and $D_q^{n+1} = D_q \circ D_q^n$ for $n \in \mathbb{N}_0$. Finally, the Kronecker delta will be denoted by δ_{mn} and the superscript t will stand for the transpose of a matrix.

2. Generalized little q-Laguerre polynomials

In what follows, we shall always assume that $0 < q < 1$.

For a real parameter a such that $0 < aq < 1$, monic little q-Laguerre polynomials $p_n(\cdot; a|q)$ are the ones fulfilling the orthogonality condition [5, 3.20.2, 3.20.4]

$$\sum_{k=0}^{\infty} \frac{(aq)^k}{(q;q)_k} p_m(q^k; a|q) p_n(q^k; a|q) = a^n q^{n^2} \frac{(aq;q)_n (q;q)_n}{(aq;q)_\infty} \delta_{mn}, \quad m, n \in \mathbb{N}_0.$$

They can be defined in terms of the q-hypergeometric series ${}_2\phi_1$ by means of ([5, 3.20.1, 3.20.4])

$$p_n(x; a|q) = (-1)^n q^{n(n-1)/2} (aq;q)_n \, {}_2\phi_1 \left(\left. \begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \right| q; qx \right), \quad n \in \mathbb{N}_0. \tag{2.1}$$

Observe that the above representation works perfectly for each real value of the parameter a, except when $a \in \{q^{-1}, q^{-2}, \ldots\}$. Our intention is to accomplish the extension of monic little q-Laguerre polynomials for all $a \in \mathbb{R}$.

Starting with (2.1), and using that $(q^{-n}; q)_k$ vanishes for $k > n$, we get

$$\begin{aligned} p_n(x; a|q) &= (-1)^n q^{n(n-1)/2} (aq;q)_n \sum_{k=0}^{\infty} \frac{(q^{-n};q)_k (0;q)_k}{(aq;q)_k} \frac{(qx)^k}{(q;q)_k} \\ &= (-1)^n q^{n(n-1)/2} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (aq^{k+1};q)_{n-k}}{(q;q)_k} (qx)^k. \end{aligned}$$

Taking into account [5, 0.3.3] that for $0 \le k \le n$

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-k(2n-k+1)/2},$$

we obtain

Definition 2.1. Let $a \in \mathbb{R}$. For each $n \in \mathbb{N}_0$ we define the nth degree generalized monic little q-Laguerre polynomial $p_n(\cdot; a|q)$ by

$$p_n(x; a|q) = \sum_{k=0}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q (aq^{k+1}; q)_{n-k} q^{(n-k)(n-k-1)/2} (-x)^k. \tag{2.2}$$

In a similar way, we readily obtain the extension of the very classical monic Laguerre polynomials $L_n^{(\alpha)}$, usually defined for $\alpha > -1$, to arbitrary real values of the parameter α, yielding an expression that reads

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^n \binom{n}{k} (\alpha+1+k)_{n-k} (-x)^k, \quad \alpha \in \mathbb{R}, n \in \mathbb{N}_0.$$

From (2.2), and using

$$\lim_{q \uparrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}, \qquad \lim_{q \uparrow 1} \frac{(q^{\alpha+1+k};q)_{n-k}}{(1-q)^{n-k}} = (\alpha+1+k)_{n-k},$$

we obtain $\lim_{q \uparrow 1} (1-q)^{-n} p_n((1-q)x; q^\alpha|q) = L_n^{(\alpha)}(x)$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}_0$, so generalized monic little q-Laguerre polynomials can be interpreted as q-analogues of generalized monic Laguerre polynomials.

Working directly with the representation (2.2), one can verify that generalized monic little q-Laguerre polynomials satisfy the same three term recurrence relation as the classical little q-Laguerre polynomials [5, 3.20.4]. Concretely,

Proposition 2.1. *Let $a \in \mathbb{R}$. The generalized monic little q-Laguerre polynomials fulfil the three term recurrence relation*

$$p_{n+1}(x; a|q) = \left(x - A_n^{(q,a)}\right) p_n(x; a|q) - B_n^{(q,a)} p_{n-1}(x; a|q), \quad n \geq 0,$$

where $A_n^{(q,a)} = q^n(1 + a - aq^n(1 + q))$, $B_n^{(q,a)} = aq^{2n-1}(1 - q^n)(1 - aq^n)$, and $p_{-1}(\cdot; a|q)$ is the null polynomial.

Due to the fact that $B_N^{(q,q^{-N})}$ vanishes for each positive integer N, no orthogonality results can be deduced for $a \in \{q^{-1}, q^{-2}, \ldots\}$. The main result of this paper consists precisely in an orthogonality statement for these special values of the parameter a. In order to achieve this aim, the next section is devoted to giving some preparatory results.

3. Two key tools for the main result

Using (2.2) it is an easy matter to show that the q-derivative operator D_q acts on the generalized little q-Laguerre polynomials in the same way that it acts on the classical little q-Laguerre ones [5, 3.20.7, 3.20.4].

Proposition 3.1. *Monic generalized little q-Laguerre polynomials verify the q-difference relation*

$$D_q p_n(x; a|q) = \frac{(1 - q^n)}{(1 - q)} p_{n-1}(x; aq|q), \quad a \in \mathbb{R}, n \geq 0. \tag{3.3}$$

Iterating expression (3.3) it follows the first key tool, that reads

Corollary 3.1. *Let $a \in \mathbb{R}$. For a nonnegative integer n*

$$D_q^k p_n(x; a|q) = \frac{(q^{n-k+1}; q)_k}{(1 - q)^k} p_{n-k}(x; aq^k|q), \quad 0 \leq k \leq n + 1.$$

Now we will show that for $n \geq N$, the point $x = 0$ is a zero of precise order N of the polynomials $p_n(\cdot; q^{-N}|q)$.

Proposition 3.2. *Fixed a positive integer N, we have*

$$p_n(x; q^{-N}|q) = x^N p_{n-N}(x; q^N|q), \quad n \geq N.$$

Proof. For a fixed positive integer N, (2.2) yields, for each $n \in \mathbb{N}_0$,

$$p_n(x; q^{-N}|q) = \sum_{k=0}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q (q^{k+1-N}; q)_{n-k} q^{(n-k)(n-k-1)/2} (-x)^k.$$

Fixed $n \geq N$, for each $k \leq N - 1$ it is clear that $(q^{k+1-N}; q)_{n-k} = 0$. Therefore, for $n \geq N$

$$\begin{aligned} p_n(x; q^{-N}|q) &= \sum_{k=N}^{n} (-1)^n \begin{bmatrix} n \\ k \end{bmatrix}_q (q^{k+1-N}; q)_{n-k} q^{(n-k)(n-k-1)/2} (-x)^k \\ &= x^N \sum_{k=0}^{n-N} (-1)^{n-N} \begin{bmatrix} n \\ N + k \end{bmatrix}_q (q^{k+1}; q)_{n-N-k} q^{(n-N-k)(n-N-k-1)/2} (-x)^k. \end{aligned}$$

Using that

$$\begin{bmatrix} n \\ N + k \end{bmatrix}_q (q^{k+1}; q)_{n-N-k} = \begin{bmatrix} n - N \\ k \end{bmatrix}_q (q^{k+1+N}; q)_{n-N-k},$$

we simplify the above equation to

$$\begin{aligned} p_n(x; q^{-N}|q) &= x^N \sum_{k=0}^{n-N} (-1)^{n-N} \begin{bmatrix} n - N \\ k \end{bmatrix}_q (q^{k+1+N}; q)_{n-N-k} q^{(n-N-k)(n-N-k-1)/2} (-x)^k \\ &= x^N p_{n-N}(x; q^N|q). \quad \square \end{aligned}$$

The previous result implies that $p_n^{(k)}(0; q^{-N}|q) = 0$ for $0 \leq k \leq N - 1$ and $n \geq N$. Taking into account that

$$D_q^k f(0) = \frac{(q; q)_k}{(1 - q)^k} \frac{f^{(k)}(0)}{k!}, \quad k \geq 0, \tag{3.4}$$

for a function f analytic in a neighborhood of 0 (see [6,7, 6.8]), we can also establish that $x = 0$ is a "q-zero" of precise order N of the polynomials $p_n(\cdot; q^{-N}|q)$ (the second key tool).

Corollary 3.2. *Let N be a positive integer. For each $n \geq N$,*

$$D_q^k p_n(0; q^{-N}|q) = 0, \quad 0 \leq k \leq N-1. \tag{3.5}$$

4. q-Sobolev orthogonality of little q-Laguerre polynomials with negative integer parameters

In Section 2 we have shown that no orthogonality results can be deduced from Favard's Theorem for the family of generalized little q-Laguerre polynomials $\{p_n(\cdot; q^{-N}|q)\}_{n=0}^{\infty}$ when N is a positive integer. We will fill up this gap by considering a suitable modification of our previous result [8, Theorem 3], using as starting point in our considerations one of the kind suggestions of the referees of that paper.

Theorem 4.1. *For each positive integer N, there exists a symmetric and positive definite matrix A of order N such that the family of generalized little q-Laguerre polynomials $\{p_n(\cdot; q^{-N}|q)\}_{n=0}^{\infty}$ is orthogonal with respect to the inner product $(\cdot,\cdot)_q^{(N;A)}$ defined by*

$$\begin{aligned}(p_1, p_2)_q^{(N;A)} &= \left(p_1(0), D_q p_1(0), \ldots, D_q^{N-1} p_1(0)\right) A \left(p_2(0), D_q p_2(0), \ldots, D_q^{N-1} p_2(0)\right)^t \\ &\quad + \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \left(D_q^N p_1(q^k)\right)\left(D_q^N p_2(q^k)\right), \quad p_1, p_2 \in \mathbb{P}.\end{aligned} \tag{4.6}$$

Proof. Let $\{l_j(\cdot; q)\}_{j=0}^{N-1} \subset \mathbb{P}_{N-1}$ be the set of polynomials defined by

$$l_j(x; q) = \frac{(1-q)^j}{(q;q)_j} x^j, \quad 0 \leq j \leq N-1.$$

Since both $\{l_j(\cdot; q)\}_{j=0}^{N-1}$ and $\{p_n(\cdot; q^{-N}|q)\}_{j=0}^{N-1}$ are bases of $\mathbb{P}_{N-1}$, and taking into account that $D_q^k l_j(0; q) = \delta_{kj}$ (see (3.4)), we have

$$p_j(x; q^{-N}|q) = \sum_{k=0}^{N-1} D_q^k p_j(0; q^{-N}|q) l_k(x; q), \quad 0 \leq j \leq N-1.$$

Therefore, we can assure that the matrix $C = (D_q^k p_j(0; q^{-N}|q))_{j,k=0}^{N-1}$ is nonsingular. Also, if $D = (\kappa_j \delta_{jk})_{j,k=0}^{N-1}$ is a nonsingular diagonal matrix of order N ($\kappa_j \in \mathbb{R}\setminus\{0\}$), then the symmetric matrix $A = C^{-1}D^2(C^{-1})^t = (C^{-1}D)(C^{-1}D)^t$ is positive definite.

In order to state the orthogonality we will consider three cases:

(i) If $0 \leq m, n \leq N-1$, then $D_q^N p_m(x; q^{-N}|q) = D_q^N p_n(x; q^{-N}|q) = 0$. Therefore

$$\begin{aligned}(p_m(\cdot; q^{-N}|q), p_n(\cdot; q^{-N}|q))_q^{(N;A)} &= \left(\left(D_q^k p_m(0; q^{-N}|q)\right)_{k=0}^{N-1} C^{-1}\right) D^2 \left(\left(D_q^k p_n(0; q^{-N}|q)\right)_{k=0}^{N-1} C^{-1}\right)^t \\ &= (\delta_{mk})_{k=0}^{N-1} D^2 ((\delta_{nk})_{k=0}^{N-1})^t = \kappa_n^2 \delta_{mn}.\end{aligned}$$

(ii) If $0 \leq m \leq N-1$ and $n \geq N$, then $D_q^N p_m(x; q^{-N}|q) = 0$ and also (Corollary 3.2) $D_q^k p_n(0; q^{-N}|q) = 0$ for $0 \leq k \leq N-1$ so clearly we have

$$(p_m(\cdot; q^{-N}|q), p_n(\cdot; q^{-N}|q))_q^{(N;A)} = 0.$$

(iii) Finally, if $m, n \geq N$, then $D_q^k p_m(0; q^{-N}|q) = D_q^k p_n(0; q^{-N}|q) = 0$ for $0 \leq k \leq N-1$. Using Corollary 3.1 and the orthogonality condition for the classical little q-Laguerre polynomials, it follows that

$$\begin{aligned}(p_m(\cdot; q^{-N}|q), p_n(\cdot; q^{-N}|q))_q^{(N;A)} &= \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \left(D_q^N p_m(q^k; q^{-N}|q)\right)\left(D_q^N p_n(q^k; q^{-N}|q)\right) \\ &= \frac{(q^{m-N+1}; q)_N}{(1-q)^N} \frac{(q^{n-N+1}; q)_N}{(1-q)^N} \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} p_{m-N}(q^k; 1|q) p_{n-N}(q^k; 1|q) \\ &= \left(\frac{(q^{n-N+1}; q)_N}{(1-q)^N}\right)^2 q^{(n-N)^2} \frac{(q;q)_{n-N}^2}{(q;q)_\infty} \delta_{mn} \\ &= \left(\frac{(q;q)_n}{(1-q)^n}\right)^2 q^{(n-N)^2} (1-q)^{2(n-N)} e_q(q) \delta_{mn}. \quad \square\end{aligned}$$

The choice $\kappa_j = ((q;q)_j/(1-q)^j)q^{(j-N)^2/2}(1-q)^{(j-N)}\sqrt{e_q(q)}$ for $0 \le j \le N-1$ would imply

$$\|p_n(\cdot;q^{-N}|q)\|_q^{(N;A)} = \frac{(q;q)_n}{(1-q)^n} q^{(n-N)^2/2}(1-q)^{(n-N)}\sqrt{e_q(q)}, \quad n \in \mathbb{N}_0.$$

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