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Combinatorial sums and implicit Riordan arrays

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Abstract

In this paper we present the theory of implicit Riordan arrays, that is, Riordan arrays which require the application of the Lagrange Inversion Formula to be dealt with. We show several examples in which our approach gives explicit results, both in finding closed expressions for sums and, especially, in solving classes of combinatorial sum inversions.

Keywords: Generating functions; Riordan arrays; Lagrange inversion formula; Combinatorial sums inversion

1. Introduction

The concept of a *Riordan array* was first introduced by Shapiro et al. [14] to generalize the properties of the Pascal Triangle (see also Rogers [13]), and immediately the authors recognized that the set of *proper Riordan arrays* has a group structure. In [15], Sprugnoli used Riordan arrays to develop a method for proving combinatorial identities, and then he showed how Riordan arrays can be used to perform combinatorial sum inversions [16]. The main advantage of Riordan arrays is the fact that they provide a human or computer-free approach to these problems (see [7]), in contrast to methods created for computer-based, or automated, proofs (see Petkovšek, Wilf and Zeilberger [11]).

A Riordan array is an infinite, lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$ defined by two formal power series $(d(t), h(t))$ such that $d_{n,k} = [t^n]d(t)h(t)^k$. When $d(t)$ is invertible (i.e., $d(0) \neq 0$), and $h(t)$ has a compositional inverse (i.e., $h(0) = 0$ and $h'(0) \neq 0$), the Riordan array is called *proper*: in the present paper we will be mainly interested in proper Riordan arrays. We observe explicitly that the formal power series $d(t)(h(t))^k$ is the generating function of the k-th column of the Riordan array $(d_{n,k})_{n,k\in\mathbb{N}}$. It is easy to prove that the set of proper Riordan arrays is a group with the usual operation of row-by-column product; this translates into the following rule: if $E = (f(t), g(t))$ and $D = (d(t), h(t))$, then

$$D \star E = (d(t), h(t)) \star (f(t), g(t)) = (d(t)f(h(t)), g(h(t))). \tag{1}$$

Furthermore, we have the following *summation rule*: if $(d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$, then

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t)f(h(t)), \tag{2}$$

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where $f(t)$ is the generating function of the sequence $(f_k)_{k\in\mathbb{N}}$. This theorem allows us to translate a combinatorial sum into the extraction of a coefficient from a generating function, thus showing the close connection between Riordan arrays and the *method of coefficients* as exposed in [9]. The previous formula is also called the *transformation* of $f(t)$ by the Riordan array $(d(t), h(t))$, and it will play a basic role in our developments.

Because of the group structure of proper Riordan arrays, if we denote by $(\overline{d}_{n,k})_{n,k\in\mathbb{N}}$ the inverse of the proper Riordan array $(d_{n,k})_{n,k\in\mathbb{N}}$, we obviously have

$$g_n = \sum_{k=0}^{n} d_{n,k} f_k \Leftrightarrow f_n = \sum_{k=0}^{n} \overline{d}_{n,k} g_k,$$

and this reduces the problem of many combinatorial sum inversions to the inversion of a proper Riordan array, a problem strictly related to the Lagrange Inversion Formula (see, e.g., Goulden and Jackson [4]). Our main reference for sum inversion will be Riordan's book [12] (see also Egorychev and Zima [2]); an approach similar to the one used in this paper is connected to the concept of a *Schauder basis*, as exposed by Huang in [6]; actually, a Schauder basis $(d(t)h(t)^i)_{i\in\mathbb{N}}$ is just the Riordan array $(d(t), h(t))$. See the end of Section 2.

When dealing with the general problem of finding a closed form for a combinatorial sum or of inverting it, we have often to do with *implicit* generating functions. Suppose we have a sequence $(c_n)_{n\in\mathbb{N}}$ defined by an expression like $c_n = [t^n]F(t)\phi(t)^n$. Since the expression depends on n, just as the coefficient we have to extract, $F(t)\phi(t)^n$ cannot be considered the generating function of $(c_n)_{n\in\mathbb{N}}$. In order to have an explicit generating function, we should apply the Lagrange Inversion Formula in the so-called *diagonalization* form (see Theorem 2):

$$\sum_{n\geq 0} c_n t^n = \left[\frac{F(w)}{1 - t\phi'(w)} \middle| \, w = t\phi(w) \right] = \left[\frac{F(w)\phi(w)}{\phi(w) - w\phi'(w)} \middle| \, w = t\phi(w) \right],$$

where we have to solve the functional equation $w = t\phi(w)$ in $w = w(t)$. This is not always possible, and consequently we call *implicit* the generating function of $(c_n)_{n\in\mathbb{N}}$. A connection between Riordan arrays and the Lagrange Inversion Formula can be found in Peart and Woan's paper [10].

In this paper we present the theory of implicit Riordan arrays, that is, Riordan arrays which require the application of the Lagrange Inversion Formula to be dealt with. In general, we cannot hope to find explicit solutions to implicit problems, but we show several examples in which our approach gives explicit results, both in finding closed expressions for sums and, especially, in solving classes of combinatorial sum inversions.

2. The problem of inverting combinatorial sums

In his book [12] *Combinatorial Identities*, Riordan considers the general problem of inverting combinatorial sums. Let $a_n = \sum_{k=0}^{n} d_{n,k} b_k$, $(n \in \mathbb{N})$ be a system of infinite identities, in which the sequence $(b_k)_{k\in\mathbb{N}}$ is unknown, while the sequence $(a_k)_{k\in\mathbb{N}}$ and the array $d_{n,k}$ are known: how is it possible to determine the first sequence? In other words, is it possible to invert the sums and get $b_n = \sum_{k=0}^{n} \overline{d}_{n,k} a_k$? If we substitute the b_k's in the first system of identities, we immediately see that the two infinite arrays $(d_{n,k})_{n,k\in\mathbb{N}}$ and $(\overline{d}_{n,k})_{n,k\in\mathbb{N}}$ should be the inverse of each other, so the problem of inversion is equivalent to the inversion of the infinite array $(d_{n,k})_{n,k\in\mathbb{N}}$. Besides, as we shall see, sometimes we can drop the hypothesis that D be a proper Riordan array and assume that it is a general Riordan array.

When $(d_{n,k})_{n,k\in\mathbb{N}}$ is a proper Riordan array, the problem can be attacked in a systematic way by the underlying theory, as we are now going to show. Actually, all the cases considered by Riordan can be reduced to this case, that is, to the inversion of explicit or implicit Riordan arrays. Since proper Riordan arrays have a group structure (with identity $I = (1,\ t)$), in principle there is no theoretical problem: if $D = (d(t), h(t))$ then $\bar{D} = (\overline{d}(t), \overline{h}(t))$ exists and is unique. In practice, as Riordan's book shows, things are quite different and the inversion of proper Riordan arrays is not an obvious problem. In general, if $D = (d(t), h(t))$ and $E = (f(t), g(t))$ are two proper Riordan arrays, their product is defined as in formula (1), and when $E = \bar{D} = (\overline{d}(t), \overline{h}(t))$ the result should be the identity $I = (1,\ t)$. In this case we find $d(t)\overline{d}(h(t)) = 1$ and $\overline{h}(h(t)) = t$, so that the Lagrange Inversion Formula implies, after an obvious change of variables,

$$\overline{d}(t) = \left[\frac{1}{d(w)} \middle| \, t = h(w) \right] \quad \text{and} \quad \overline{h}(t) = [w | t = h(w)].$$

Surely, the simplest inversion is

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

Here $d_{n,k} = \binom{n}{k}$ and so $(d_{n,k})_{n,k\in\mathbb{N}}$ is the Pascal triangle, that is, the proper Riordan array $P = (1/(1-t), t/(1-t))$, as is immediately proven:

$$d_{n,k} = [t^n]\frac{1}{1-t}\left(\frac{t}{1-t}\right)^k = [t^{n-k}]\frac{1}{(1-t)^{k+1}} = \binom{-k-1}{n-k}(-1)^{n-k} = \binom{n}{n-k} = \binom{n}{k}.$$

Therefore, we find

$$\overline{d}(t) = [1-w \mid w = t(1-w)] = \left[1-w \mid w = \frac{t}{1+t}\right] = \frac{1}{1+t}$$

and obviously $\overline{h}(t) = t\overline{d}(t)$; now, we have

$$\overline{d}_{n,k} = [t^n]\frac{1}{1+t}\left(\frac{t}{1+t}\right)^k = [t^{n-k}]\frac{1}{(1+t)^{k+1}} = \binom{-k-1}{n-k} = (-1)^{n-k}\binom{n}{k}.$$

Actually, when the starting sum can be expressed as a generating function relation, the method can be made more direct:

Theorem 1. *Let us suppose that $(d_{n,k})_{n,k\in\mathbb{N}}$ is the proper Riordan array $(d(t), h(t))$ and $a(t), b(t)$ are the generating functions of the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$. Then the inverse relation of $a_n = \sum_{k=0}^{n} d_{n,k} b_k$ is obtained by extracting the coefficient of t^n in the relation*

$$b(t) = \left[\frac{a(w)}{d(w)} \middle| t = h(w)\right].$$

Proof. By the summation formula for Riordan arrays, the starting sum can be written: $a(t) = d(t)b(h(t))$ or $b(h(t)) = a(t)/d(t)$. Now the formula follows when we set $y = h(t)$ and change variables to adjust notation. ■

In the previous example, we have $b(t) = [a(w)(1-w) \mid t = w/(1-w)]$, that is, by computing $w = t/(1+t)$, $b(t) = \frac{1}{1+t} a\left(\frac{t}{1+t}\right)$, which is the right part of the above inversion, because this transformation corresponds to the Riordan array $\bar{P} = \left(\binom{n}{k}(-1)^{n-k}\right)_{n,k\in\mathbb{N}}$.

The direct computation of $\overline{d}(t), \overline{h}(t)$ or the previous theorem can be sufficient to solve (at least in principle) the problem of explicit inversion. Before examining implicit inversion, let us give a third approach to explicit inversion, which directly gives the form of $\overline{d}_{n,k}$. To this purpose, we need to introduce the Lagrange Inversion Formula in the forms which will be used in what follows. Our basic reference is Goulden and Jackson [4].

Theorem 2. *Let $\phi(t)$ be a formal power series with $\phi(0) \neq 0$; there exists a unique $w = w(t)$ such that $w(0) = 0$ and $w = t\phi(w)$. Besides, we have*

(1) *if $F(t)$ is any formal Laurent series, then*

$$[t^n]F(w(t)) = [t^n]\left[F(w) \mid w = t\phi(w)\right] = \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n$$
$$= [t^n]F(t)\phi(t)^{n-1}(\phi(t) - t\phi'(t));$$

(2) *if $(c_n)_{n\in\mathbb{N}}$ is the sequence defined by $c_n = [t^n]f(t)\phi(t)^n$, where $f(t)$ is a formal power series, then the generating function of the sequence is given by the following diagonalization rule:*

$$C(t) = \left[\frac{f(w)}{1-t\phi'(w)} \middle| w = t\phi(w)\right] = \left[\frac{f(w)\phi(w)}{\phi(w) - w\phi'(w)} \middle| w = t\phi(w)\right].$$

Proof. See Goulden and Jackson [4, p. 27] for (2) and the first part of (1). For the second part of (1), let us consider the "residue" theorem as formulated by Goulden and Jackson:

$$[t^{-1}]f(t)g'(t) = -[t^{-1}]f'(t)g(t)$$

and apply it to part (1):

$$\begin{aligned}[t^n]F(w(t)) &= \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n = \frac{1}{n}[t^{-1}]F'(t)\left(\frac{\phi(t)}{t}\right)^n \\ &= -\frac{1}{n}[t^{-1}]F(t)\frac{d}{dt}\left(\frac{\phi(t)}{t}\right)^n = -[t^{-1}]F(t)\left(\frac{\phi(t)}{t}\right)^{n-1}\left(\frac{t\phi'(t)-\phi(t)}{t^2}\right) \\ &= -[t^n]F(t)\phi(t)^{n-1}(t\phi'(t)-\phi(t)).\end{aligned}$$

This is what we were looking for. ■

As an application of this theorem, we give a formula for the generic element in the inverse of any proper Riordan array:

Theorem 3. *Let $D = (d(t), h(t))$ be a proper Riordan array; the generic element $\overline{d}_{n,k}$ of the inverse proper Riordan array $\bar{D} = (\overline{d}(t), \overline{h}(t))$ is given by*

$$\overline{d}_{n,k} = [t^{n-k}]\frac{h'(t)}{d(t)}\left(\frac{t}{h(t)}\right)^{n+1}.$$

Proof. The bivariate generating function of $\bar{D}$ is given by

$$\frac{\overline{d}(t)}{1-w\overline{h}(t)} = \left[\frac{1}{d(y)}\,\frac{1}{1-wy}\,\middle|\, t = h(y)\right].$$

From this generating function we extract the appropriate coefficient by means of Theorem 2:

$$\begin{aligned}\overline{d}_{n,k} &= [t^n][w^k]\left[\frac{1}{d(y)}\,\frac{1}{1-wy}\,\middle|\, t=h(y)\right] = [t^n][w^k]\left[\frac{1}{d(y)}\sum_{k=0}^{\infty} w^k y^k\,\middle|\, t=h(y)\right] \\ &= [t^n]\left[\frac{y^k}{d(y)}\,\middle|\, y = \frac{ty}{h(y)}\right] = [t^n]\frac{t^k}{d(t)}\,\frac{t^{n-1}}{h(t)^{n-1}}\left(\frac{t}{h(t)} - t\frac{h(t)-th'(t)}{h(t)^2}\right),\end{aligned}$$

which is equivalent to the formula in the assertion.

This result can be used to express Theorem 1 in another form:

Theorem 4. *Let us suppose that the sum $a_n = \sum_{k=0}^n d_{n,k}b_k$ corresponds to the generating function relation $a(t) = d(t)b(h(t))$; that is, $(d_{n,k})$ is the Riordan array $(d(t), h(t))$. Then the inverse sum is*

$$b_n = \sum_{k=0}^{n}\left([t^{n-k}]\frac{h'(t)}{d(t)}\left(\frac{t}{h(t)}\right)^{n+1}\right)a_k.$$

Proof. This is a simple consequence of Theorem 3 and of the fundamental property of sums containing a Riordan array. ■

Example. Some inversions in the book of Riordan are immediate applications of this theorem. Let us consider inversion (2.2.2); that is,

$$a_n = \sum_{k=0}^{n}\left(\binom{p+qk-k}{n-k} + q\binom{p+qk-k}{n-k-1}\right)b_k \Leftrightarrow b_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{p+qn-k}{n-k}a_k;$$

here the Riordan array is found by developing the binomial coefficients:

$$[t^{n-k}](1+t)^{p+qk-k} + q[t^{n-k-1}](1+t)^{p+qk-k} = [t^n](1+qt)(1+t)^p(t(1+t)^{q-1})^k,$$

thus obtaining

$$D = ((1+qt)(1+t)^p,\ t(1+t)^{q-1}).$$

We now apply Theorem 4, which gives

$$\begin{aligned}
b_n &= \sum_{k=0}^{n} \left([t^{n-k}] \frac{(1+t)^{q-1} + t(q-1)(1+t)^{q-2}}{(1+qt)(1+t)^p(1+t)^{(n+1)(q-1)}} \right) a_k \\
&= \sum_{k=0}^{n} \left([t^{n-k}] \frac{(1+qt)(1+t)^{q-2}}{(1+qt)(1+t)^p(1+t)^{nq-n+q-1}} \right) a_k \\
&= \sum_{k=0}^{n} \left([t^{n-k}](1+t)^{-p-nq+n-1} \right) a_k = \sum_{k=0}^{n} \binom{-p-nq+n-1}{n-k} a_k \\
&= \sum_{k=0}^{n} \binom{p+nq-n+1+n-k-1}{n-k} (-1)^{n-k} a_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{p+qn-k}{n-k} a_k .
\end{aligned}$$

Inversion (2.6.3) can be proved in the same way.

Example. Inversions (3.1.3) and (3.1.3a) in Riordan's book [12] involve the use of exponential generating functions. Let us prove the second one, which is rather interesting:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (x+n)(x+k)^{n-k-1} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (x+n)^{n-k} a_k .$$

By observing that $(x+n) = (x+k) + (n-k)$, the sum is transformed as follows:

$$\frac{a_n}{n!} = \sum_{k=0}^{n} \left(\frac{(x+k)^{n-k}}{(n-k)!} + \frac{(x+k)^{n-k-1}}{(n-k-1)!} \right) \frac{b_k}{k!}.$$

Let us denote by $\widehat{a}(t)$ the exponential generating function of a sequence (a_k), which is the same as the ordinary generating function of the sequence $(a_k/k!)$. Here, the Riordan array is found by observing that the two terms between parentheses are

$$[t^{n-k}]\mathrm{e}^{(x+k)t} + [t^{n-k-1}]\mathrm{e}^{(x+k)t} = [t^n](1+t)\mathrm{e}^{xt}(t\mathrm{e}^t)^k;$$

hence

$$D = ((1+t)\mathrm{e}^{xt}, t\mathrm{e}^t).$$

Therefore, we have

$$\begin{aligned}
\frac{b_n}{n!} &= \sum_{k=0}^{n} \left([t^{n-k}] \frac{\mathrm{e}^t + t\mathrm{e}^t}{(1+t)\mathrm{e}^{xt}\mathrm{e}^{(n+1)t}} \right) \frac{a_k}{k!} = \sum_{k=0}^{n} \left([t^{n-k}]\mathrm{e}^{-xt}\mathrm{e}^{-nt} \right) \frac{a_k}{k!} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \frac{(x+n)^{n-k}}{(n-k)!} \frac{a_k}{k!}.
\end{aligned}$$

This is equivalent to the right part of the inversion, as desired. Inversion (3.1.3) is proved in a similar way.

Example. Let us briefly consider the three examples 3.2, 3.3 and 3.4 in Huang's paper [6]; as observed in the Section 1, the Schauder bases $\left(X^i/(1-X)^{p+i+1}\right)_{i\in\mathbb{N}}$, $\left(X^i \mathrm{e}^{(p+i)X}\right)_{i\in\mathbb{N}}$ and $\left(X^i (X/(\mathrm{e}^X-1))^{p+i}\right)_{i\in\mathbb{N}}$ correspond to the three Riordan arrays:

$$\left(\frac{1}{(1-t)^{p+1}}, \frac{t}{1-t} \right), \quad \left(\mathrm{e}^{pt}, t\mathrm{e}^t\right), \quad \left(\frac{1}{(\mathrm{e}^t-1)^p}, \frac{t}{\mathrm{e}^t-1} \right),$$

and Theorem 4 can be used to find the inverse arrays. Obviously, they coincide with the arrays found by Huang.

3. Implicit inversions

When the expression to be inverted depends on n, a direct application of the Riordan array concepts is no longer feasible, and the rule of diagonalization should be applied. A typical example is the following:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k = \sum_{k=0}^{n} \binom{n}{n-k} b_k = [t^n](1+t)^n b(t).$$

Because of the n, the relation $a(t) = (1+t)^n b(t)$ is not a genuine generating function equality. By the diagonalization rule, we have

$$[t^n](1+t)^n b(t) = [t^n]\left[\frac{b(w)}{1-t} \,\middle|\, w = t(1+w) \right] = [t^n][(1+w)b(w) \mid w = t(1+w)].$$

Now, the expression between the substitution brackets represents a generating function, because it does not depend on n; therefore, we have

$$a(t) = [(1+w)b(w) \mid w = t(1+w)] \qquad a\left(\frac{w}{1+w}\right) = (1+w)b(w),$$

and we have an expression only depending on w. By changing the indeterminate,

$$b(t) = \frac{1}{1+t} a\left(\frac{t}{1+t}\right).$$

Finally, this is a transformation of $a(t)$ by means of the proper Riordan array:

$$D = \left(\frac{1}{1+t}, \frac{t}{1+t}\right) \qquad d_{n,k} = [t^n]\frac{1}{1+t}\left(\frac{t}{1+t}\right)^k = (-1)^{n-k}\binom{n}{k}$$

and we conclude with the inverse relation:

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

We can translate this approach into a general result:

Theorem 5. *Let us suppose that the sum $a_n = \sum_{k=0}^{n} d_{n,k} b_k$ corresponds to the implicit Riordan array $D = (f(t)\phi(t)^n, t)$, that is, to the relation $a_n = [t^n] f(t)\phi(t)^n b(t)$, with $\phi(0) \neq 0$; then the inverse sum is*

$$b_n = \sum_{k=0}^{n} \left([t^{n-k}] \frac{\phi(t) - t\phi'(t)}{f(t)\phi(t)^{k+1}} \right) a_k.$$

Proof. By applying the rule of diagonalization, we have

$$a(t) = \left[\frac{f(w)b(w)}{1 - t\phi'(w)} \,\middle|\, w = t\phi(w) \right].$$

We now substitute $w/\phi(w)$ for t in both members, and find

$$a\left(\frac{w}{\phi(w)}\right) = \frac{f(w)b(w)\phi(w)}{\phi(w) - w\phi'(w)},$$

which is equivalent to

$$b(w) = \frac{\phi(w) - w\phi'(w)}{f(w)\phi(w)} a\left(\frac{w}{\phi(w)}\right).$$

This formula proves that $b(w)$ is the transformation of $a(w)$ by the Riordan array

$$\left(\frac{\phi(w) - w\phi'(w)}{f(w)\phi(w)}, \frac{w}{\phi(w)} \right),$$

whose generic element is

$$\overline{d}_{n,k} = [w^n]\frac{\phi(w) - w\phi'(w)}{f(w)\phi(w)}\left(\frac{w}{\phi(w)}\right)^k = [w^{n-k}]\frac{\phi(w) - w\phi'(w)}{f(w)\phi(w)^{k+1}}.$$

The formula we are looking for now follows from the Riordan array rule $b_n = \sum_{k=0}^{n} \overline{d}_{n,k} a_k$. ■

Example. As an example, let us consider inversion (2.4.1) in Riordan's book. We start from

$$a_n = \sum_k \binom{n}{k} b_{n-ck},$$

and observe that $b_{n-ck} = [t^n]b(t)(t^c)^k$; consequently, we consider b_{n-ck} as the generic element of the (not proper) Riordan array $(b(t),\ t^c)$. The sum corresponds to

$$a_n = [t^n]b(t)[(1+y)^n \mid y = t^c] = [t^n]b(t)(1+t^c)^n.$$

Therefore, we can apply Theorem 5 with $f(t) = 1$ and $\phi(t) = 1 + t^c$:

$$\begin{aligned} b_n &= \sum_{k=0}^{n}\left([t^{n-k}]\frac{(1+t^c) - ctt^{c-1}}{(1+t^c)^{k+1}}\right) a_k = \sum_{k=0}^{n}\left([t^{n-k}]\frac{1-(c-1)t^c}{(1+t^c)^{k+1}}\right) a_k \\ &= \sum_{k=0}^{n}\left(\binom{-k-1}{(n-k)/c} - (c-1)\binom{-k-1}{(n-k-c)/c}\right) a_k. \end{aligned}$$

Let us now set $k = n - ch$ or $n - k = ch$:

$$b_n = \sum_{h=0}^{n/c}\left(\binom{n-ch+h}{h} + (c-1)\binom{n-ch+h-1}{h-1}\right) a_{n-ch},$$

as found by Riordan.

Example. In Egorychev [1, p. 91], we find the following problem: what is the value of f_k if we know that

$$\sum_{k=1}^{n}\binom{n-1}{k-1} n^{n-k}(k+p)!f_k = (2p)!n^{n+p}?$$

If we set $b_k = (k+p)!f_k$, we have to solve the inversion

$$a_n = \sum_{k=1}^{n}\binom{n-1}{k-1} n^{n-k} b_k,$$

and eventually substitute to a_k its value $(2p)!k^{k+p}$. The sum can be rewritten:

$$a_n = \sum_{k=0}^{n}\binom{n}{k} n^{n-k-1} k b_k = \sum_{k=0}^{n}\binom{n}{k} n^{n-k-1}(n-(n-k))b_k.$$

By introducing exponential generating functions, we have

$$\frac{a_n}{n!} = \sum_{k=0}^{n}\left(\frac{n^{n-k}}{(n-k)!} - \frac{n^{n-k-1}}{(n-k-1)!}\right)\frac{b_k}{k!} = [t^n]\mathrm{e}^{nt}\widehat{b}(t) - [t^n]t\mathrm{e}^{nt}\widehat{b}(t).$$

Hence,

$$\frac{a_n}{n!} = [t^n](1-t)\mathrm{e}^{nt}\widehat{b}(t),$$

and we apply Theorem 5 with $f(t) = 1 - t$ and $\phi(t) = \mathrm{e}^t$:

$$\frac{b_n}{n!} = \sum_{k=0}^{n}\left([t^{n-k}]\frac{\mathrm{e}^t - t\mathrm{e}^t}{(1-t)\mathrm{e}^{(k+1)t}}\right)\frac{a_k}{k!} = \sum_{k=0}^{n}(-1)^{n-k}\frac{k^{n-k}}{(n-k)!}\frac{a_k}{k!}$$

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{n-k} a_k.$$

We now substitute their values to a_k and to b_n:

$$(n+p)!f_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{n-k} (2p)! k^{k+p}$$

and derive Egorychev's solution:

$$f_n = \frac{(2p)!}{(n+p)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{n+p}.$$

We can go a bit further and find a closed form for the last sum. The generating function of the sequence $(k^p)_{k\in\mathbb{N}}$ is (see, e.g., Graham, Knuth and Patashnik [5, p. 337])

$$\sum_{k\geq 0} k^p t^k = \sum_{k=0}^{\infty} \left\{ {p \atop k} \right\} \frac{k! t^k}{(1-t)^{k+1}},$$

so we can apply the Riordan array transformation:

$$\begin{aligned} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{n+p} &= [t^n] \frac{1}{1+t} \left[\sum_{k=0}^{\infty} \left\{ {n+p \atop k} \right\} \frac{k! y^k}{(1-y)^{k+1}} \,\middle|\, y = \frac{t}{1+t} \right] \\ &= [t^n] \frac{1}{1+t} \sum_{k=0}^{\infty} \left\{ {n+p \atop k} \right\} k! \frac{t^k}{(1+t)^k} \, \frac{(1+t)^{k+1}}{1^{k+1}} \\ &= [t^n] \sum_{k=0}^{\infty} \left\{ {n+p \atop k} \right\} k! t^k = \left\{ {n+p \atop n} \right\} n!. \end{aligned}$$

Consequently, we conclude that

$$f_n = \frac{(2p)!n!}{(n+p)!} \left\{ {n+p \atop n} \right\},$$

which is not present in Egorychev.

We can now give a general result for the inversion of sums:

Theorem 6. *Let us suppose that the sum $a_n = \sum_{k=0}^{n} d_{n,k} b_k$ corresponds to the implicit Riordan array $D = (f(t)\phi(t)^n, h(t))$, that is, to the relation $a_n = [t^n] f(t)\phi(t)^n b(h(t))$, with $\phi(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$. Then the inverse sum is*

$$b_n = \sum_{k=0}^{n} \left([t^{n-k}] \frac{(\phi(t) - t\phi'(t))h'(t)}{f(t)\phi(t)^{k+1}} \frac{t^{n+1}}{h(t)^{n+1}} \right) a_k.$$

Proof. By setting $w = t\phi(w)$ we eliminate the dependence on n:

$$a_n = [t^n] \left[\frac{f(w)b(h(w))}{1 - w\phi'(w)/\phi(w)} \,\middle|\, w = t\phi(w) \right] = [t^n] \left[\frac{\phi(w)f(w)b(h(w))}{\phi(w) - w\phi'(w)} \,\middle|\, w = t\phi(w) \right].$$

This is a generating function identity, and therefore we get

$$a\left(\frac{w}{\phi(w)}\right) = \frac{\phi(w)f(w)b(h(w))}{\phi(w) - w\phi'(w)} \quad \text{or} \quad b(h(w)) = \frac{\phi(w) - w\phi'(w)}{\phi(w)f(w)} a\left(\frac{w}{\phi(w)}\right).$$

We now set $y = h(w)$ in order to obtain an expression for $b(y)$:

$$b(y) = \left[\frac{\phi(w) - w\phi'(w)}{\phi(w)f(w)} a\left(\frac{w}{\phi(w)}\right) \,\middle|\, w = \frac{yw}{h(w)} \right].$$

An application of the Lagrange inversion formula (Theorem 2) gives us

$$b_n = [t^n]\frac{\phi(t) - t\phi'(t)}{\phi(t)f(t)}a\left(\frac{t}{\phi(t)}\right)\frac{h'(t)t^{n+1}}{h(t)^{n+1}},$$

which is equivalent to the formula in the assertion. ■

Example. Let us consider inversions of Abel's type in Riordan's book; the ingenious ways used by Riordan to prove them can be substituted by a more systematic approach. Let us consider the sums

$$a_n = \sum_{k=0}^{n}\binom{n}{k}(x + pn + qk)^{n-k}b_k;$$

they are equivalent to

$$\frac{a_n}{n!} = \sum_{k=0}^{n}\frac{(x + pn + qk)^{n-k}}{(n-k)!}\frac{b_k}{k!},$$

and therefore we have

$$\frac{a_n}{n!} = [t^n]\mathrm{e}^{xt}(e^{pt})^n\widehat{b}(t\mathrm{e}^{qt}).$$

The previous theorem, with $f(t) = \mathrm{e}^{xt}$, $\phi(t) = \mathrm{e}^{pt}$ and $h(t) = t\mathrm{e}^{qt}$, allows us to invert:

$$\begin{aligned}\frac{b_n}{n!} &= \sum_{k=0}^{n}\left([t^{n-k}]\frac{(1-pt)\mathrm{e}^{pt}(1+qt)\mathrm{e}^{qt}}{\mathrm{e}^{xt}\mathrm{e}^{(k+1)pt}\mathrm{e}^{(n+1)qt}}\right)\frac{a_k}{k!}\\ &= \sum_{k=0}^{n}\left([t^{n-k}](1-pt)(1+qt)\mathrm{e}^{-(x+pk+qn)t}\right)\frac{a_k}{k!}\\ &= \sum_{k=0}^{n}(-1)^{n-k}\left(1 + (p-q)\frac{n-k}{x+pk+qn} - pq\frac{(n-k)(n-k-1)}{(x+pk+qn)^2}\right)\frac{(x+pk+qn)^{n-k}}{(n-k)!}\frac{a_k}{k!}.\end{aligned}$$

The quantity between parentheses can be easily developed, and eventually we have

$$\frac{b_n}{n!} = \sum_{k=0}^{n}(-1)^{n-k}\left((p+q)^2nk + x^2 + x(p+q)(n+k) + pq(n-k)\right)\frac{(x+pk+qn)^{n-k-2}}{(n-k)!}\frac{a_k}{k!}$$

or

$$\frac{b_n}{n!} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F(x,p,n,q,k)(x+pk+qn)^{n-k-2}a_k.$$

Inversions (3.1.2), (3.1.3), (3.1.3a), (3.1.4) and (3.1.5) in Riordan's book [12] are particular cases of this formula. In fact we have the following correspondences:

(3.1.2)	$p = 1$	$q = -1$	$F(x,p,n,q,k) = x^2 - n + k$
(3.1.3)	$p = 0$	$q = 1$	$F(x,p,n,q,k) = (x+k)(x+n)$
(3.1.3*a*)	$p = 1$	$q = 0$	$F(x,p,n,q,k) = (x+k)(x+n)$
(3.1.4)	$p = 1$	$q = 1$	$F(x,p,n,q,k) = (x+2n)(x+n+k)$
(3.1.5)	$x = 0$	$p = q = 1$	$F(x,p,n,q,k) = 4nk + n - k.$

Obviously, other inversions are obtained by assigning different values to x, p, q, and the reader can try with some cases of interest.

4. Implicit summation

The diagonalization rule allows us to solve many combinatorial sums by using the method of Riordan arrays, that is Eq. (2). As we have shown in [15], this equation can take on several special forms, which are worthy of being considered on their own. We quote without proof the following cases.

The formula for *partial sums*:

$$\sum_{k=0}^{n} f_k = [t^n]\frac{f(t)}{1-t};$$

the *Euler transformation*:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = [t^n]\frac{1}{1-t} f\left(\frac{t}{1-t}\right);$$

the so-called *rule A*:

$$\sum_{k=0}^{n} \binom{n+ak}{m+bk} f_k = [t^n]\frac{t^m}{(1-t)^{m+1}} \cdot f\left(\frac{t^{b-a}}{(1-t)^b}\right) \quad b > a;$$

and the so-called *rule B* for simple binomial coefficients:

$$\sum_{k=0}^{n} \binom{n+ak}{m+bk} f_k = [t^m](1+t)^n f(t^{-b}(1+t)^a) \quad b < 0.$$

Let us consider the following partial sum ((1.84) in Gould's collection [3]):

$$S_n = \sum_{k=0}^{n} \binom{2n-1}{k} = 4^{n-1} + \binom{2n-1}{n} \quad (n \geq 1),$$

where the index k is limited by the sum, not by the binomial coefficient. The formula for partial sums now gives

$$S_n = \sum_{k=0}^{n} \binom{2n-1}{k} = [t^n]\frac{(1+t)^{2n-1}}{1-t} = [t^n]\left[\frac{1}{(1+w)(1-w)} \cdot \frac{1+w}{1-w}\,\middle|\, w = t(1+w)^2\right].$$

If we solve the functional equation $w = t(1+w)^2$, we have

$$S_n = [t^n]\left[\frac{1}{(1-w)^2}\,\middle|\, w = \frac{1-2t-\sqrt{1-4t}}{2}\right] = \frac{1}{2}[t^n]\frac{1-2t+\sqrt{1-4t}}{1-4t} = 4^{n-1} + \binom{2n-1}{n}.$$

An analogous pattern follows the identity of Szily (3.38) in Gould's collection:

$$S_n = \sum_{k=-n}^{n} (-1)^k \binom{2n}{n-k}\binom{2r}{r-k} = \binom{2n}{n}\binom{2r}{r}\binom{n+r}{n}^{-1} = \frac{(2n)!(2r)!}{(n+r)!n!r!}.$$

By using rule B, we find

$$S_n = \sum_{k=-n}^{n} (-1)^k \binom{2n}{n-k}\binom{2r}{r-k} = [t^n](1+t)^{2n}\left[\frac{(-1)^r(1-u)^{2r}}{u^r}\,\middle|\, u = t\right]$$

$$= (-1)^r[t^n](1+t)^{2n}\frac{(1-t)^{2r}}{t^r} = (-1)^r[t^n]\left[\frac{(1-w)^{2r}}{w^r} \cdot \frac{1+w}{1-w}\,\middle|\, w = t(1+w)^2\right].$$

By solving the functional equation, we find the values

$$\frac{(1-w)^2}{w} = \frac{1-4t}{t}; \qquad \frac{1+w}{1-w} = \frac{1}{\sqrt{1-4t}},$$

and this allows us to find the sum:

$$S = (-1)^r [t^n](1-4t)^{r-1/2} = (-1)^r \binom{r-1/2}{n+r}(-4)^{n+r} = (-1)^n \binom{r-1/2}{n+r} 4^{n+r}.$$

This expression can be simplified by observing that

$$\begin{aligned}\binom{r-1/2}{n+r} &= \frac{(r-1/2)(r-3/2)\cdots(-n+1/2)}{(n+r)!}\\ &= \frac{(2r-1)(2r-3)\cdots(-2n+1)}{2^{n+r}(n+r)!} = \frac{(2r)!(2n)!(-1)^n}{4^{n+r}(n+r)!}.\end{aligned}$$

Therefore we conclude:

$$S_n = \frac{(2r)!(2n)!}{r!n!(n+r)!} = \binom{2r}{r}\binom{2n}{n}\binom{n+r}{n}^{-1}.$$

The following identity is rather complicated ((3.57) in Gould's collection):

$$S_n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}\binom{2n+2x}{k+x} = \binom{n+x-1/2}{2n+x}(-4)^{2n+x}.$$

We begin by using rule B:

$$S_n = \sum_{k=0}^{2n} (-1)^k \binom{2n}{2n-k}\binom{2n+2x}{k+x} = [t^{2n}](1+t)^{2n}\left[\frac{(1-u)^{2n+2x}}{u^x}\,\middle|\, u=t\right] = [t^{2n}](1-t^2)^{2n}\frac{(1-t)^{2x}}{t^x}$$

and then we set $m = 2n$:

$$S_n = [t^m](1-t^2)^m \frac{(1-t)^{2x}}{t^x} = [t^m]\left[\left(\frac{(1-w)^2}{w}\right)^x \cdot \frac{1-w^2}{1+w^2}\,\middle|\, w = t(1-w^2)\right].$$

Now we solve the functional equation:

$$tw^2 + w - t = 0, \qquad w = \frac{\sqrt{1+4t^2}-1}{2t}$$

and compute the relevant quantities:

$$\frac{(1-w^2)}{w} = \frac{\sqrt{1+4t^2}-2t}{t}, \qquad \frac{1-w^2}{1+w^2} = \frac{1}{\sqrt{1+4t^2}}.$$

Thus we have

$$S_n = [t^{2n}]\frac{1}{\sqrt{1+4t^2}}\left(\frac{\sqrt{1+4t^2}-2t}{t}\right)^x = [t^{2n+x}]\frac{(\sqrt{1+4t^2}-2t)^x}{\sqrt{1+4t^2}}.$$

At this point we apply the Lagrange Inversion Formula in the following particular way (see [8]): we set $F = \sqrt{1+4t^2}-2t$ and $t = w/\phi$; then we find

$$\phi F + 2w = \sqrt{\phi^2+4w^2}, \qquad \phi = \frac{4wF}{1-F^2}.$$

Since we should have $F(0) = 1$, we set $\phi = F$ and find $1-\phi^2 = 4w$. Consequently, we have

$$\phi = \sqrt{1-4w} = F; \qquad w = t\sqrt{1-4w}; \qquad \sqrt{1+4t^2} = F + 2t = \sqrt{1-4w} + \frac{2w}{\sqrt{1-4w}}.$$

Returning to S_n, we find

$$S_n = [t^{2n+x}]\left[\frac{(\sqrt{1-4w})^x}{\sqrt{1-4w}+2w/\sqrt{1-4w}}\,\middle|\, w = t\sqrt{1-4w}\right].$$

By applying case (1) in Theorem 2, we obtain

$$\begin{aligned}S_n &= [t^{2n+x}]\frac{(\sqrt{1-4t})^x}{\sqrt{1-4t}+2t/\sqrt{1-4t}}(\sqrt{1-4t})^{2n+x-1}\left(\sqrt{1-4t}+\frac{2t}{\sqrt{1-4t}}\right)\\ &= [t^{2n+x}](\sqrt{1-4t})^{2n+2x-1} = \binom{n+x-1/2}{2n+x}(-4)^{2n+x}.\end{aligned}$$

We could illustrate the method with many other examples, but we conclude with formula (1.119) in Gould's collection, a special case of a general formula due to Abel:

$$S_n = \sum_{k=1}^{n}\binom{n}{k}(yk)^{n-k}(x-yk)^{k-1} = x^{n-1}.$$

The sum can be easily transformed:

$$S_n = n!\sum_{k=0}^{n}\frac{(yk)^{n-k}}{(n-k)!}\cdot\frac{(x-yk)^{k-1}}{k!}.$$

The first factor corresponds to a Riordan array:

$$\frac{(yk)^{n-k}}{(n-k)!} = [t^{n-k}]\mathrm{e}^{ykt} \rightsquigarrow D = \left(1, t\mathrm{e}^{yt}\right),$$

while for the second we have

$$\frac{(x-yk)^{k-1}}{k!} = \frac{1}{x}\left([t^k]\mathrm{e}^{(x-yk)t} + y[t^{k-1}]\mathrm{e}^{(x-yk)t}\right) = \frac{1}{x}\left([t^k](1+yt)\mathrm{e}^{xt}\mathrm{e}^{-ykt}\right);$$

this corresponds to the generating function:

$$\sum_{k\geq 0}\frac{(x-yk)^{k-1}}{k!}t^k = \frac{1}{x}\left[\frac{(1+yw)\mathrm{e}^{xw}}{1+yw}\,\middle|\, w = t\mathrm{e}^{-yw}\right] = \frac{1}{x}\left[\mathrm{e}^{xw}\,\middle|\, w = te^{-yw}\right].$$

Finally, we can develop the sum by performing a substitution inside another substitution:

$$S_n = \frac{n!}{x}[t^n]\left[\left[\mathrm{e}^{xw}\,\middle|\, w = u\mathrm{e}^{-yw}\right]\,\middle|\, u = t\mathrm{e}^{yt}\right] = \frac{n!}{x}[t^n]\mathrm{e}^{xt} = x^{n-1}.$$

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