A survey on Riordan arrays

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Outline

1. Some history
2. Main properties of Riordan arrays
3. Riordan arrays and binary words avoiding a pattern
4. Riordan arrays, combinatorial sums and recursive matrices

A previous seminar

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- I remember with pleasure my seminar at INRIA on October 10, 1994: *Riordan arrays and their applications*

References -1-

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3. R. Sprugnoli. Riordan arrays and combinatorial sums. *Discrete Mathematics*, 132: 267–290, 1994.
4. D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri. On some alternative characterizations of Riordan arrays. *Canadian Journal of Mathematics*, 49(2): 301–320, 1997.

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1. T. X. He and R. Sprugnoli. Sequence characterization of Riordan arrays.*Discrete Mathematics*, 309: 3962–3974, 2009.

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The bibliography on the subject is vast and still growing.

Definition in terms of $d(t)$ and $h(t)$

- A *Riordan array* is a pair

$$D = \mathcal{R}(d(t),\ h(t))$$

in which $d(t)$ and $h(t)$ are formal power series such that $d(0) \neq 0$ and $h(0) = 0$; if $h'(0) \neq 0$ the Riordan array is called *proper*.

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- The pair defines an infinite, lower triangular array $(d_{n,k})_{n,k \in N}$ where:

$$d_{n,k} = [t^n]d(t)(h(t))^k$$

An example: the Pascal triangle

$$P = \mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$$

$$d_{n,k} = [t^n]\frac{1}{1-t} \cdot \frac{t^k}{(1-t)^k} = [t^{n-k}](1-t)^{-k-1} = \binom{n}{k}$$

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

An example: the Catalan triangle

$$C = \mathcal{R}\left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right)$$

$$d_{n,k} = [t^n]d(t)(h(t))^k = [t^{n+1}]\left(\frac{1-\sqrt{1-4t}}{2}\right)^{k+1} = \frac{k+1}{n+1}\binom{2n-k}{n-k}$$

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	2	2	1			
3	5	5	3	1		
4	14	14	9	4	1	
5	42	42	28	14	5	1

The Group structure

Product: $\mathcal{R}(d(t),\ h(t)) * \mathcal{R}(a(t),\ b(t)) = \mathcal{R}(d(t)a(h(t)),\ b(h(t)))$

Identity: $\mathcal{R}(1,\ t)$

Inverse: $\mathcal{R}(d(t),\ h(t))^{-1} = \mathcal{R}\left(\frac{1}{d(\overline{h}(t))},\ \overline{h}(t)\right)$

$$h(\overline{h}(t)) = \overline{h}(h(t)) = t$$

Pascal triangle: product and inverse

$$P = \mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$$

$$P * P = \mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t}\right) * \mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t}\right) =$$

$$= \mathcal{R}\left(\frac{1}{1-t}\frac{1-t}{1-2t}, \frac{t}{1-t}\frac{1-t}{1-2t}\right) = \mathcal{R}\left(\frac{1}{1-2t}, \frac{t}{1-2t}\right).$$

$$P^{-1} = \mathcal{R}\left(\frac{1}{1+t}, \frac{t}{1+t}\right)$$

Subgroups

APPELL

$$\mathcal{R}(d(t),\ t) * \mathcal{R}(a(t),\ t) = \mathcal{R}(d(t)a(t),\ t)$$

$$\mathcal{R}(d(t),\ t)^{-1} = \mathcal{R}\left(\frac{1}{d(t)},\ t\right)$$

LAGRANGE

$$\mathcal{R}(1,\ h(t)) * \mathcal{R}(1,\ b(t)) = \mathcal{R}(1,\ h(b(t)))$$

$$\mathcal{R}(1,\ h(t))^{-1} = \mathcal{R}(1,\ \overline{h}(t))$$

RENEWAL $\quad d(t) = h(t)/t$

$HITTING - TIME$ $\quad d(t) = \dfrac{th'(t)}{h(t)}$

Inversion of Riordan arrays

$$\mathcal{R}(d(t),\ h(t))^{-1} = \mathcal{R}\left(\frac{1}{d(\overline{h}(t))},\ \overline{h}(t)\right)$$

Every Riordan array is the product of an Appell and a Lagrange Riordan array

$$\mathcal{R}(d(t),\ h(t)) = \mathcal{R}(d(t),\ t) * \mathcal{R}(1,\ h(t))$$

From this fact we obtain the formula for the inverse Riordan array

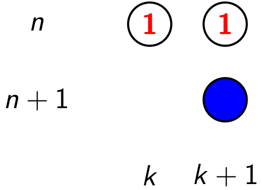
Pascal triangle: construction by columns

$d(t)h(t)^k$ is the g.f. of column k

$$\frac{1}{1-t}, \quad \frac{t}{(1-t)^2}, \quad \frac{t^2}{(1-t)^3}, \ldots$$

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	**6**	**4**	1	
5	1	5	10	**10**	5	1

Pascal triangle: construction by rows



$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

The A and Z sequences

An alternative definition, is in terms of the so-called A-sequence and Z-sequence, with generating functions $A(t)$ and $Z(t)$ satisfying the relations:

$$h(t) = tA(h(t)), \quad d(t) = \frac{d_0}{1 - tZ(h(t))} \quad \text{with} \quad d_0 = d(0).$$

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots$$

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots$$

Pascal triangle: A-sequence $1, 1, 0, 0, \cdots \Longrightarrow A(t) = 1 + t$

The A-sequence for the Catalan triangle

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	2	2	1					
3	**5**	**5**	**3**	**1**				
4	14	**14**	9	4	1			
5	42	42	28	14	5	1		
6	132	132	**90**	**48**	**20**	**6**	**1**	
7	429	429	297	**165**	75	27	7	1

A-sequence $1, 1, 1, 1, \cdots \Longrightarrow A(t) = \frac{1}{1-t}$

Rogers' Theorem - 1978

The A-sequence is unique and only depends on $h(t)$

$$h(t) = tA(h(t))$$

Pascal $\quad h(t) = t(1 + h(t))$

$$h_P(t) = \frac{t}{1-t}$$

Catalan $\quad h(t) = t\frac{1}{1-h(t)}$

$$h_C(t) = \frac{1-\sqrt{1-4t}}{2}.$$

The B-sequence: $B(t) = A(t)^{-1}$

$d_{n,k}$ linearly depends on the elements of row $n+1$

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{k+j+1} = \binom{n}{k}$$

A-approach to R.a.'s

$$\text{Product} \quad A_3(t) = A_2(t) A_1\left(\frac{t}{A_2(t)}\right)$$

$$\text{Inverse} \quad A^*(t) = \left[\frac{1}{A(y)} \;\middle|\; y = tA(y)\right]$$

$$A_{P*C}(t) = \frac{1}{1-t}\left[1+y \;\middle|\; y = t(1-t)\right] = \frac{1+t-t^2}{1-t}$$

$$A_{C*P}(t) = (1+t)\left[\frac{1}{1-y} \;\middle|\; y = \frac{t}{1+t}\right] = (1+t)^2$$

$$A_{P^{-1}}(t) = \left[\frac{1}{1+y} \;\middle|\; y = t(1+y)\right] = 1-t$$

Pascal triangle: the A-matrix (not unique)

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$P^{[0]}(t) = 1 \qquad P^{[1]}(t) = 1 + t$$

$$A(t) = \frac{P^{[0]}(t) + \sqrt{P^{[0]}(t)^2 + 4tP^{[1]}(t)}}{2}$$

$$A(t) = \frac{1 + \sqrt{1 + 4t + 4t^2}}{2} = 1 + t$$

$$\begin{array}{ccc}
n-1 & 1 & 1 \\
n & 1 & \\
n+1 & & \bullet \\
 & k & k+1
\end{array}$$

The A-matrix in general

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j} d_{n-i,k+j} + \sum_{j\geq 0}\rho_j d_{n+1,k+j+2}.$$

Matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ is called the A-matrix of the Riordan array. If, for $i \geq 0$:

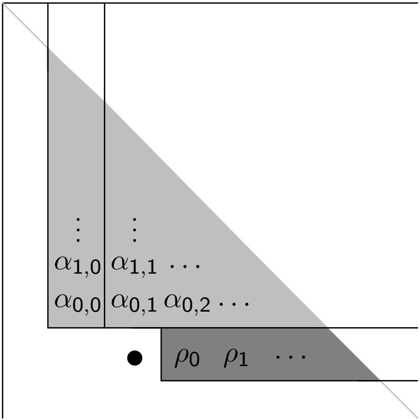
$$P^{[i]}(t) = \alpha_{i,0} + \alpha_{i,1}t + \alpha_{i,2}t^2 + \alpha_{i,3}t^3 + \dots$$

and $Q(t)$ is the generating function for the sequence $(\rho_j)_{j\in\mathbb{N}}$, then we have:

$$\frac{h(t)}{t} = \sum_{i\geq 0} t^i P^{[i]}(h(t)) + \frac{h(t)^2}{t} Q(h(t)).$$

$$A(t) = \sum_{i\geq 0} t^i A(t)^{-i} P^{[i]}(t) + tA(t)Q(t).$$

A graphical representation of the A-matrix



Binary words avoiding a pattern

- We consider the language of binary words with no occurrence of a pattern $\mathfrak{p} = p_0 \cdots p_{h-1}$.

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- The fundamental notion is that of the *autocorrelation vector* of bits $c = (c_0, \ldots, c_{h-1})$ associated to a given $\mathfrak{p}$.

The pattern $\mathfrak{p} = 00011$

0	0	0	1	1	Tails

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0	0	0	1	1	1

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0	0	0	1	1	Tails	
0	0	0	1	1		1
	0	0	0	1	1	0

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0	0	0	1	1	Tails		
0	0	0	1	1			1
	0	0	0	1	1		0
		0	0	0	1	1	0

The pattern $\mathfrak{p} = 00011$

0	0	0	1	1				Tails
0	0	0	1	1				1
	0	0	0	1	1			0
		0	0	0	1	1		0
			0	0	0	1	1	0

The pattern $\mathfrak{p} = 00011$

0	0	0	1	1	Tails				
0	0	0	1	1					1
	0	0	0	1	1				0
		0	0	0	1	1			0
			0	0	0	1	1		0
				0	0	0	1	1	0

The autocorrelation vector is then $c = (1, 0, 0, 0, 0)$

The bivariate generating function

Let $F^{[\mathfrak{p}]}_{n,k}$ denotes the number of words excluding the pattern and having n bits 1 and k bits 0, then we have

$$F^{[\mathfrak{p}]}(x,y) = \sum_{n,k\geq 0} F^{[\mathfrak{p}]}_{n,k} x^n y^k = \frac{C^{[\mathfrak{p}]}(x,y)}{(1-x-y)C^{[\mathfrak{p}]}(x,y) + x^{n_1^{\mathfrak{p}}} y^{n_0^{\mathfrak{p}}}},$$

where $n_1^{[\mathfrak{p}]}$ and $n_0^{[\mathfrak{p}]}$ correspond to the number of ones and zeroes in the pattern and $C^{[\mathfrak{p}]}(x,y)$ is the bivariate autocorrelation polynomial.

An example with $\mathfrak{p} = 110011$

We have $C^{[\mathfrak{p}]}(x,y) = 1 + x^2y^2 + x^3y^2$, and:

$$F^{[\mathfrak{p}]}(x,y) = \frac{1 + x^2y^2 + x^3y^2}{(1-x-y)(1+x^2y^2+x^3y^2) + x^4y^2}.$$

n/k	0	1	2	3	4	5	6	7
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	**2**	3	4	5	6	7	8
2	**1**	3	**6**	10	15	21	28	36
3	**1**	4	10	**20**	35	56	84	120
4	**1**	5	14	33	**67**	122	205	324
5	**1**	6	19	50	114	**232**	432	750
6	**1**	7	25	72	181	404	**822**	1552
7	**1**	8	32	100	273	660	1451	**2952**

...the lower and upper triangular parts

n/k	0	1	2	3	4	5
0	**1**					
1	**2**	**1**				
2	**6**	3	**1**			
3	**20**	10	4	**1**		
4	**67**	33	14	5	**1**	
5	**232**	114	50	19	6	**1**

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0	**1**					
1	**2**	**1**				
2	**6**	3	**1**			
3	**20**	10	4	**1**		
4	**67**	35	15	5	**1**	
5	**232**	122	56	21	6	**1**

Matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$

- Let $R^{[\mathfrak{p}]}_{n,k} = F^{[\mathfrak{p}]}_{n,n-k}$ with $k \leq n$. More precisely, $R^{[\mathfrak{p}]}_{n,k}$ counts the number of words avoiding $\mathfrak{p}$ with n bits one and $n-k$ bits zero.

Matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$

- Let $R_{n,k}^{[\mathfrak{p}]} = F_{n,n-k}^{[\mathfrak{p}]}$ with $k \leq n$. More precisely, $R_{n,k}^{[\mathfrak{p}]}$ counts the number of words avoiding $\mathfrak{p}$ with n bits one and $n-k$ bits zero.
- Let $\bar{\mathfrak{p}} = \bar{p}_0 \dots \bar{p}_{h-1}$ be the conjugate pattern.

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- Let $\bar{\mathfrak{p}} = \bar{p}_0 \dots \bar{p}_{h-1}$ be the conjugate pattern.
- We obviously have $R^{[\bar{\mathfrak{p}}]}_{n,k} = F^{[\bar{\mathfrak{p}}]}_{n,n-k} = F^{[\mathfrak{p}]}_{n-k,n}$, therefore, the matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ represent the lower and upper triangular part of the array $\mathcal{F}^{[\mathfrak{p}]}$, respectively.

Riordan patterns

- When matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ are both Riordan arrays?

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Riordan patterns

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- D. Merlini and R. Sprugnoli. Algebraic aspects of some Riordan arrays related to binary words avoiding a pattern. *Theoretical Computer Science*, 412 (27), 2988-3001, 2011.
- We say that $\mathfrak{p} = p_0 \ldots p_{h-1}$ is a Riordan pattern if and only if

$$C^{[\mathfrak{p}]}(x,y) = C^{[\mathfrak{p}]}(y,x) = \sum_{i=0}^{\lfloor (h-1)/2 \rfloor} c_{2i} x^i y^i, \ |n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]}| \in \{0,1\}.$$

Main Theorem -1-

The matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ are both Riordan arrays $\mathcal{R}^{[\mathfrak{p}]} = (d^{[\mathfrak{p}]}(t), h^{[\mathfrak{p}]}(t))$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]} = (d^{[\bar{\mathfrak{p}}]}(t), h^{[\bar{\mathfrak{p}}]}(t))$ if and only if $\mathfrak{p}$ is a Riordan pattern. Moreover we have:

$$d^{[\mathfrak{p}]}(t) = d^{[\bar{\mathfrak{p}}]}(t) = [x^0]F\left(x, \frac{t}{x}\right) = \frac{1}{2\pi i}\oint F\left(x, \frac{t}{x}\right)\frac{dx}{x}$$

and

$$h^{[\mathfrak{p}]}(t) = \frac{1 - \sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,1} t^{i+1} - \sqrt{(1 - \sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,1} t^{i+1})^2 - 4\sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,0} t^{i+1}(\sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,2} t^{i+1} + 1)}}{2(\sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,2} t^{i+1} + 1)}$$

Main Theorem -2-

... where $\delta_{i,j}$ is the Kronecker delta,

$$\sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,0} t^i = \sum_{i=0}^{n_1^{\mathfrak{p}}-1} c_{2i} t^i - \delta_{-1, n_0^{\mathfrak{p}} - n_1^{\mathfrak{p}}} t^{n_1^{\mathfrak{p}}-1},$$

$$\sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,1} t^i = -\sum_{i=0}^{n_1^{\mathfrak{p}}-1} c_{2(i+1)} t^i - \delta_{0, n_0^{\mathfrak{p}} - n_1^{\mathfrak{p}}} t^{n_1^{\mathfrak{p}}-1},$$

$$\sum_{i=0}^{n_1^{\mathfrak{p}}-1} \alpha_{i,2} t^i = \sum_{i=0}^{n_1^{\mathfrak{p}}-1} c_{2(i+1)} t^i - \delta_{1, n_0^{\mathfrak{p}} - n_1^{\mathfrak{p}}} t^{n_1^{\mathfrak{p}}-1},$$

and the coefficients c_i are given by the autocorrelation vector of $\mathfrak{p}$.
An analogous formula holds for $h^{[\bar{\mathfrak{p}}]}(t)$.

A Corollary

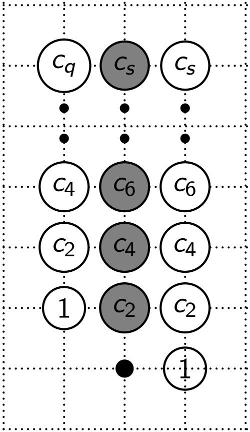
Let $\mathfrak{p}$ be a Riordan pattern. Then the Riordan array $\mathcal{R}^{[\mathfrak{p}]}$ is characterized by the A-matrix defined by the following relation:

$$R^{[\mathfrak{p}]}_{n+1,k+1} = R^{[\mathfrak{p}]}_{n,k} + R^{[\mathfrak{p}]}_{n+1,k+2} - R^{[\mathfrak{p}]}_{n+1-n_1^{\mathfrak{p}},k+1+n_0^{\mathfrak{p}}-n_1^{\mathfrak{p}}} +$$

$$-\sum_{i\geq 1} c_{2i}\left(R^{[\mathfrak{p}]}_{n+1-i,k+1} - R^{[\mathfrak{p}]}_{n-i,k} - R^{[\mathfrak{p}]}_{n+1-i,k+2}\right),$$

where the c_i are given by the autocorrelation vector of $\mathfrak{p}$.

The A-matrix corresponding to a Riordan pattern



The coefficients in the gray circles are negative, $s = 2n_1^{\mathfrak{p}}$, $q = 2(n_1^{\mathfrak{p}} - 1)$. Moreover, we have to consider the contribution of $-R^{[\mathfrak{p}]}_{n+1-n_1^{\mathfrak{p}},k+1+n_0^{\mathfrak{p}}-n_1^{\mathfrak{p}}}$.

The case $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 1$

By specializing the main result to the cases $|n_1^{\mathfrak{p}} - n_0^{\mathfrak{p}}| \in \{0, 1\}$ and by setting $C^{[\mathfrak{p}]}(t) = C^{[\mathfrak{p}]}(\sqrt{t}, \sqrt{t}) = \sum_{i \geq 0} c_{2i} t^i$, we have the following explicit generating functions:

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_0^{\mathfrak{p}}})}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) - \sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_0^{\mathfrak{p}}})}}{2C^{[\mathfrak{p}]}(t)}.$$

The case $n_1^{[\mathfrak{p}]} - n_0^{[\mathfrak{p}]} = 0$

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{(C^{[\mathfrak{p}]}(t) + t^{n_0^{\mathfrak{p}}})^2 - 4tC^{[\mathfrak{p}]}(t)^2}},$$

$$h^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t) + t^{n_0^{\mathfrak{p}}} - \sqrt{(C^{[\mathfrak{p}]}(t) + t^{n_0^{\mathfrak{p}}})^2 - 4tC^{[\mathfrak{p}]}(t)^2}}{2C^{[\mathfrak{p}]}(t)}.$$

The case $n_0^{[\mathfrak{p}]} - n_1^{[\mathfrak{p}]} = 1$

$$d^{[\mathfrak{p}]}(t) = \frac{C^{[\mathfrak{p}]}(t)}{\sqrt{C^{[\mathfrak{p}]}(t)^2 - 4tC^{[\mathfrak{p}]}(t)(C^{[\mathfrak{p}]}(t) - t^{n_1^{\mathfrak{p}}})}},$$

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An example with $\mathfrak{p} = 00011$

n/k	0	1	2	3	4	5
0	1					
1	2	1				
2	6	3	1			
3	18	10	**4**	1		
4	**58**	32	15	5	1	
5	192	**106**	**52**	21	6	1

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1-4t+4t^3}}$$

$$h^{[\mathfrak{p}]}(t) = \frac{1-\sqrt{1-4t+4t^3}}{2(1-t^2)}$$

$$\mathbf{R}^{[\mathfrak{p}]}_{\mathbf{n+1,k+1}} = \mathbf{R}^{[\mathfrak{p}]}_{\mathbf{n,k}} + \mathbf{R}^{[\mathfrak{p}]}_{\mathbf{n+1,k+2}} - \mathbf{R}^{[\mathfrak{p}]}_{\mathbf{n-1,k+2}}.$$

The A-sequence for $\mathfrak{p} = 00011$

- For $\mathfrak{p} = 00011$, we find after setting $R(t) = \sqrt{1 + 4t^4 - 4t^3}$:

$$A(t) = \frac{\left(2t^3 - t^2 - t - 1 - (t^2 + t + 1)R(t)\right)\left(2t^3 - \sqrt{2}\sqrt{2t^6 + 8t^4 - 12t^3 + 4 - (4 - 4t^3)R(t)}\right)}{8t^4(t-1)(t+1)}$$

$$= 1+t+t^2+t^4+t^5+2t^7+t^8-t^9+5t^{10}-t^{11}-4t^{12}+16t^{13}-14t^{14}-8t^{15}+57t^{16}-83t^{17}+15t^{18}+197t^{19}+O(t^{20}).$$

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- In general, the Riordan arrays for binary words avoiding $\mathfrak{p}$ are characterized by a complex A-sequence, while the A-matrix is quite simple. However, the presence of negative coefficients leads to non trivial combinatorial interpretations.

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- In general, the Riordan arrays for binary words avoiding $\mathfrak{p}$ are characterized by a complex A-sequence, while the A-matrix is quite simple. However, the presence of negative coefficients leads to non trivial combinatorial interpretations.
- S. Bilotta, D. Merlini, E. Pergola, R. Pinzani. Pattern $1^{j+1}0^j$ avoiding binary words. To appear in *Fundamenta Informaticae*.

Formulas relative to whole classes of patterns

- $\mathfrak{p} = 1^{j+1}0^j$

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1 - 4t + 4t^{j+1}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1 - \sqrt{1 - 4t + 4t^{j+1}}}{2}$$

- $\mathfrak{p} = 0^{j+1}1^j$

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1 - 4t + 4t^{j+1}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1 - \sqrt{1 - 4t + 4t^{j+1}}}{2(1 - t^j)}$$

- $\mathfrak{p} = 1^j0^j$ and $\mathfrak{p} = 0^j1^j$

$$d^{[\mathfrak{p}]}(t) = \frac{1}{\sqrt{1 - 4t + 2t^j + t^{2j}}}, \quad h^{[\mathfrak{p}]}(t) = \frac{1 + t^j - \sqrt{1 - 4t + 2t^j + t^{2j}}}{2}$$

- $\mathfrak{p} = (10)^j1$

$$d^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i}{\sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}}, \quad h^{[\mathfrak{p}]}(t) = \frac{\sum_{i=0}^{j} t^i - \sqrt{1 - 2\sum_{i=1}^{j} t^i - 3\left(\sum_{i=1}^{j} t^i\right)^2}}{2\sum_{i=0}^{j} t^i}$$

Riordan array summation

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(h(t))$$

Partial sum theorem:

$$\sum_{k=0}^{n} f_k = [t^n] \frac{f(t)}{1-t}$$

Euler transformation:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = [t^n] \frac{1}{1-t} f\left(\frac{t}{1-t}\right).$$

A simple example: Harmonic numbers

$$\mathcal{G}\left(\frac{1}{n}\right) = \ln\frac{1}{1-t}$$

$$\mathcal{G}\left(\sum_{k=1}^{n}\frac{1}{k}\right) = \mathcal{G}(H_n) = \frac{1}{1-t}\ln\frac{1}{1-t}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} =$$

$$= [t^n]\frac{1}{1-t}\left[\ln\frac{1}{1+w}\;\middle|\; w = \frac{t}{1-t}\right] =$$

$$= [t^n]\frac{1}{1-t}\ln\frac{1}{1-t} = H_n.$$

General rules for binomial coefficients

$$\sum_k \binom{n+ak}{m+bk} f_k = [t^n]\frac{t^m}{(1-t)^{m+1}} f\left(\frac{t^{b-a}}{(1-t)^b}\right) \qquad b > a$$

$$\sum_k \binom{n+ak}{m+bk} f_k = [t^m](1+t)^n f(t^{-b}(1+t)^a) \qquad b < 0$$

$$\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = [t^n]\frac{t^m}{(1-t)^{m+1}}\left[\frac{\sqrt{1+4y}-1}{2y} \;\Big|\; y = \frac{t}{(1-t)^2}\right] =$$

$$= [t^{n-m}]\frac{1}{(1-t)^{m+1}}\left(\sqrt{1+\frac{4t}{(1-t)^2}}-1\right)\frac{(1-t)^2}{2t} = [t^{n-m}]\frac{1}{(1-t)^m} = \binom{n-1}{m-1}.$$

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- R. Sprugnoli. Riordan Array Proofs of Identities in Gould's Book.

Recursive matrices

- A. Luzon, D. Merlini, M. A. Moron and R. Sprugnoli. Identities induced by Riordan arrays. *Linear Algebra and its Applications*, 436 (3), 631-647, 2012.

Recursive matrices

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$$D = \mathcal{X}(d(t),\ h(t))$$

$$d_{n,k} = [t^n]d(t)h(t)^k \quad n, k \in \mathbb{Z}$$

Recursive matrices

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$$D = \mathcal{X}(d(t),\ h(t))$$

$$d_{n,k} = [t^n]d(t)h(t)^k \quad n, k \in \mathbb{Z}$$

- The introduction of recursive matrices simply extends the properties of Riordan arrays.

The Pascal recursive matrix

$n \backslash k$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-6	1	0	0	0	0	0	0	0	0	0	0	0	0
-5	-5	1	0	0	0	0	0	0	0	0	0	0	0
-4	10	-4	1	0	0	0	0	0	0	0	0	0	0
-3	-10	6	-3	1	0	0	0	0	0	0	0	0	0
-2	5	-4	3	-2	1	0	0	0	0	0	0	0	0
-1	-1	1	-1	1	-1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	0	0	0	0	0	0	1	1	0	0	0	0	0
2	0	0	0	0	0	0	1	2	1	0	0	0	0
3	0	0	0	0	0	0	1	3	3	1	0	0	0
4	0	0	0	0	0	0	1	4	6	4	1	0	0
5	0	0	0	0	0	0	1	5	10	10	5	1	0
6	0	0	0	0	0	0	1	6	15	20	15	6	1

The Catalan recursive matrix

	−4	−3	−2	−1	0	1	2	3	4	5
−6	0	0	0	0	0	0	0	0	0	0
−5	0	0	0	0	0	0	0	0	0	0
−4	1	0	0	0	0	0	0	0	0	0
−3	−3	1	0	0	0	0	0	0	0	0
−2	0	−2	1	0	0	0	0	0	0	0
−1	−1	−1	−1	1	0	0	0	0	0	0
0	−3	−2	−1	0	1	0	0	0	0	0
1	−9	−5	−2	0	1	1	0	0	0	0
2	−28	−14	−5	0	2	2	1	0	0	0
3	−90	−42	−14	0	5	5	3	1	0	0
4	−297	−132	−42	0	14	14	9	4	1	0
5	−1001	−429	−132	0	42	42	28	14	5	1
6	−3432	−1430	−429	0	132	132	90	48	20	6

Generalized Sums

Identities with three parameters $k, n, m \in \mathbb{Z}$

$$d_{n+m,k+m} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n,k+j} = \sum_{j=0}^{n-k} h_{j+m}^{(m)} d_{n-j,k}$$

$$a_j^{(m)} = [t^j]A(t)^m$$

$$h_{j+m}^{(m)} = [t^{j+m}]h(t)^m = [t^j](h(t)/t)^m$$

Generalized Sums for the Catalan triangle

$$\sum_{j=0}^{n-k} \binom{m+j-1}{j} \frac{k+j+1}{n+1} \binom{2n-j-k}{n-j-k} =$$

$$= \frac{k+m+1}{n+m+1} \binom{2n+m-k}{n-k}.$$

$$\sum_{j=0}^{n-k} \frac{m}{m+2j} \binom{m+2j}{j} \frac{k+1}{n-j+1} \binom{2n-2j-k}{n-j-k} =$$

$$= \frac{k+m+1}{n+m+1} \binom{2n+m-k}{n-k}$$

Specializing the parameters

$$n \mapsto n, \;\; m \mapsto n, \;\; k \mapsto 0$$

$$\sum_{j=0}^{n} \frac{j+1}{n+1} \binom{n+j-1}{j} \binom{2n-j}{n-j} = \frac{n+1}{2n+1} \binom{3n}{n}$$

$$\sum_{j=0}^{n} \frac{n}{n+2j} \binom{n+2j}{j} \frac{1}{n-j+1} \binom{2n-2j}{n-j} = \frac{n+1}{2n+1} \binom{3n}{n}$$

$$n \mapsto 2n, \;\; m \mapsto n, \;\; k \mapsto n$$

$$\sum_{j=0}^{n} \frac{n+j+1}{2n+1} \binom{n+j-1}{j} \binom{3n-j}{n-j} = \frac{2n+1}{3n+1} \binom{4n}{n}$$

$$\sum_{j=0}^{n} \frac{n}{n+2j} \binom{n+2j}{j} \frac{n+1}{2n-j+1} \binom{3n-2j}{n-j} = \frac{2n+1}{3n+1} \binom{4n}{n}$$

Work in progress: the complementary Riordan array

	−4	−3	−2	−1	0	1	2	3	4	5
−6	0	0	0	0	0	0	0	0	0	0
−5	0	0	0	0	0	0	0	0	0	0
−4	**1**	0	0	0	0	0	0	0	0	0
−3	**-3**	**1**	0	0	0	0	0	0	0	0
−2	**0**	**-2**	**1**	0	0	0	0	0	0	0
−1	**-1**	**-1**	**-1**	**1**	0	0	0	0	0	0
0	−3	−2	−1	0	1	0	0	0	0	0
1	−9	−5	−2	0	1	1	0	0	0	0
2	−28	−14	−5	0	2	2	1	0	0	0
3	−90	−42	−14	0	5	5	3	1	0	0
4	−297	−132	−42	0	14	14	9	4	1	0
5	−1001	−429	−132	0	42	42	28	14	5	1

$$D^{\perp} = \mathcal{R}(d(\overline{h}(t))\overline{h}'(t),\ \overline{h}(t)) = \mathcal{R}(\frac{1-2t}{1-t}, t(1-t))$$

End of the seminar

Thank you for your attention and for the invitation

Exercise: find the identities induced by Pascal triangle.

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- $d_{n+m,k+m} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n,k+j} = \sum_{j=0}^{n-k} h_{j+m}^{(m)} d_{n-j,k}$

Exercise: find the identities induced by Pascal triangle.

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$$a_j^{(m)} = [t^j](1+t)^m = \binom{m}{j}$$

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Well! You have proved Vandermonde's identity

Exercise: find the identities induced by Pascal triangle.

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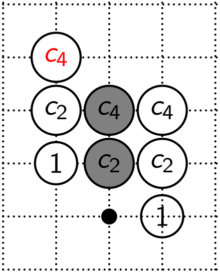
Well! You have proved Vandermonde's identity

$$\binom{n+m}{k+m} = \sum_{j=0}^{n}\binom{m+j-1}{j}\binom{n-j}{k}.$$

.

Exercise: find $A^{[\mathfrak{p}]}(t)$ for $\mathfrak{p} = 10101$

$$C^{[\mathfrak{p}]}(x,y) = 1 + xy + x^2y^2 \Rightarrow Q(t) = 1, \quad P^{[0]}(t) = P^{[1]}(t) = 1 - t + t^2$$



Moreover, we have to consider the contribution of $-R^{[\mathfrak{p}]}_{n+1-n_1^{\mathfrak{p}},k+1+n_0^{\mathfrak{p}}-n_1^{\mathfrak{p}}} = -R^{[\mathfrak{p}]}_{n-2,k}$.

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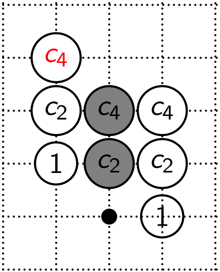

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$$A(t) = \sum_{i \geq 0} t^i A(t)^{-i} P^{[i]}(t) + tA(t)Q(t) = 1 - t + t^2 + tA(t)^{-1}(1 - t + t^2) + tA(t)$$

$$A(t) = \frac{1 - t + t^2 - \sqrt{1 + 2t - 5t^2 + 6t^2 - 3t^4}}{2(1-t)} = 1 + t + 3t^3 - 3t^4 + 12t^5 - 30t^6 + 93t^7 - 282t^8 + O(t^9)$$