FIBONACCI NUMBERS AND ASSOCIATED MATRICES

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by

Ashley Marie Meinke

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Thesis written by
Ashley Marie Meinke
B.S., Kent State University, 2009
M.S., Kent State University, 2011

Approved by

______________________________, Advisor
Aloysius Bathi Kasturiarachi

______________________________, Chair, Department of Mathematics
Andrew Tonge

______________________________, Dean, College of Arts and Sciences
Timothy Moerland

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Fibonacci Numbers and Associated Matrices

Ashley M. Meinke

1 Introduction

In this thesis, we approach the study of Fibonacci numbers using the theory of matrices. Fibonacci numbers are widely studied and the formulas to derive them are well-known. Such formulas include Binet's formula,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

and Cassini's formula,

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n,$$

where f_n is the n^{th} Fibonacci number. What makes this approach unique is the use of the theory of matrices to derive the Fibonacci numbers and in particular, the formulas of Binet and Cassini. We use the theory of diagonalizing a matrix and examine the eigenvalues of certain 2×2 generating matrices to derive Binet-type formulas for the Lucas, generalized Fibonacci and generalized weighted Fibonacci numbers. We derive Cassini-type formulas for Lucas, generalized Fibonacci and generalized weighted Fibonacci numbers by computing determinants of certain matrices. We extend these results to Tribonacci and generalized Tribonacci numbers with a similar 3×3 matrix approach. In all cases, we do a thorough analysis of the recursive sequences versus the derived Binet-type formulas. In fact, through numerical computations, we show the results of the recursive sequences and the Binet-type formulas are the same.

Our approach places an emphasis on showing the connection between the theory of matrices and the Fibonacci numbers. As with all mathematics, there are many ways to arrive at the same result and our approach illustrates a non-standard way to derive Fibonacci numbers and other related sequences. We begin with a historical introduction, examining Fibonacci's life and mathematical work and discuss the presence of these numbers even before Fibonacci's time. We also provide a sampling of some of the interesting properties that Fibonacci numbers possess. We delve into the necessary matrix theory in a later section.

1.1 Historical Background: Fibonacci and Hemachandra

Fibonacci (1170-1250), formally known as Leonardo of Pisa, was an Italian mathematician during the Middle Ages. Fibonacci is best known for his “Fibonacci numbers”, which will be described in detail in Section 1.2: Fibonacci Numbers. Fibonacci's mathematical background began during his many visits to North Africa, where he was introduced to the early works of algebra, arithmetic and geometry. He also traveled to countries located in the Mediterranean and studied the mathematical systems that they were practicing. His travels led him to the realization that Europe was lacking on the mathematical scene.

At the age of thirty, Fibonacci published his first book entitled *Liber Abaci*, which means “Book of Calculation” or “Book of Counting”. It is said that the work of Egyptian mathematician Abu Kamil inspired Fibonacci's work in *Liber Abaci.* The beginning of his book has a statement about the Hindu-Arabic number system: “There are nine figures of the Indians: 9 8 7 6 5 4 3 2 1 . With these nine figures, and with this sign 0...any number may be written, as will be demonstrated below.” [2] The problems in the book were able to illustrate for the first time the advantages of the new Hindu-Arabic numeral system. During Fibonacci's time, *Liber Abaci* was considered to be a complete source for arithmetical knowledge. The publication of this book inspired additional research in algebra and arithmetic and continued to serve as a key mathematics source for hundreds of years.

Other publications by Fibonacci include *Practica Geometriae* (1220), which means “Practice of Geometry”, *Flos* (1225), which means “blossom” or “flower” and *Liber Quadratorum* (1225), which means “Book of Square Numbers.” *Practica Geometriae* was a collection of geometry and trigonometry results. *Flos*, on the other hand, was a small book that looked at a variety of indeterminate problems. One such result in *Flos* was Fibonacci's acceptance of a negative value as the solution to a quadratic equation. The inspiration for Fibonacci to write *Liber Quadratorum* came during the mathematical competitions held at the court of Emperor Frederick II. *Liber Quadratorum* is devoted to second degree diophantine equations, which are defined as indeterminate polynomial equations that allow only integer values as solutions.

Fibonacci did not go beyond the work of the Arabic mathematicians before him, but his work gave a new outlook on ancient problems and often times he was able to give original proofs. His knowledge of the mathematics that existed before his time was very apparent and provided him with a strong mathematical background. Fibonacci's ability to solve a wide variety of mathematical problems in a very imaginative, ingenious manner have hailed him as one of the greatest mathematician of the Middle Ages.

We now briefly shift the discussion to Indian mathematicians and the role they played in the "so-called" Fibonacci numbers. While the Fibonacci numbers are named after Leonardo Fibonacci, who was described in detail above, it is interesting to note that knowledge about these numbers actually occurred long before his time. The sequence of numbers: 0, 1 ,2 ,3 ,5 ,8 ,13 ,..., are the "so-called" Fibonacci numbers and they originated in ancient India. Singh claims that Indian mathematician *Ācārya Pingala* was the first to possess knowledge of the Fibonacci numbers. It is speculated that he lived sometime around 400 B.C. *Ācārya Virahānka*, who lived between 600 and 800 A.D., is said to have been the first Indian mathematician to give a written representation of the Fibonacci numbers. Another significant figure in the role of Fibonacci numbers is *Gopāla*, who was born sometime before 1135 A.D. He discusses both the Fibonacci numbers themselves and *Virahānka's* explicit representation of these numbers.

Singh states that "the concept of the sequence of these numbers in India is at least as old as the origin of the metrical sciences of Sanskrit and Prakrit poetry." [10] The basis of Sanskrit poetry is the number of *mātrās*, also called "mora", which mean "syllabic instant" or "measure." Units having one matra are called *laghu*, meaning "light" and units having two matras are called *guru*, meaning "heavy." A laghu is denoted by $\vert$ and a guru is denoted by S. The symbols $\vert$ and S are used in metric the same as the numbers 1 and 2 are used in combinatorics. There are three types of meters in Sanskrit and Prakrit poetry: *varna-vrttas*, *mātrā-vrttas* and *gana-vrttas*. We will discuss the *mātrā-vrttas* and their relationship to the Fibonacci numbers.

"*Mātrā-vrttas* are meters in which the number of morae remains constant and the number of letters is arbitrary." [10] Table 1 shows the different combinations of *laghu* and *guru* for $n = 1, 2, 3, 4, 5$ and 6 mora. Note that the variations of n mora are obtained by taking the $n - 2$ column of mora and adding an S to each variation and taking the $n - 1$ column and adding a $\vert$ to each variation. Thus, the number of meter variations having n *mātrā* is given by,

$$\phi(n) = \phi(n-2) + \phi(n-1), \; n \geq 2. \tag{1}$$

	1 mora	2 mora	3 mora	4 mora	5 mora	6 mora	
	$\vert$	S	$\vert\, S$	$S\ S$	$\vert\, S\ S$	$S\ S\ S$	$S\ S\ \vert\vert$
		$\vert\vert$	$S\ \vert$	$\vert\vert\ S$	$S\ \vert\ S$	$\vert\vert\ S\ S$	$\vert\vert\ S\ \vert\vert$
			$\vert\vert\vert$	$\vert\ S\ \vert$	$\vert\vert\vert\ S$	$\vert\ S\ \vert\ S$	$\vert\ S\ \vert\vert\vert$
				$S\ \vert\vert$	$S\ S\ \vert$	$S\ \vert\vert\ S$	$S\ \vert\vert\vert\vert$
				$\vert\vert\vert\vert$	$\vert\vert\ S\ \vert$	$\vert\vert\vert\vert\ S$	$\vert\vert\vert\vert\vert\vert$
					$\vert\ S\ \vert\vert$	$\vert\ S\ S\ \vert$	
					$S\ \vert\vert\vert$	$S\ \vert\ S\ \vert$	
					$\vert\vert\vert\vert\vert$	$\vert\vert\vert\ S\ \vert$	
Total	1	2	3	5	8	13	

Table 1: Mora and the Fibonacci Numbers

Ārchārya Hemachandra, a great Jain writer, mentions the idea of the number of variations of *mātrā-vrttas* in his writing *Chandonuśāsana* (c. 1150). Hemachandra lived at Ahnilvad Patan in Gujratm, where he was supported by two kings. He is known for contributing to Sanskrit and Prakrit literature and is claimed to have written six books. Three of these books are said to be in existence today. In *Chandonuśāsana,* his rule is translated and quoted from [9] as follows:

"Sum of the last and the last one but one numbers (of variations) is (the number of variations) of the *mātrā-vrttas* coming afterwards."

He continues,

"From amongst the numbers 1, 2, etc. those which are last and the last but one, are added (and) the sum, kept thereafter, gives the number of variations of the *mātrā-vrttas*. For example, the sum of 2 and 1, the last and the last but one, is 3 (which) is kept afterwards and is the number of variations (of metre) having 3 *mātrās*. The sum of 3 and 2 is 5 (which) is kept afterwards and is the number of variations (of the metre) having 4 *mātrās*...Thus: 1, 2, 3, 5, 8, 13, 21, 34 and so on, also."

Hence, the "so-called" Fibonacci numbers are born. The connection between these numbers and their relationship to Indian poetry continued to be examined even after Hemachandra's time. Figure 1 displays sketches of Indian mathematician *Ārchārya Hemachandra* and Italian mathematician, Fibonacci.

Figure 1: *Ārchārya Hemachandra* (left) and Fibonacci (right)
Sources: http://www.jainworld.com/literature/story28.htm (Hemachandra)
http://www.fibonacci.name/ (Fibonacci)

1.2 Fibonacci Numbers

The Fibonacci sequence originated with Fibonacci's famous "rabbit problem," which was discussed in his book *Liber Abaci*. The problem was stated in his book as follows:

"How pairs of rabbits will be produced each month, beginning with a single pair, if every month each 'productive' pair bears a new pair which becomes productive from the second month on?" [4]

The problem assumes that the rabbits are immortal and that the young rabbit pairs grow into adult rabbit pairs each month. Another important note is that the rabbits are not productive until they are two months of age. Table 2 summarizes the number of adult pairs, young pairs and the total number of rabbits each month.

Observe that each new adult entry is the previous adult entry plus the previous young adult entry. Each new young pair entry is the same as the previous adult pair entry, since adult pairs produce one young pair each month. The numbers in the total column are called the Fibonacci numbers and when continued forever, produce what is known as the Fibonacci sequence.

The Fibonacci sequence is an example of a recursive sequence, which is defined as a sequence in which "every term can be represented as a linear combination of preceding terms." [1] According to [1], the first known recursive sequence is in fact the Fibonacci sequence. It is interesting to note that Fibonacci did not derive an explicit formula for his special sequence, but it is speculated that the recursive behavior of the sequence was known to him. In 1634, mathematician Albert Girard wrote the formula for the Fibonacci sequence in his work *L'Arithmetique de Simon Stevin de Bruges.* Eduoard Lucas, a French number theorist, rediscovered the sequence in the late nineteenth century and is attributed to naming it the "Fibonacci sequence."

The Fibonacci sequence begins with two seeds: $f_0 = 0$ and $f_1 = 1$. The Fibonacci numbers are obtained by the sequence,

$$f_n = f_{n-2} + f_{n-1} \text{ for } n \geq 2. \tag{2}$$

Using (2), we obtain the following sequence of numbers:

$f_0 = 0,\ f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ f_6 = 8,\ f_7 = 13,\ f_8 = 21, ...$

Growth of Rabbit Colony			
Months	Adult Pairs	Young Pairs	Total
1	1	1	2
2	2	1	3
3	3	2	5
4	5	3	8
5	8	5	13
6	13	8	21
7	21	13	34
8	34	21	55
9	55	34	89
10	89	55	144
11	144	89	233
12	233	144	377

Table 2: Growth of Rabbit Colony [2]

The Fibonacci numbers have a multitude of interesting properties. Most notably, the Fibonacci numbers hold a special relationship with the golden ratio, an irrational number defined as $\frac{1+\sqrt{5}}{2}$. This number has been studied for centuries and will be discussed in detail in Section 2.5: More About the Golden Ratio. The Fibonacci numbers also hold many fascinating number theoretical properties. Looking at (2), one can observe the following divisibility properties of f_n : Every third Fibonacci number is divisible by $f_3 = 2$, every fourth Fibonacci number is divisible by $f_4 = 3$, every fifth Fibonacci number is divisible by $f_5 = 5$ and the pattern continues. Theorem 1 is a generalization of this idea. To prove Theorem 1, we need a Fibonacci Identity, given in Identity 1. We prove Identity 1 using matrix theory in Section 2.2.

Identity 1 $f_{m+n} = f_m f_{n+1} + f_{m-1} f_n$ for $m \geq 1$ and $n \geq 0$.

Theorem 1 $f_m \mid f_{mn}$ for $m, n \in \mathbb{N}$.

Proof of Theorem 1:
When $n = 1$, the statement is clearly true.
Assume $f_m \mid f_{mn}$ holds for all $k \leq n$.
We want to show $f_m \mid f_{m(n+1)}$.
Using Identity 1, we have $f_{m(n+1)} = f_{mn+m} = f_{mn} f_{m+1} + f_{mn-1} f_m$.
Clearly, $f_m \mid f_{mn-1} f_m$ and by the induction hypothesis, $f_m \mid f_{mn}$.
Therefore, $f_m \mid f_{m(n+1)}$, so the result holds for $n + 1$.
Therefore, by the Principle of Mathematical Induction, Theorem 1 holds for all $n \in \mathbb{N}$. ■

An examination of the Fibonacci numbers will also show that "the units digit of the Fibonacci numbers is cyclic with a periodicity of 60." [3] This means that the units digit in f_0 is the same as the units digit of f_{60}, f_{120}, f_{180}, etc. The same result is true for f_1, f_{61}, f_{121}, etc. The Lucas numbers, which will be discussed at length in Section 2.3 Lucas Numbers, share a similar property. In particular, the period of the Lucas numbers is 12. Table 3 illustrates this idea for Fibonacci numbers and we generalize this result with proof below.

n	f_{60n}	Units digit is 0:
0	f_0	**0**
1	f_{60}	154800875592**0**
2	f_{120}	535835925499096664087184**0**
3	f_{180}	1854770768947198621219013852139970776**0**
n	f_{60n+1}	Units digit is 1:
0	f_1	**1**
1	f_{61}	250473078196**1**
2	f_{121}	867000739850794865805192**1**
3	f_{181}	3001082145496345390753066714782948988**1**

Table 3: Periodicity of the Units Digit of the Fibonacci Numbers

We now prove a more general result.

Theorem 2 $f_{60n+i} \equiv f_i \pmod{10}$, where $i, n \geq 0$ are integers.

Proof of Theorem 2:
When $n = 0$: $f_i \equiv f_i \pmod{10}$ is clearly true.
Assume the result is true for all integers $k \leq n$.
Hence, $f_{60n+i} \equiv f_i \pmod{10}$.
Observe from Table 2 that $f_{60} \equiv 0 \pmod{10}$.
Also, since $f_{59} = 956,722,026,041$ we have $f_{59} \equiv 1 \pmod{10}$.
Then, $f_{60(n+1)+i} = f_{(60n+i)+60}$.
Replace $n = k - m$ in Identity 1 to obtain $f_k = f_m f_{k-m+1} + f_{m-1} f_{k-m}$.
Let $k = 60n + i + 60$ and $m = 60$.
Then, $f_{(60n+i)+60} = f_{60} f_{60n+i+1} + f_{59} f_{60n+i}$.
Using the fact that $f_{60} \equiv 0 \pmod{10}$ and $f_{59} \equiv 1 \pmod{10}$, we have that
$f_{60} f_{60n+i+1} + f_{59} f_{60n+i} \equiv 0 \cdot f_{60n+i+1} + 1 \cdot f_i \pmod{10}$.
Therefore, $f_{60(n+1)+i} \equiv f_i \pmod{10}$.
Theorem 2 holds for $n + 1$.
Therefore, by the Principle of Mathematical Induction, Theorem 2 holds for all $n \geq 0$. ■

An interesting unsolved problem about Fibonacci numbers involves prime Fibonacci numbers. It is unknown if there exist infinitely many prime Fibonacci numbers; however, research gives some insight into this problem. Vajda shows that the only nonprime n value for which f_n is prime is $n = 4$. According to [3], "For f_n $(n \neq 4)$ to be a prime it is necessary for n to be a prime." This does not mean, however, that *every* f_n such that n is prime is a prime number. For instance, $f_{19} = 4181 = 37 \cdot 113$ shows that $n = 19$ is prime but f_{19} is not. However, this statement shows that when examining prime Fibonacci numbers, we only need to consider f_n such that n is prime. Another interesting fact about Fibonacci numbers is that they can be used to prove there exist infinitely many prime numbers. We give a nontraditional proof of this fact below. First, we need a theorem.

Theorem 3 If $\gcd(m, n) = 1$, then $\gcd(f_m, f_n) = 1$.

See Chapter 16 of [6] for a multitude of other divisibility properties and for proof of this theorem.

Theorem 4 There exist infinitely many prime numbers.

Proof of Theorem 4:

Suppose there exist a finite number of primes.

Call these primes: $p_1, ..., p_k$.

Now consider the corresponding Fibonacci numbers: $f_{p_1}, ..., f_{p_k}$.

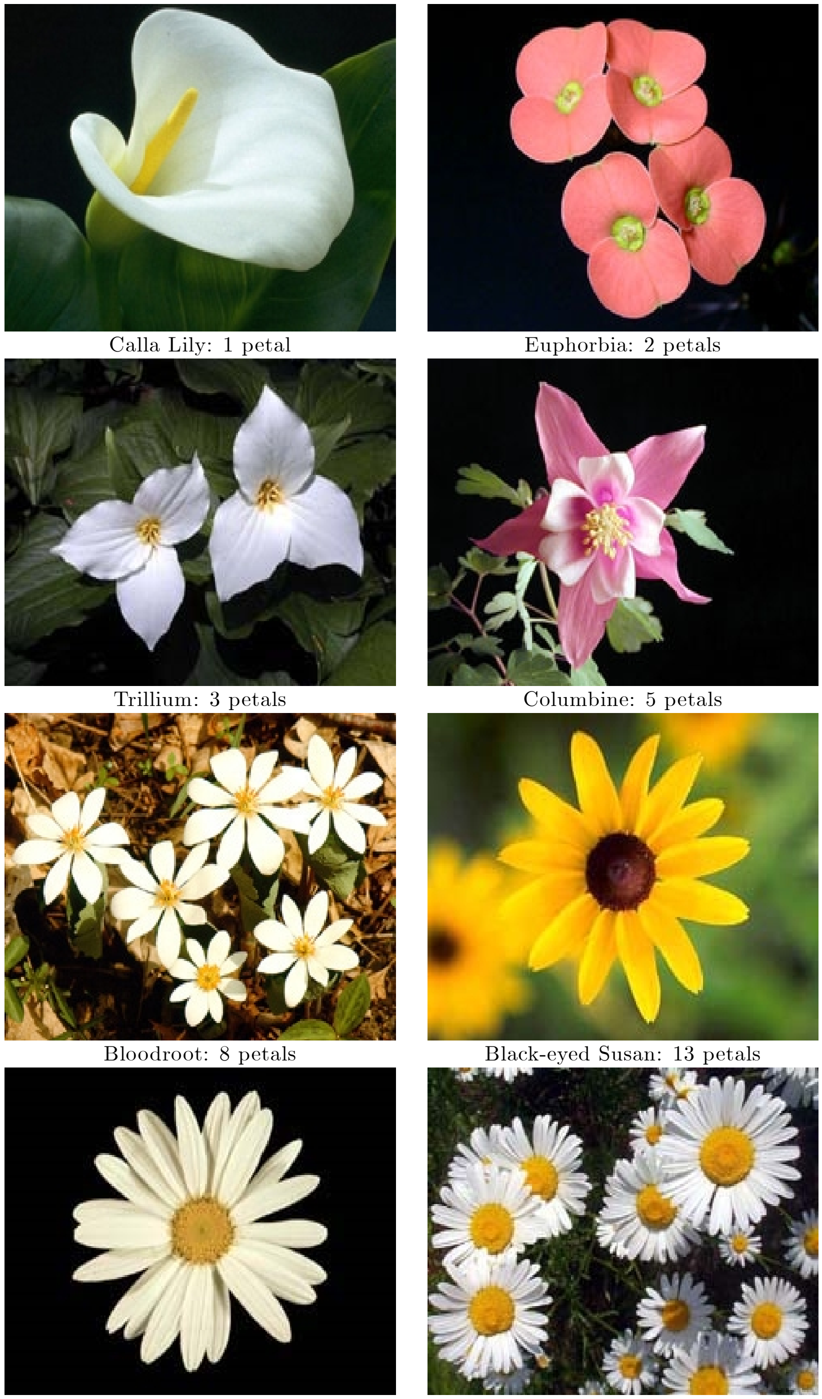
By Theorem 3, $\gcd(f_{p_i}, f_{p_j}) = 1$ for $i \neq j$.

Since there are only k primes, each f_{p_i} has exactly one prime factor, that is, each is a prime.

However, this is a contradiction since $f_{19} = 4181 = 37 \cdot 113$.

Therefore, there exist infinitely many prime numbers. $\blacksquare$

In addition, the Fibonacci numbers make appearances in nature and have applications in other fields besides mathematics. In nature, pinecones and sunflowers have growth patterns that are Fibonacci numbers. The seeds of sunflowers occur in spirals, one set of spirals going clockwise and one set going counterclockwise. Studies show that the number of spirals of most sunflowers occur as consecutive Fibonacci numbers. The most common number of spirals are 34 in one direction and 55 in the other. Consecutive Fibonacci numbers also appear as the number of spirals formed by the scales of pinecones. The Fibonacci numbers are also a driving force behind the genealogy of bees, flower petal patterns and the number of branches in certain trees. Figure 2 has a collection of flowers whose petal amounts occur as Fibonacci numbers.



Calla Lily: 1 petal

Euphorbia: 2 petals

Trillium: 3 petals

Columbine: 5 petals

Bloodroot: 8 petals

Black-eyed Susan: 13 petals

Shasta Daisy: 21 petals

Field Daisies: 34 petals

Figure 2: Flower Petals and Fibonacci Numbers

Source: http://britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

Fibonacci numbers become useful in the fields of computer science, optics and music. In particular, the piano's keyboard illustrates a beautiful example of how

Fibonacci numbers and music are related. In music, an octave is an interval between two pitches, each of which is represented by the same musical note. The difference is that the frequency of the lower note is half that of the higher note. On the piano's keyboard, an octave consists of five black keys and eight white keys, totaling 13 keys. In addition, the black keys are divided into a group of two and a group of three keys. Figure 3 exhibits the characteristics of a piano's keyboard and its relation to the Fibonacci numbers.

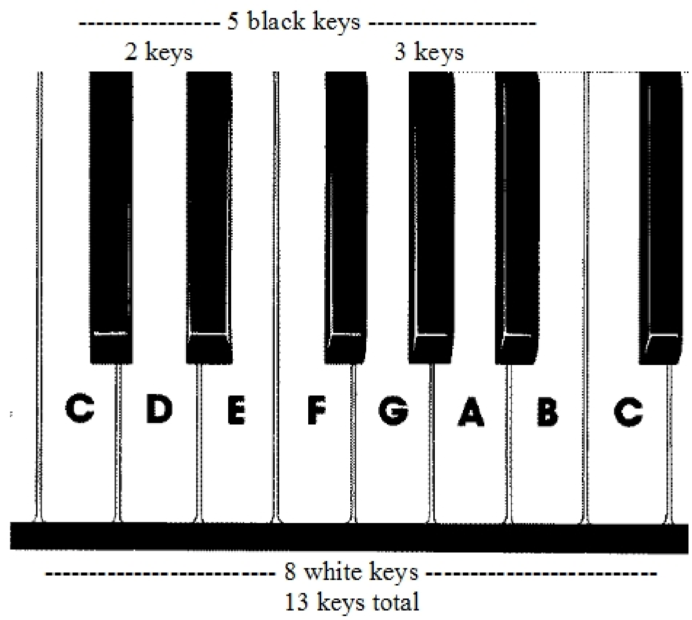


Figure 3: The Piano's Keyboard
Source: http://www.playpiano.com/playnotes.htm (piano image)

Numerous studies of Fibonacci numbers have been done since Fibonacci's time. The beauty of Fibonacci numbers can be admired by a wide variety of audiences, ranging from amateur to more experienced mathematicians. Other great mathematicians who have made significant progress in the study of Fibonacci numbers are German mathematical astronomer Johann Kepler, Jacques Binet, Gabriel Lame and Eugene Catalan.

2 Linear Algebra and Fibonacci Numbers

We now begin our study of Fibonacci numbers from a matrix theory point of view. We first examine the theory of diagonalizing a matrix, which plays a vital role during this approach. We then use this theory to derive Binet and Cassini-type formulas for the Fibonacci, Lucas, generalized Fibonacci and generalized weighted Fibonacci. We then extend these results to the Tribonacci and generalized Tribonacci numbers.

2.1 The Theory of Diagonalizing a Matrix

A common problem in linear algebra is the following: "For a square matrix A, does there exist an invertible matrix S such that $S^{-1}AS$ is diagonal?" [7] The simplicity of diagonal matrices and the useful properties they possess make them an interesting class of matrices to study. We begin the section with some definitions and then discuss restrictions on matrix A that guarantee it to be diagonalizable. We use [5] as a reference for the definitions and theorems.

Let M_n denote the set of all $n \times n$ matrices with entries in $\mathbb{C}$.

Definition A matrix $B \in M_n$ is said to be *similar* to a matrix $A \in M_n$ if there exists a nonsingular matrix $S \in M_n$ such that $B = S^{-1}AS$.

Definition If the matrix $A \in M_n$ is similar to a diagonal matrix, then A is said to be *diagonalizable.*

Definition The *Hermitian adjoint* A^* of $A \in M_n$ is defined by $A^* = \overline{A}^T$, where $\overline{A}$ is the component-wise conjugate.

Definition A matrix $U \in M_n$ is said to be *unitary* if $U^*U = I$, where I is the $n \times n$ identity matrix.

Definition The *standard inner product* of $x, y \in \mathbb{C}^n$ is given by $\langle x, y \rangle = y^*x$.

In fact, not all matrices are diagonalizable. Schur's Unitary Triangularization Theorem, however, gives us a triangular matrix representation for all $A \in M_n$.

Theorem (Schur) Given $A \in M_n$ with eigenvalues $\lambda_1, ..., \lambda_n$ in any prescribed order, there is a unitary matrix $U \in M_n$ such that $U^*AU = T = [t_{ij}]$ is upper triangular, with diagonal entries $t_{ii} = \lambda_i, 1 \leq i \leq n$. That is, every square matrix A is unitarily equivalent to a triangular matrix whose diagonal entries are the eigenvalues of A in any prescribed order. Furthermore, if $A \in M_n(\mathbb{R})$ and if all the eigenvalues of A are real, then U may be chosen to be real and orthogonal.

Proof of Schur's Theorem:

Let $\lambda_1, ..., \lambda_n$ be eigenvalues of the corresponding eigenvectors $y_1, ..., y_n$.
Let $x_1 = \frac{y_1}{\langle y_1, y_1 \rangle^{1/2}}$ be the normalized eigenvector corresponding to λ_1.
(i.e. $Ax_1 = \lambda_1 x_1$)

The nonzero vector x_1 may be extended to a basis $x_1, v_2, ..., v_n$ of $\mathbb{C}^n$.
We apply the Gram-Schmidt orthonormalization process to the basis.

Let $w_2 = v_2 - \langle v_2, x_1 \rangle x_1$ so that v_2 is orthogonal to x_1 and choose $z_2 = \frac{w_2}{\langle w_2, w_2 \rangle^{1/2}}$ so that z_2 is normalized and orthogonal to x_1.

We continue this process.
If we assume that $x_1, z_2, ..., z_{k-1}$ have been constructed,
we let $w_k = v_k - \langle v_k, z_{k-1} \rangle z_{k-1} - \langle v_k, z_{k-2} \rangle z_{k-2} - \cdots - \langle v_k, x_1 \rangle x_1$
be such that w_k is orthogonal to $x_1, z_2, ..., z_{k-1}$.
Then, normalize w_k to obtain $z_k = \frac{w_k}{\langle w_k, w_k \rangle^{1/2}}$.
We continue until we obtain the orthonormal basis $x_{1,} z_2, ..., z_n$.

Construct a unitary matrix $U_1 \in M_n$ by $U_1 = \begin{bmatrix} | & | & & | \\ x_1 & z_2 & \cdots & z_n \\ | & | & & | \end{bmatrix}_{n \times n}$.

$$U_1^* A U_1 = \begin{bmatrix} - \overline{x_1} - \\ - \overline{z_2} - \\ \vdots \\ - \overline{z_n} - \end{bmatrix} A \begin{bmatrix} | & | & & | \\ x_1 & z_2 & \cdots & z_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - \overline{x_1} - \\ - \overline{z_2} - \\ \vdots \\ - \overline{z_n} - \end{bmatrix} \begin{bmatrix} | & & & \\ \lambda_1 x_1 & * & \cdots & * \\ | & & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & * \\ 0 & A_1 \end{bmatrix},$$

where $A_1 \in M_{n-1}$ has eigenvalues $\lambda_2, ..., \lambda_n$.

Let x_2 be a normalized eigenvector corresponding to λ_2. Choose a basis containing x_2 and repeat the Gram-Schmidt orthonormalization process to get the orthonormal set $x_2, z_3', ..., z_n'$ which is a basis for A_1.

Construct a unitary matrix $U_2 \in M_{n-1}$ s.t. $U_2^* A_1 U_2 = \begin{bmatrix} \lambda_2 & * \\ 0 & A_2 \end{bmatrix}_{n-1 \times n-1}$.

Let $V_2 = \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix}_{n \times n}$.

Claim: The matrices V_2 and $U_1 V_2$ are unitary.

Proof of Claim:

$$V_2^* V_2 = \begin{bmatrix} 1 & 0 \\ 0 & U_2^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1 \times n-1} \end{bmatrix} = I_{n \times n}.$$

Therefore, V_2 is unitary.

$$(U_1 V_2)^* U_1 V_2 = V_2^* U_1^* U_1 V_2 = V_2^* I_{n \times n} V_2 = V_2^* V_2 = I_{n \times n}.$$

Therefore, U_1V_2 is unitary, so the Claim holds.

Then $(U_1V_2)^* AU_1V_2 = V_2^*U_1^*AU_1V_2 = V_2^* \begin{bmatrix} \lambda_1 & * \\ 0 & A_1 \end{bmatrix} V_2 = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & A_2 \end{bmatrix}$.

Continuing this process, we obtain unitary matrices $U_i \in M_{n-i+1}$, where $1 \leq i \leq n-1$ and unitary matrices $V_j \in M_n$, where $2 \leq j \leq n-1$.

Then $U^*AU = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & * & \vdots \\ \vdots & 0 & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$, where $U = U_1V_2V_3 \cdots V_{n-1}$ is the desired unitary matrix.

Note: Suppose all the eigenvalues of A are real. Then we can choose all corresponding eigenvectors to be real and the above calculations will yield a unitary matrix U that has real entries, which verifies the final statement in the theorem. ■

We can place certain restrictions on matrix A that will guarantee A to be diagonalizable. In particular, Theorem 5 illustrates the relationship between eigenvalues, linearly independent vectors and diagonalization.

Theorem 5 Let $A \in M_n$. Then A is diagonalizable if and only if there is a set of n linearly independent vectors, each of which is an eigenvector of A.

Proof of Theorem 5:
Assume A is diagonalizable.
Then there exists $S \in M_n$ such that S is invertible and $S^{-1}AS = D$, where D is a diagonal matrix.
Let the diagonal entries of D be $\lambda_1, ..., \lambda_n$, where $\lambda_i, 1 \leq i \leq n$, are not necessarily distinct.
Let $\mathbf{s}_1, ..., \mathbf{s}_n$ be the column vectors of S.

Then $SD = \begin{bmatrix} | & & | \\ \mathbf{s}_1 & \ldots & \mathbf{s}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$.

$SD = \begin{bmatrix} | & & | \\ \lambda_1\mathbf{s}_1 & \ldots & \lambda_n\mathbf{s}_n \\ | & & | \end{bmatrix}$.

We know that $S^{-1}AS = D$, so $AS = SD$.

Then, $\begin{bmatrix} | & & | \\ A\mathbf{s}_1 & \ldots & A\mathbf{s}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\mathbf{s}_1 & \ldots & \lambda_n\mathbf{s}_n \\ | & & | \end{bmatrix}$.

Therefore, $A\mathbf{s}_i = \lambda_i\mathbf{s}_i$ for each column vector $\mathbf{s}_i, 1 \leq i \leq n$.
We know that S is invertible, so it has linearly independent column vectors.
Therefore, A has n linearly independent eigenvectors.
Conversely, assume A has n linearly independent eigenvectors.
Denote these eigenvectors by $\mathbf{s}_1, ..., \mathbf{s}_n$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n$.
Since each $\mathbf{s}_i$ is an eigenvector of A, we have $A\mathbf{s}_i = \lambda_i\mathbf{s}_i$.

Then, $\begin{bmatrix} | & & | \\ A\mathbf{s}_1 & \ldots & A\mathbf{s}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\mathbf{s}_1 & \ldots & \lambda_n\mathbf{s}_n \\ | & & | \end{bmatrix}$.

$$AS = \begin{bmatrix} | & & | \\ \mathbf{s}_1 & \ldots & \mathbf{s}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} = SD.$$

Since $\mathbf{s}_i$ are linearly independent, S is invertible.
We have $S^{-1}AS = D$.
Therefore, A is diagonalizable. ■

Diagonalizing a matrix becomes a useful technique in later sections as we derive the Fibonacci numbers. In particular, as in the above notation, we denote an S matrix to be the "symmetrizer" matrix. This allows us to construct different representations of the same matrix, yielding important results.

2.2 Using Matrices to Derive the Fibonacci Numbers

Then,

This section describes the construction of the Fibonacci numbers using a 2×2 matrix calculation. We also derive the well-known Binet and Cassini formulas in this section.

Let $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and define $x_{n,n+1} = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ for $n \geq 0$.

Observe that,

$$Fx_{01} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = x_{12},$$

$$Fx_{12} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} = x_{23}, \text{ etc.}$$

We have $Fx_{n,n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix}$.

Note that,

$$F^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \text{ for } n \geq 1. \tag{3}$$

Proof of (3):
When $n = 1 : F = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Assume $F^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$ for all $k \leq n$.

Multiplying both sides by F, we obtain:

$F^{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+1} + f_n & f_n + f_{n-1} \\ f_{n+1} & f_n \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}$.

Therefore, (3) holds for $n + 1$. By the Principle of Mathematical Induction, (3) holds for all $n \geq 1$. ■

In general,

$$F^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix} \text{ for } n \geq 0. \tag{4}$$

Proof of (4):
Using (3), we have $F^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix}$. ■

We can now prove Identity 1.

Recall that Identity 1 states the following:
$f_{m+n} = f_m f_{n+1} + f_{m-1} f_n$ for $m \geq 1$ and $n \geq 0$.

Proof of Identity 1:

Using (3), we have that

$$F^{m+n} = \begin{bmatrix} f_{m+n+1} & f_{m+n} \\ f_{m+n} & f_{m+n-1} \end{bmatrix} = \begin{bmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{bmatrix} \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = F^m F^n.$$

We have, $\begin{bmatrix} f_{m+n+1} & f_{m+n} \\ f_{m+n} & f_{m+n-1} \end{bmatrix} = \begin{bmatrix} f_{m+1}f_{n+1} + f_m f_n & f_{m+1}f_n + f_m f_{n-1} \\ f_m f_{n+1} + f_{m-1}f_n & f_m f_n + f_{m-1}f_{n-1} \end{bmatrix}$.

Setting corresponding entries equal, we obtain $f_{m+n} = f_m f_{n+1} + f_{m-1}f_n$. ∎

We now illustrate the relationship between the Fibonacci numbers and the "golden ratio", which has many applications in the arts and sciences. The golden ratio, an irrational number, is defined as $\frac{1+\sqrt{5}}{2}$ and has been a curiosity for many throughout the centuries. Our matrix approach shows that the eigenvalues of F are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. We continue our discussion of the golden ratio in Section 2.5: More About the Golden Ratio.

$$\det(F - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0.$$

We have $\lambda = \frac{1\pm\sqrt{5}}{2}$, so the spectrum of F is $\sigma(F) = \left\{\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}\right\}$.

Table 4 contains a list of properties of λ_1 and λ_2 which are used throughout the thesis.

$\lambda_1 + \lambda_2 = 1$
$\lambda_1 - \lambda_2 = \sqrt{5}$
$\lambda_1\lambda_2 = -1$
$\lambda_1 + 1 = \lambda_1^2$
$\lambda_2 + 1 = \lambda_2^2$
$\frac{1}{\sqrt{5}}(\lambda_1 + 2) = \lambda_1$
$\frac{-1}{\sqrt{5}}(\lambda_2 + 2) = \lambda_2$
$2\lambda_1 - 1 = \sqrt{5}$
$1 - 2\lambda_2 = \sqrt{5}$
$\lambda_1^2 - \lambda_1 - 1 = 0$
$\lambda_2^2 - \lambda_2 - 1 = 0$

Table 4: Properties of λ_1 and λ_2

Next, we find eigenvectors for λ_1 and λ_2.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then, $x_1 + x_2 = \lambda_1 x_1$.
$x_1 = \lambda_1 x_2$

Let $x_2 = 1$.
Then $x_1 = \lambda_1$ and $\lambda_1 + 1 = \lambda_1^2$.
We have, $\lambda_1^2 - \lambda_1 - 1 = 0$ which is true from Table 4.

Thus, an eigenvector for λ_1 is $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$.

Similarly, we obtain $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ as an eigenvector for λ_2.

Next, to get another representation for the matrix F^n, we diagonalize F^n.

Let $S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$ be the symmetrizer matrix. Note that $\det(S) = \lambda_1 - \lambda_2 = \sqrt{5}$.

$S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$.

In general,

$$S^{-1} F^n S = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}. \tag{5}$$

Proof of (5):

Note that $S^{-1}F^nS = (S^{-1}FS)(S^{-1}FS) \cdots (S^{-1}FS)(S^{-1}FS) = (S^{-1}FS)^n$.

Therefore, $S^{-1}F^nS = (S^{-1}FS)^n = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$. ■

From (5), we have that $F^n = S \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} S^{-1}$.

Then, $F^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$.

We have,

$$F^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1\lambda_2(\lambda_2^n - \lambda_1^n) \\ \lambda_1^n - \lambda_2^n & \lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1}) \end{bmatrix}. \tag{6}$$

Matching entries of (6) and (3), which are both representations of F^n, we have the relations in $(a) - (d)$ below.

a.) $f_{n+1} = \frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})$

b.) $f_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$

c.) $f_n = \frac{1}{\sqrt{5}}\lambda_1\lambda_2(\lambda_2^n - \lambda_1^n) = -\frac{1}{\sqrt{5}}(\lambda_2^n - \lambda_1^n) = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$

d.) $f_{n-1} = \frac{1}{\sqrt{5}}\lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1}) = -\frac{1}{\sqrt{5}}(\lambda_2^{n-1} - \lambda_1^{n-1}) = \frac{1}{\sqrt{5}}(\lambda_1^{n-1} - \lambda_2^{n-1})$.

Note: We also use the property that $\lambda_1\lambda_2 = -1$ from Table 4 in the calculations above.

Therefore, $f_n = \frac{1}{\sqrt{5}}\,(\lambda_1^n - \lambda_2^n)$ for $n \geq 0$. Recall,

Replacing $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ in the above equation, we obtain Binet's Formula:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \text{ for } n \geq 0. \tag{7}$$

An interesting historical note is that French mathematician De Moivre knew of (7) in the year 1718. Jacques Philippe Marie Binet, another Frenchman, rediscovered it in 1843.

We can also derive another well-known formula by calculating the determinant of matrix F^n.

We have $\det(F^n) = f_{n+1}f_{n-1} - f_n^2 = [\det(F)]^n = (-1)^n$.

Therefore,

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n \text{ for } n \geq 1. \tag{8}$$

Equation (8) is known as Cassini's Formula, named after Italian mathematician Giovanni Domenico Cassini.

We verify that (2) and (7) give the same results up to $n = 300$ in Table 5.

<table>
<tr><th>n</th><th>$f_n = f_{n-2} + f_{n-1},\ \ n \geq 2$</th><th>$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], n \geq 0$</th></tr>
<tr><td colspan="3"></td></tr>
<tr><td>0</td><td>Define $f_0 = 0$</td><td>0</td></tr>
<tr><td>1</td><td>Define $f_1 = 1$</td><td>1</td></tr>
<tr><td>2</td><td colspan="2">1</td></tr>
<tr><td>3</td><td colspan="2">2</td></tr>
<tr><td>4</td><td colspan="2">3</td></tr>
<tr><td>5</td><td colspan="2">5</td></tr>
<tr><td>6</td><td colspan="2">8</td></tr>
<tr><td>7</td><td colspan="2">13</td></tr>
<tr><td>8</td><td colspan="2">21</td></tr>
<tr><td>9</td><td colspan="2">34</td></tr>
<tr><td>10</td><td colspan="2">55</td></tr>
<tr><td>20</td><td colspan="2">6765</td></tr>
<tr><td>30</td><td colspan="2">832040</td></tr>
<tr><td>40</td><td colspan="2">102334155</td></tr>
<tr><td>50</td><td colspan="2">12586269025</td></tr>
<tr><td>100</td><td colspan="2">354224848179261915075</td></tr>
<tr><td>200</td><td colspan="2">280571172992510140037611932413038677189525</td></tr>
<tr><td>300</td><td colspan="2">222232244629420445529739893461...</td></tr>
<tr><td></td><td colspan="2">...909967206666939096499764990979600</td></tr>
</table>

Table 5: Generating Fibonacci Numbers Using (2) and (7)

2.3 Lucas Numbers

The Fibonacci sequence is not the only sequence that arises from two starting seeds. Choosing different starting seeds will generate entirely different Fibonacci-like sequences. In fact, Edouard Lucas, a French mathematician, investigated a sequence that begins with the seeds $l_0 = 2$ and $l_1 = 1$.

The Lucas numbers are generated by the sequence,

$$l_n = l_{n-2} + l_{n-1} \text{ for } n \geq 2. \tag{9}$$

Using (9), we obtain the following sequence of numbers:

$l_2 = 3, l_3 = 4,\ l_4 = 7,\ l_5 = 11,\ l_6 = 18,\ l_7 = 29, ...$

As with the Fibonacci calculation, we use a 2×2 matrix approach to derive the Lucas numbers. The Lucas numbers and Fibonacci numbers are closely related and have many interesting properties. The identities of the Lucas numbers given in (10) – (12) are needed during the matrix calculation.

$$f_{n-1} + f_{n+1} = l_n \text{ for } n \geq 1. \tag{10}$$

Proof of (10):
When $n = 1 : f_0 + f_2 = 1 = l_1$.
Suppose (10) holds for all $k \leq n$.
Then, $l_{n+1} = l_n + l_{n-1} = f_{n-1} + f_{n+1} + f_{n-2} + f_n = f_n + f_{n+2}$.
Therefore, (10) holds for $n+1$. By the Principle of Mathematical Induction, (10) holds for all $n \geq 1$. $\blacksquare$

$$f_n + l_n = 2f_{n+1} \text{ for } n \geq 0. \tag{11}$$

Proof of (11):
Subtracting (10) – (2), we obtain: $l_n - f_{n+1} = f_{n+1} - f_n$.
Therefore, $f_n + l_n = 2f_{n+1}$. $\blacksquare$

$$l_{n+1} - f_{n+1} = 2f_n \text{ for } n \geq 0. \tag{12}$$

Proof of (12):
Using (11), we have $l_{n+1} - f_{n+1} = 2f_{n+2} - f_{n+1} - f_{n+1}$.
Then, $l_{n+1} - f_{n+1} = 2(f_{n+2} - f_{n+1})$.
Using (2), $l_{n+1} - f_{n+1} = 2(f_{n+1} + f_n - f_{n+1})$.
Therefore, $l_{n+1} - f_{n+1} = 2f_n$. $\blacksquare$

Let $L = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

$$LF^n = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+1} + 2f_n & f_n + 2f_{n-1} \\ 2f_{n+1} - f_n & 2f_n - f_{n-1} \end{bmatrix}$$

Using (11) and (12), we can write,

$$LF^n = \begin{bmatrix} l_{n+1} & l_n \\ l_n & l_{n-1} \end{bmatrix} \text{ for } n \geq 1. \tag{13}$$

Using the representation of F^n in (6) and properties in Table 4, we have

$$LF^n = \begin{bmatrix} \lambda_1^{n+1} + \lambda_2^{n+1} & \lambda_1^n + \lambda_2^n \\ \lambda_1^n + \lambda_2^n & \lambda_1^{n-1} + \lambda_2^{n-1} \end{bmatrix} \text{ for } n \geq 1. \tag{14}$$

Matching entries of (13) and (14), the Lucas numbers are given by the sequence,

$$l_n = \lambda_1^n + \lambda_2^n \text{ for } n \geq 0. \tag{15}$$

Observe that (15) is a Binet-type formula for the Lucas numbers.

We can also derive a Cassini-type formula for the Lucas numbers by computing the determinant of the matrix LF^n.

$\det(LF^n) = \det(L)\det(F^n)$

We have, $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{vmatrix} = \begin{vmatrix} l_{n+1} & l_n \\ l_n & l_{n-1} \end{vmatrix}$.

Therefore,

$$-5(-1)^n = l_{n+1}l_{n-1} - l_n^2 \text{ for } n \geq 1. \tag{16}$$

We verify that (9) and (15) give the same results up to $n = 300$ in Table 6.

n	$l_n = l_{n-2} + l_{n-1},\ n \geq 2$	$l_n = \lambda_1^n + \lambda_2^n, n \geq 0$
0	Define $l_0 = 2$	2
1	Define $l_1 = 1$	1
2	3	
3	4	
4	7	
5	11	
6	18	
7	29	
8	47	
9	76	
10	123	
20	15127	
30	1860498	
40	228826127	
50	28143753123	
100	792070839848372253127	
200	627376215338105766356982006981782561278127	
300	4969264057837466763937914368824...	
	...6823089806748952203469952020000 2	

Table 6: Generating Lucas Numbers Using (9) and (15)

2.4 Generalized Fibonacci Numbers

We now generalize to an additive sequence which begins with any two starting seeds, g_0 and g_1. The generalized Fibonacci sequence is given by,

$$g_n = g_{n-2} + g_{n-1} \text{ for } n \geq 2. \tag{17}$$

We derive a formula for the generalized Fibonacci numbers using 2×2 matrices, similar to the approach used in Sections 2.2 and 2.3.

Let $G = \begin{bmatrix} g_1 & g_0 \\ g_0 & g_1 - g_0 \end{bmatrix}$.

Observe that,

$$GF^n = \begin{bmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{bmatrix} \text{ for } n \geq 1. \tag{18}$$

Proof of (18):

When $n = 1$: $GF = \begin{bmatrix} g_1 & g_0 \\ g_0 & g_1 - g_0 \end{bmatrix} \begin{bmatrix} 1\,1 \\ 1\,0 \end{bmatrix} = \begin{bmatrix} g_2\, g_1 \\ g_1\, g_0 \end{bmatrix}$.

Suppose (18) holds for all $k \leq n$.

Multiplying GF^n on the right by F, we obtain:

$GF^{n+1} = \begin{bmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{bmatrix} \begin{bmatrix} 1\,1 \\ 1\,0 \end{bmatrix} = \begin{bmatrix} g_{n+2}\, g_{n+1} \\ g_{n+1} \quad g_n \end{bmatrix}$.

Therefore, (18) holds for $n+1$. By the Principle of Mathematical Induction, (18) holds for all $n \geq 1$. ■

Now, use the representation of F^n in (5) to calculate GF^n.

$GF^n = \frac{1}{\sqrt{5}} \begin{bmatrix} g_1 & g_0 \\ g_0 & g_1 - g_0 \end{bmatrix} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1\lambda_2(\lambda_2^n - \lambda_1^n) \\ \lambda_1^n - \lambda_2^n & \lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1}) \end{bmatrix}$

Matching entries of the above matrix and representation of GF^n in (18), we obtain the relations in $(a) - (d)$ below.

a.) $g_n = \frac{1}{\sqrt{5}}[g_0\left(\lambda_1^{n+1} - \lambda_2^{n+1}\right) + (g_1 - g_0)\left(\lambda_1^n - \lambda_2^n\right)]$
$g_n = \frac{1}{\sqrt{5}}\left[\lambda_1^n[g_0\lambda_1 + (g_1 - g_0)] + \lambda_2^n[(g_0 - g_1) - g_0\lambda_2]\right]$
Observe that $g_0\lambda_1 + (g_1 - g_0) = g_1 - g_0\lambda_2$
and $(g_0 - g_1) - g_0\lambda_2 = g_0\lambda_1 - g_1$.

Then $g_n = \frac{1}{\sqrt{5}}[(g_1 - g_0\lambda_2)\lambda_1^n + (g_0\lambda_1 - g_1)\,\lambda_2^n]$.

b.) $g_{n-1} = \frac{1}{\sqrt{5}}\left[g_0\lambda_1\lambda_2(\lambda_2^n - \lambda_1^n) + (g_1 - g_0)\,\lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1})\right]$
$g_{n-1} = \frac{1}{\sqrt{5}}\left[g_0\lambda_1^n - g_0\lambda_2^n + (g_0 - g_1)(\lambda_2^{n-1} - \lambda_1^{n-1})\right]$
$g_{n-1} = \frac{1}{\sqrt{5}}[\lambda_1^{n-1}[g_0\lambda_1 + (g_1 - g_0)] + \lambda_2^{n-1}[(g_0 - g_1) - g_0\lambda_2]]$

$g_{n-1} = \frac{1}{\sqrt{5}}[(g_1 - g_0\lambda_2)\lambda_1^{n-1} + (g_0\lambda_1 - g_1)\,\lambda_2^{n-1}]$.

c.) $g_n = \frac{1}{\sqrt{5}}[g_1\lambda_1\lambda_2(\lambda_2^n - \lambda_1^n) + g_0\lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1})]$

$g_n = \frac{1}{\sqrt{5}}[g_1\lambda_1^n - g_1\lambda_2^n + g_0\lambda_1^{n-1} - g_0\lambda_2^{n-1}]$

$g_n = \frac{1}{\sqrt{5}}[\lambda_1^{n-1}(g_1\lambda_1 + g_0) - \lambda_2^{n-1}(g_1\lambda_2 + g_0)]$

Using the fact that $\lambda_1\lambda_2 = -1$, it can be shown that

$g_1\lambda_1 + g_0 = (g_1 - g_0\lambda_2)\,\lambda_1$ and $-(g_1\lambda_2 + g_0) = (g_0\lambda_1 - g_1)\,\lambda_2$.

This allows us to rewrite $g_n = \frac{1}{\sqrt{5}}[(g_1 - g_0\lambda_2)\lambda_1^n + (g_0\lambda_1 - g_1)\,\lambda_2^n]$.

d.) $g_{n+1} = \frac{1}{\sqrt{5}}[g_1(\lambda_1^{n+1} - \lambda_2^{n+1}) + g_0\,(\lambda_1^n - \lambda_2^n)]$

Using the same argument as in part (c), it can be shown that

$g_{n+1} = \frac{1}{\sqrt{5}}[(g_1 - g_0\lambda_2)\lambda_1^{n+1} + (g_0\lambda_1 - g_1)\,\lambda_2^{n+1}]$.

The sequence for generalized Fibonacci numbers is given by:

$$g_n = \frac{1}{\sqrt{5}}[(g_1 - g_0\lambda_2)\lambda_1^n + (g_0\lambda_1 - g_1)\,\lambda_2^n] \text{ for } n \geq 0. \tag{19}$$

We now find the Cassini-type formula for the generalized Fibonacci numbers by computing the determinant of the matrix GF^n.

$\det(GF^n) = \det(G)\det(F^n)$

This implies that $\begin{vmatrix} g_1 & g_0 \\ g_0 & g_1 - g_0 \end{vmatrix} \begin{vmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{vmatrix} = \begin{vmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{vmatrix}$.

Therefore,

$$(g_1^2 - g_0g_1 - g_0^2)(-1)^n = g_{n-1}g_{n+1} - g_n^2 \text{ for } n \geq 1. \tag{20}$$

Using the starting seeds $g_0 = -1$ and $g_1 = 3$, we compute the generalized Fibonacci numbers using both the recursive definition in (17) and the derived formula in (19). We verify that (17) and (19) give the same results up to $n = 300$ in Table 7.

n	$g_n = g_{n-2} + g_{n-1},\ \ n \geq 2$	$g_n = \frac{1}{\sqrt{5}}[(g_1 - g_0\lambda_2)\lambda_1^n + (g_0\lambda_1 - g_1)\,\lambda_2^n],\ n \geq 2$
0	Let $g_0 = -1$	
1	Let $g_1 = 3$	
2	2	
3	5	
4	7	
5	12	
6	19	
7	31	
8	50	
9	81	
10	131	
20	16114	
30	1981891	
40	243756479	
50	29980065026	
100	843751548703230576199	
200	668310997804732606953150759954744089524274	
300	529349653311098221157193908675...	
	...450769774300542076731827708328599	

Table 7: Generating Generalized Fibonacci Using (17) and (19)

2.5 More About the Golden Ratio

The golden ratio, given by $\lambda_1 = \frac{1+\sqrt{5}}{2}$, has been called many names throughout history. Some of these names include the golden number, golden proportion, golden mean, golden cut, golden section, divine proportion, the Fibonacci number and the mean of Phidias. Greek mathematicians defined it as the "division of a line in mean and extreme ratio." [4] Since ancient times, the golden ratio has been of interest to many people in varying disciplines. The interest in the golden ratio goes back to at least 2600 B.C. when the Egyptians were constructing the Great Pyramid. Theories suggest that the Egyptians were aware of the golden ratio and used it during the building of the Great Pyramid. Calculations show that the ratio between the base and hypotenuse of the right triangle inside the Great Pyramid is approximately 0.61762 which is very close to the reciprocal of the golden ratio. Thousands of years later in 1497, Italian mathematician Luca Pacioli wrote *De Divina Proportione*, which is thought to be the first book written about the golden ratio.

The golden ratio can be used to construct what are called "golden rectangles," which are considered the most aesthetically-pleasing rectangles. What makes these rectangles special is that the ratio of the length to the width is the golden ratio.

The golden ratio has also appeared in Greek sculptures, paintings and pottery and in ancient furniture and architectural design. Famous painters Georges Seurat and Leonardo da Vinci were known to include golden rectangles and golden ratios in their paintings. Architectural structures that used the golden rectangles include the *Parthenon* in Athens, Greece, and the *Cathedral of Chartes* and *Tower of Saint Jacques* in Paris, France. One can also construct "golden right angled triangles" which are right triangles that have λ_1 as the hypotenuse and $\sqrt{\lambda_1}$ and 1 as side lengths. Another fascinating fact is that certain proportions of human anatomy have shown evidence of the golden ratio and golden rectangle. Ancient Greeks noted that the human "head fits nicely into a golden rectangle." [6] In addition, the golden ratio appears in the human hand and human bones. Because of these special relationships, we sometimes refer to the golden ratio as "the number of our physical body." [6]

We have shown that the eigenvalues of matrix F are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. We have also shown that these values are used to derive the generalized Fibonacci numbers. The relationship between the generalized Fibonacci numbers and the golden ratio does not stop there. In fact, we will show that $\frac{g_{n+1}}{g_n} \to \lambda_1$ as $n \to \infty$.

Let $x_n = \frac{g_n}{g_{n+1}}$. Assume $g_0 \geq 0$, $g_1 > 0$. Then (x_n) is well-defined.

For $n \geq 1$, $x_n = \frac{g_n}{g_{n+1}} = \frac{g_n}{g_n+g_{n-1}} = \frac{1}{1+\frac{g_{n-1}}{g_n}} = \frac{1}{1+x_{n-1}}$.

Consider $x_n - x_{n+1} = \frac{1}{1+x_{n-1}} - \frac{1}{1+x_n}$.

Then, $x_n - x_{n+1} = \frac{x_n - x_{n-1}}{(1+x_{n-1})(1+x_n)}$.

Since $x_0 = \frac{g_0}{g_1} \geq 0$ and $x_1 = \frac{g_1}{g_2} = \frac{g_1}{g_0+g_1} = c > 0$, we show by induction that $x_n > 0$ for all $n \geq 1$.

Assume that $x_n > 0$ for all $k \leq n$.

Then, $x_{n+1} = \frac{1}{1+x_n} > 0$.

Therefore, $x_n > 0$ for all $n \geq 1$.

We conclude that, $(1 + x_{n-1})(1 + x_n) > 1 + c$ for $n \geq 2$.

Then, $|x_n - x_{n+1}| < \frac{|x_n - x_{n-1}|}{(1+c)}$ for all $n \geq 2$,

$|x_2 - x_1| = \left|\frac{g_2}{g_3} - \frac{g_1}{g_2}\right| = \left|\frac{g_0+g_1}{g_0+2g_1} - \frac{g_1}{g_0+g_1}\right| = \frac{\left|g_0^2+g_0g_1-g_1^2\right|}{(g_0+2g_1)(g_0+g_1)} = k > 0.$

We begin our iteration.

$|x_2 - x_3| < \frac{|x_2-x_1|}{(1+c)} = \frac{k}{(1+c)}$,

$|x_3 - x_4| < \frac{|x_2-x_3|}{(1+c)} < \frac{k}{(1+c)^2}$,

$\vdots$

Then, $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ for $n \geq 2$ which we verify by induction.

Assume $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ for all $k \leq n$.

Then, $|x_{n+1} - x_{n+2}| < \frac{|x_{n+1}-x_n|}{(1+c)} < \frac{k}{(1+c)^n}$.
Therefore, $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ for $n \geq 2$.
Finally, we estimate the difference between arbitrary terms x_m and x_n.
For $m < n$,
$|x_m - x_n| = |(x_m - x_{m+1}) + (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + (x_{n-1} - x_n)|$.
Then, $|x_m - x_n| \leq |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + \cdots + |x_{n-2} - x_{n-1}| + |x_{n-1} - x_n|$.
Using $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ we obtain,
$|x_m - x_n| \leq \frac{k}{(1+c)^{m-1}} + \frac{k}{(1+c)^m} + \cdots + \frac{k}{(1+c)^{n-3}} + \frac{k}{(1+c)^{n-2}}$,
$|x_m - x_n| < \frac{\frac{k}{(1+c)^{m-1}}}{\left[1-\frac{1}{(1+c)}\right]} = \frac{k}{c(1+c)^{m-2}}$.

If $\epsilon > 0$, choose N such that $\frac{k}{c(1+c)^{N-2}} < \epsilon$.
Then for all $n > m \geq N$, $|x_m - x_n| < \frac{k}{c(1+c)^{m-2}} \leq \frac{k}{c(1+c)^{N-2}} < \epsilon$.
Thus, (x_n) is Cauchy, so $\lim_{n\to\infty} x_n = G$ exists.
Considering $x_n = \frac{1}{1+x_{n-1}}$ and taking limits we obtain, $G = \frac{1}{1+G}$.
This yields $G^2 + G - 1 = 0$ whose solutions are $\frac{-1\pm\sqrt{5}}{2}$.
Since $(x_n) > 0$ for all $n \geq 1$, we have $G = \frac{-1+\sqrt{5}}{2} > 0$.
Therefore for $n \geq 1$, $\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \lim_{n\to\infty} \frac{1}{\frac{g_n}{g_{n+1}}} = \lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{G} = \frac{1+\sqrt{5}}{2} = \lambda_1$.
Therefore,

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \frac{1+\sqrt{5}}{2}. \tag{21}$$

In this last step we used the following Theorem:

Theorem 6 If (y_n) is a sequence of nonzero numbers converging to a nonzero limit y, then $\left(\frac{1}{y_n}\right)$ converges to $\frac{1}{y}$.

Proof of Theorem 6:
First, define $\alpha = \frac{1}{2}|y| > 0$.
Since $\lim(y_n) = y$, there exists $K_1 \in \mathbb{N}$ such that if $n \geq K_1$, then $|y_n - y| < \alpha$.
Write $|y| = |y_n - y - y_n| \leq |y_n - y| + |y_n|$.
It follows that $-\alpha \leq |y_n - y| \leq |y_n| - |y|$ for $n \geq K_1$.
Then, we have $\frac{1}{2}|y| = |y| - \alpha \leq |y_n|$ for $n \geq K_1$.
Then, $\frac{1}{|y_n|} \leq \frac{2}{|y|}$ for $n \geq K_1$.
We have the estimate, $\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y-y_n}{y_n y}\right| = \frac{1}{|y_n y|}|y - y_n| \leq \frac{2}{|y|^2}|y - y_n|$ for $n \geq K_1$.

Now, given $\epsilon > 0$, there exists $K_2 \in \mathbb{N}$ such that if $n \geq K_2$ then $|y - y_n| < \frac{1}{2}\epsilon |y|^2$.
Therefore, if $K = \sup\{K_1, K_2\}$, then $\left|\frac{1}{y_n} - \frac{1}{y}\right| < \epsilon$.

Since $\epsilon > 0$ is arbitrary, we have $\lim \left(\frac{1}{y_n}\right) = \frac{1}{y}$. ■

Recall (19) is given by,

$$g_n = \frac{1}{\sqrt{5}}[(g_1 - g_0\lambda_2)\lambda_1^n + (g_0\lambda_1 - g_1)\lambda_2^n] \text{ for } n \geq 0.$$

We now reprove (21) using (19).

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1^{n+1} + (g_0\lambda_1 - g_1)\lambda_2^{n+1}}{(g_1 - g_0\lambda_2)\lambda_1^n + (g_0\lambda_1 - g_1)\lambda_2^n}$$

$$= \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1 + (g_0\lambda_1 - g_1)\lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n}{(g_1 - g_0\lambda_2) + (g_0\lambda_1 - g_1)\left(\frac{\lambda_2}{\lambda_1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1 + (g_0\lambda_1 - g_1)\lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^n}{(g_1 - g_0\lambda_2) + (g_0\lambda_1 - g_1)\left(\frac{\lambda_2}{\lambda_1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1 + (-1)^n(g_0\lambda_1 - g_1)\lambda_2 \left(\frac{-1+\sqrt{5}}{1+\sqrt{5}}\right)^n}{(g_1 - g_0\lambda_2) + (-1)^n(g_0\lambda_1 - g_1)\left(\frac{-1+\sqrt{5}}{1+\sqrt{5}}\right)^n}$$

Let $a = -1 + \sqrt{5}$. Then $a + 2 = 1 + \sqrt{5}$ and $0 < \dfrac{a}{a+2} < 1$.

$$\text{Then, } \lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1 + (-1)^n(g_0\lambda_1 - g_1)\lambda_2 \left(\frac{a}{a+2}\right)^n}{(g_1 - g_0\lambda_2) + (-1)^n(g_0\lambda_1 - g_1)\left(\frac{a}{a+2}\right)^n},$$

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1 + (-1)^n(g_0\lambda_1 - g_1)\lambda_2 \left(\frac{a}{a+2}\right)^n}{(g_1 - g_0\lambda_2) + (-1)^n(g_0\lambda_1 - g_1)\left(\frac{a}{a+2}\right)^n}.$$

Using the known theorem: If $0 < b < 1$, then $\lim(b^n) = 0$, we have,

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1 + (-1)^n(g_0\lambda_1 - g_1)\lambda_2 \left(\frac{a}{a+2}\right)^n}{(g_1 - g_0\lambda_2) + (-1)^n(g_0\lambda_1 - g_1)\left(\frac{a}{a+2}\right)^n}.$$

$$\text{Therefore, } \lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \lim_{n\to\infty} \frac{(g_1 - g_0\lambda_2)\lambda_1}{(g_1 - g_0\lambda_2)} = \lambda_1, \text{ provided } g_1 \neq g_0\lambda_2.$$

Recall that we previously discussed the connection between the Fibonacci numbers and the piano's keyboard. The idea of a Fibonacci ratio can also be examined when looking at major sixths, which are six tones apart, and minor sixths, which are $5\frac{1}{2}$ tones apart. The interval from the notes C to A is a major sixth. The note C makes 264 vibrations per second and the note A makes 440 vibrations per second. Note that $\frac{264}{440} = \frac{3}{5}$, which is a Fibonacci ratio. In particular, this is the reciprocal of $\frac{f_{n+1}}{f_n}$ for $n = 4$. Also, the interval from E and C is a minor sixth. The ratio of vibrations per second for each of these notes is $\frac{330}{528} = \frac{5}{8}$, which is the reciprocal of $\frac{f_{n+1}}{f_n}$ for $n = 5$. This shows that the golden ratio and Fibonacci ratios play an important role in many areas and make appearances in some of the most unusual places.

2.6 Generalized Weighted Fibonacci Numbers

We describe the generalized weighted Fibonacci numbers and use a 2×2 matrix calculation to derive Binet and Cassini-type formulas. Begin with seeds g_0, g_1 and weights a_0, a_1. Then the generalized weighted Fibonacci is constructed as follows:

$w_0 = g_0$
$w_1 = g_1$
$w_2 = a_0g_0 + a_1g_1$
$w_3 = a_0g_1 + a_1(a_0g_0 + a_1g_1) = a_0w_1 + a_1w_2$
$w_4 = a_0(a_0g_0 + a_1g_1) + a_1[a_0g_1 + a_1(a_0g_0 + a_1g_1)] = a_0w_2 + a_1w_3$
$\vdots$

The generalized weighted Fibonacci sequence is defined as,

$$w_n = a_0w_{n-2} + a_1w_{n-1} \text{ for } n \geq 2. \tag{22}$$

Let $W = \begin{bmatrix} a_1 & a_0 \\ 1 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} a_0g_0 + a_1g_1 & g_1 \\ g_1 & g_0 \end{bmatrix}$.

Also, define $x_{n,n+1} = \begin{bmatrix} w_{n+1} \\ w_n \end{bmatrix}$ for $n \geq 0$.

Observe that,

$$W^n \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} = \begin{bmatrix} w_{n+1} \\ w_n \end{bmatrix} \text{ for } n \geq 1. \tag{23}$$

We also have,

$$W^n G = \begin{bmatrix} w_{n+2} & w_{n+1} \\ w_{n+1} & w_n \end{bmatrix} \text{ for } n \geq 0. \tag{24}$$

Proof of (24):

When $n = 1 : WG = \begin{bmatrix} a_1 & a_0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 g_0 + a_1 g_1 & g_1 \\ g_1 & g_0 \end{bmatrix}$

$$WG = \begin{bmatrix} a_0 g_1 + a_1(a_0 g_0 + a_1 g_1) & a_0 g_0 + a_1 g_1 \\ a_0 g_0 + a_1 g_1 & g_1 \end{bmatrix}$$

Then $WG = \begin{bmatrix} w_3 & w_2 \\ w_2 & w_1 \end{bmatrix}$, which is true.

Suppose (24) holds for all $k \leq n$.

Then $W^{n+1} G = \begin{bmatrix} a_1 & a_0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_{n+2} & w_{n+1} \\ w_{n+1} & w_n \end{bmatrix}$

$$W^{n+1} G = \begin{bmatrix} a_0 w_{n+1} + a_1 w_{n+2} & a_0 w_n + a_1 w_{n+1} \\ w_{n+2} & w_{n+1} \end{bmatrix}$$

$$W^{n+1} G = \begin{bmatrix} w_{n+3} & w_{n+2} \\ w_{n+2} & w_{n+1} \end{bmatrix}$$

Therefore, (24) holds for $n+1$. By the Principle of Mathematical Induction, (24) holds for all $n \geq 1$. ■

We now diagonalize W^n and follow the approach taken in the previous sections.

$$\det(W - \tau I) = \begin{vmatrix} a_1 - \tau & a_0 \\ 1 & -\tau \end{vmatrix} = \tau^2 - a_1 \tau - a_0 = 0.$$

We have $\tau = \frac{a_1 \pm \sqrt{a_1^2 + 4a_0}}{2}$, so $\sigma(W) = \left\{ \tau_1 = \frac{a_1 + \sqrt{a_1^2 + 4a_0}}{2}, \tau_2 = \frac{a_1 - \sqrt{a_1^2 + 4a_0}}{2} \right\}$.

Table 8 gives some properties of the eigenvalues τ_1 and τ_2.

$\tau_1 + \tau_2 = a_1$
$\tau_1 \tau_2 = -a_0$
$\tau_1 - \tau_2 = \sqrt{a_1^2 + 4a_0}$

Table 8: Properties of τ_1 and τ_2

We use the symmetrizer matrix $S = \begin{bmatrix} \tau_1 & \tau_2 \\ 1 & 1 \end{bmatrix}$ and $S^{-1} = \frac{1}{\sqrt{a_1^2+4a_0}} \begin{bmatrix} 1 & -\tau_2 \\ -1 & \tau_1 \end{bmatrix}$ to diagonalize W^n.

Since $S^{-1}WS = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}$, we have $S^{-1}W^nS = \begin{bmatrix} \tau_1^n & 0 \\ 0 & \tau_2^n \end{bmatrix}$.

Then, $W^n = \frac{1}{\sqrt{a_1^2+4a_0}} \begin{bmatrix} \tau_1 & \tau_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tau_1^n & 0 \\ 0 & \tau_2^n \end{bmatrix} \begin{bmatrix} 1 & -\tau_2 \\ -1 & \tau_1 \end{bmatrix}$.

$$W^n = \frac{1}{\sqrt{a_1^2+4a_0}} \begin{bmatrix} \tau_1^{n+1} & \tau_2^{n+1} \\ \tau_1^n & \tau_2^n \end{bmatrix} \begin{bmatrix} 1 & -\tau_2 \\ -1 & \tau_1 \end{bmatrix}$$

$$W^n = \frac{1}{\sqrt{a_1^2+4a_0}} \begin{bmatrix} \tau_1^{n+1} - \tau_2^{n+1} & \tau_1\tau_2^{n+1} - \tau_1^{n+1}\tau_2 \\ \tau_1^n - \tau_2^n & \tau_1\tau_2^n - \tau_1^n\tau_2 \end{bmatrix}.$$

Then for $n \geq 0$,

$$W^nG = \frac{1}{\sqrt{a_1^2+4a_0}} \begin{bmatrix} \tau_1^{n+1} - \tau_2^{n+1} & \tau_1\tau_2^{n+1} - \tau_1^{n+1}\tau_2 \\ \tau_1^n - \tau_2^n & \tau_1\tau_2^n - \tau_1^n\tau_2 \end{bmatrix} \begin{bmatrix} a_0g_0 + a_1g_1 & g_1 \\ g_1 & g_0 \end{bmatrix}. \tag{25}$$

We now match entries of (24) and (25) to obtain the relations in $(a) - (d)$ below.

a.) $w_n = \frac{1}{\sqrt{a_1^2+4a_0}}[g_0(\tau_1\tau_2^n - \tau_1^n\tau_2) + g_1(\tau_1^n - \tau_2^n)]$

$w_n = \frac{1}{\sqrt{a_1^2+4a_0}}[g_0\tau_1\tau_2^n - g_0\tau_1^n\tau_2 + g_1\tau_1^n - g_1\tau_2^n]$

$w_n = \frac{1}{\sqrt{a_1^2+4a_0}}[(g_1 - g_0\tau_2)\tau_1^n + (g_0\tau_1 - g_1)\tau_2^n]$

b.) $w_{n+1} = \frac{1}{\sqrt{a_1^2+4a_0}}[g_0(\tau_1\tau_2^{n+1} - \tau_1^{n+1}\tau_2) + g_1(\tau_1^{n+1} - \tau_2^{n+1})]$

$w_{n+1} = \frac{1}{\sqrt{a_1^2+4a_0}}[(g_1 - g_0\tau_2)\tau_1^{n+1} + (g_0\tau_1 - g_1)\tau_2^{n+1}]$

c.) $w_{n+1} = \frac{1}{\sqrt{a_1^2+4a_0}}[(a_0g_0 + a_1g_1)(\tau_1^n - \tau_2^n) + g_1(\tau_1\tau_2^n - \tau_1^n\tau_2)]$

$w_{n+1} = \frac{1}{\sqrt{a_1^2+4a_0}}[a_0g_0\tau_1^n - a_0g_0\tau_2^n + a_1g_1\tau_1^n - a_1g_1\tau_2^n + g_1\tau_1\tau_2^n - g_1\tau_1^n\tau_2]$

$w_{n+1} = \frac{1}{\sqrt{a_1^2+4a_0}}[(a_0g_0 + a_1g_1 - g_1\tau_2)\tau_1^n + (g_1\tau_1 - a_0g_0 - a_1g_1)\tau_2^n]$

It can be shown that $a_0g_0 + a_1g_1 - g_1\tau_2 = (g_1 - g_0\tau_2)\tau_1$

and $g_1\tau_1 - a_0g_0 - a_1g_1 = (g_0\tau_1 - g_1)\tau_2$.

Then we can rewrite $w_{n+1} = \frac{1}{\sqrt{a_1^2+4a_0}}[(g_1 - g_0\tau_2)\tau_1^{n+1} + (g_0\tau_1 - g_1)\tau_2^{n+1}]$.

d.) $w_{n+2} = \frac{1}{\sqrt{a_1^2+4a_0}}[(a_0g_0 + a_1g_1)(\tau_1^{n+1} - \tau_2^{n+1}) + g_1(\tau_1\tau_2^{n+1} - \tau_1^{n+1}\tau_2)]$

Using the same relationships as in part (c), we can show that

$w_{n+2} = \frac{1}{\sqrt{a_1^2+4a_0}}[(g_1 - g_0\tau_2)\tau_1^{n+2} + (g_0\tau_1 - g_1)\tau_2^{n+2}]$.

The formula for the generalized weighted Fibonacci is given by,

$$w_n = \frac{1}{\sqrt{a_1^2 + 4a_0}}[(g_1 - g_0\tau_2)\tau_1^n + (g_0\tau_1 - g_1)\tau_2^n] \text{ for } n \geq 2. \tag{26}$$

Computing the determinant of W^nG, we find a Cassini-type formula for the generalized weighted Fibonacci.

$\det(W)^n \det(G) = \det(W^nG)$

We have $\begin{vmatrix} a_1 & a_0 \\ 1 & 0 \end{vmatrix}^n \begin{vmatrix} a_0g_0 + a_1g_1 & g_1 \\ g_1 & g_0 \end{vmatrix} = \begin{vmatrix} w_{n+2} & w_{n+1} \\ w_{n+1} & w_n \end{vmatrix}$.

Therefore,

$$(-a_0)^n[a_0g_0^2 + a_1g_0g_1 - g_1^2] = w_nw_{n+2} - w_{n+1}^2 \text{ for } n \geq 0. \tag{27}$$

To illustrate the validity of the derived formula for the generalized weighted Fibonacci in (26), we look at an example. We use the weights, $a_0 = 2$ and $a_1 = 3$, and the starting seeds, $w_0 = 2$ and $w_1 = 4$. Then, for this example we have $\tau_1 = \frac{3+\sqrt{17}}{2}$ and $\tau_2 = \frac{3-\sqrt{17}}{2}$. We verify that (22) and (26) give the same results up to $n = 200$ in Table 9.

n	$w_n = a_0 w_{n-2} + a_1 w_{n-1}$	$w_n = \frac{1}{\sqrt{a_1^2+4a_0}}[(g_1 - g_0\tau_2)\tau_1^n + (g_0\tau_1 - g_1)\tau_2^n], n \geq 2$
	Let $a_0 = 2$ and $a_1 = 3$	
0	Let $w_0 = 2$	
1	Let $w_1 = 4$	
2	16	
3	56	
4	200	
5	712	
6	2536	
7	9032	
8	32168	
9	114568	
10	408040	
20	133997477672	
30	44003833267308136	
40	14450550680937703569320	
50	4745459644705250022516171496	
100	18123735678699424468880477782563575783512981141692125480	
200	264354428422732511704387576225179757004...	
	...837752308984544225096629578073572658030565...	
	...018049633198075670267804834728	

Table 9: Generating Generalized Weighted Fibonacci Using (22) and (26)

2.7 Tribonacci Numbers

A sequence related to the Fibonacci sequence can be constructed by beginning with three seeds instead of two. This sequence derives the Tribonacci numbers and is defined in the following manner. Begin with three seeds: $a_0 = 0,\ a_1 = 0$ and $a_2 = 1$.

The Tribonacci sequence is defined by,

$$a_n = a_{n-3} + a_{n-2} + a_{n-1} \text{ for } n \geq 3. \tag{28}$$

We obtain the following sequence of numbers:

$$a_0 = 0,\ a_1 = 0,\ a_2 = 1,\ a_3 = 1,\ a_4 = 2,\ a_5 = 4,\ a_6 = 7,\ a_7 = 13,\ a_8 = 24, ...$$

According to [6], this sequence was first studied in 1963 by M. Feinberg while he was a freshman at Susquehanna Township Junior High School, located in Pennsylvania.

We now derive the Tribonacci numbers using a 3×3 matrix calculation.

Let $A = \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,1 \end{bmatrix}$ and define $x_{n,n+1,n+2} = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$ for $n \geq 0$.

Observe that,

$$Ax_{012} = \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

$$Ax_{123} = \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_3 \\ a_4 \end{bmatrix}, \text{etc.}$$

We have $Ax_{n,n+1,n+2} = \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{bmatrix}.$

Note that,

$$A^n = \begin{bmatrix} a_{n-1} & a_{n-2} + a_{n-1} & a_n \\ a_n & a_{n-1} + a_n & a_{n+1} \\ a_{n+1} & a_n + a_{n+1} & a_{n+2} \end{bmatrix} \text{ for } n \geq 2. \tag{29}$$

Proof of (29):

When $n = 2 : A^2 = \begin{bmatrix} 0\,0\,1 \\ 1\,1\,1 \\ 1\,2\,2 \end{bmatrix} = \begin{bmatrix} a_1 & a_0 + a_1 & a_2 \\ a_2 & a_1 + a_2 & a_3 \\ a_3 & a_2 + a_3 & a_4 \end{bmatrix}.$

Suppose (29) holds for all $k \leq n$.

Multiplying by A, we have

$$A^{n+1} = \begin{bmatrix} a_{n-1} & a_{n-2} + a_{n-1} & a_n \\ a_n & a_{n-1} + a_n & a_{n+1} \\ a_{n+1} & a_n + a_{n+1} & a_{n+2} \end{bmatrix} \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,1 \end{bmatrix} = \begin{bmatrix} a_n & a_{n-1} + a_n & a_{n+1} \\ a_{n+1} & a_n + a_{n+1} & a_{n+2} \\ a_{n+2} & a_{n+1} + a_{n+2} & a_{n+3} \end{bmatrix}.$$

Therefore, (29) holds for $n+1$. By the Principle of Mathematical Induction, (29) holds for all $n \geq 2$. ■

In general,

$$A^n x_{012} = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} = x_{n+1,n+2,n+3} \text{ for } n \geq 1. \tag{30}$$

Proof of (30):

Using (29), we have $A^n x_{012} = \begin{bmatrix} a_{n-1} & a_{n-2}+a_{n-1} & a_n \\ a_n & a_{n-1}+a_n & a_{n+1} \\ a_{n+1} & a_n + a_{n+1} & a_{n+2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}$. ■

Next, we find the eigenvalues of A.

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$

$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -\lambda & 1 \end{vmatrix}$

It follows that $-\lambda^3 + \lambda^2 + \lambda + 1 = 0$. Denote $f(\lambda) = -\lambda^3 + \lambda^2 + \lambda + 1 = 0$.

Then,

$\lambda_1 = \frac{1}{3}\left[\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1\right] \approx 1.839286755...,$

$\lambda_2 = -\frac{1}{6}\left[\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} - 2\right] + \frac{\sqrt{3}}{6} i \left[\sqrt[3]{19+3\sqrt{33}} - \sqrt[3]{19-3\sqrt{33}}\right],$

and

$\overline{\lambda_2} = -\frac{1}{6}\left[\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} - 2\right] - \frac{\sqrt{3}}{6} i \left[\sqrt[3]{19+3\sqrt{33}} - \sqrt[3]{19-3\sqrt{33}}\right]$

are the eigenvalues of A.

Table 10 lists some properties of the eigenvalues λ_1, λ_2 and $\overline{\lambda_2}$.

$\lambda_1 + \lambda_2 + \overline{\lambda_2} = 1$
$\lambda_1 \lambda_2 \overline{\lambda_2} = 1$
$\lambda_1\lambda_2 + \lambda_1\overline{\lambda_2} + \lambda_2\overline{\lambda_2} = -1$
$f(\lambda) = -\lambda^3 + \lambda^2 + \lambda + 1 = 0$ for $\lambda = \lambda_1, \lambda_2$ and $\overline{\lambda_2}$

Table 10: Properties of λ_1,λ_2 and $\overline{\lambda_2}$

Next, we find the eigenvectors for λ_1, λ_2 and $\overline{\lambda_2}$.

$$\begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$x_2 = \lambda_1 x_1$
$x_3 = \lambda_1 x_2$
$x_1 + x_2 + x_3 = \lambda_1 x_3$
Let $x_1 = 1$.
Then $x_2 = \lambda_1$ and $x_3 = \lambda_1^2$.
This implies that $-\lambda_1^3 + \lambda_1^2 + \lambda_1 + 1 = 0$, which is true by Table 10.

Thus, the eigenvector for λ_1 is $\begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix}$.

The other eigenvectors follow similarly.

We have $\begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \overline{\lambda_2} \\ \overline{\lambda_2}^2 \end{bmatrix}$ as eigenvectors for λ_1, λ_2 and $\overline{\lambda_2}$, respectively.

Next, we aim to diagonalize A.

Let $S = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \overline{\lambda_2} \\ \lambda_1^2 & \lambda_2^2 & \overline{\lambda_2}^2 \end{bmatrix}$ be the symmetrizer matrix.

For ease of notation in the matrix calculations below, let $a = \lambda_1, b = \lambda_2$ and $c = \overline{\lambda_2}$.

Rewrite $S = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$. It can be shown that $\det(S) = (a-b)(b-c)(c-a) = k$

$S^{-1} = \frac{1}{k}(adj(S))^T$

$$S^{-1} = \begin{bmatrix} bc(c-b) & -(c-b)(c+b) & (c-b) \\ -ac(c-a) & (c-a)(c+a) & -(c-a) \\ ab(b-a) & -(b-a)(b+a) & (b-a) \end{bmatrix}$$

$$S^{-1}A^nS = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$$

$$A^n = \tfrac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}\begin{bmatrix} bc(c-b) & -(c-b)(c+b) & (c-b) \\ -ac(c-a) & (c-a)(c+a) & -(c-a) \\ ab(b-a) & -(b-a)(b+a) & (b-a) \end{bmatrix}$$

$$A^n = \frac{1}{k}\begin{bmatrix} a^n & b^n & c^n \\ a^{n+1} & b^{n+1} & c^{n+1} \\ a^{n+2} & b^{n+2} & c^{n+2} \end{bmatrix}\begin{bmatrix} bc(c-b) & -(c-b)(c+b) & (c-b) \\ -ac(c-a) & (c-a)(c+a) & -(c-a) \\ ab(b-a) & -(b-a)(b+a) & (b-a) \end{bmatrix} \text{ for } n \geq 0. \tag{31}$$

Setting corresponding entries equal in (29) and the result of A^n in (31) above, we have the relations in $(a) - (g)$ below.

a.) $a_{n-1} = \frac{1}{k}[a^{n-1}(c-b) - b^{n-1}(c-a) + c^{n-1}(b-a)]$

b.) $a_n = \frac{1}{k}\left[a^n(c-b) - b^n(c-a) + c^n(b-a)\right]$, which is a repeated entry.

c.) $a_{n+1} = \frac{1}{k}\left[a^{n+1}(c-b) - b^{n+1}(c-a) + c^{n+1}(b-a)\right]$, which is a repeated entry.

d.) $a_{n+2} = \frac{1}{k}\left[a^{n+2}(c-b) - b^{n+2}(c-a) + c^{n+2}(b-a)\right]$

Note that for $(a) - (d)$ above, we use the fact that $abc = 1$ from Table 10.

e.) $a_n + a_{n+1} = \frac{1}{k}\left[-a^{n+2}(c-b)(c+b) + b^{n+2}(c-a)(c+a) - c^{n+2}(b-a)(b+a)\right]$
To show this relation holds, we'll use (b) and (c) from above.

$a_n + a_{n+1} = \frac{1}{k}\left[a^n(c-b) - b^n(c-a) + c^n(b-a)\right] +$
$+\frac{1}{k}\left[a^{n+1}(c-b) - b^{n+1}(c-a) + c^{n+1}(b-a)\right]$
Then, $a_n + a_{n+1} = \frac{1}{k}\left[-a^{n+2}(c-b)\left(-\frac{1}{a^2} - \frac{1}{a}\right)\right] +$
$+\frac{1}{k}\left[b^{n+2}(c-a)\left(-\frac{1}{b^2} - \frac{1}{b}\right)\right] - \frac{1}{k}\left[c^{n+2}(b-a)\left(-\frac{1}{c^2} - \frac{1}{c}\right)\right]$.
It can be shown that the following hold:
i.) $-\frac{1}{a^2} - \frac{1}{a} = c + b$
ii.) $-\frac{1}{b^2} - \frac{1}{b} = c + a$
iii.) $-\frac{1}{c^2} - \frac{1}{c} = b + a$.

Using $(i) - (iii)$, we have that
$a_n + a_{n+1} = \frac{1}{k}\left[-a^{n+2}(c-b)(c+b) + b^{n+2}(c-a)(c+a) - c^{n+2}(b-a)(b+a)\right]$,
so (e) holds.
The relations in (f) and (g) follow similarly.

f.) $a_{n-1} + a_n = \frac{1}{k}\left[-a^{n+1}(c-b)(c+b) + b^{n+1}(c-a)(c+a) - c^{n+1}(b-a)(b+a)\right]$

g.) $a_{n-2} + a_{n-1} = \frac{1}{k}\left[-a^n(c-b)(c+b) + b^n(c-a)(c+a) - c^n(b-a)(b+a)\right]$

We conclude that the sequence for the Tribonacci numbers for $n \geq 0$ is given by:

$$a_n = \frac{1}{(\lambda_1 - \lambda_2)\left(\lambda_2 - \overline{\lambda_2}\right)\left(\overline{\lambda_2} - \lambda_1\right)}\left[\lambda_1^n(\overline{\lambda_2} - \lambda_2) - \lambda_2^n(\overline{\lambda_2} - \lambda_1) + \overline{\lambda_2}^n(\lambda_2 - \lambda_1)\right]. \tag{32}$$

Observe that (32) is a Binet-type formula for the Tribonacci numbers.

We can also derive a Cassini-type formula for the Tribonacci numbers by calculating $\det(A^n)$.

Note that $\det(A^n) = [\det(A)]^n = 1^n = 1$, and using (29) for $n \geq 2$, the determinant condition becomes,

$$a_n^3 + a_{n-1}^2 a_{n+2} + a_{n-2}a_{n+1}^2 - 2a_{n-1}a_n a_{n+1} - a_{n-2}a_n a_{n+2} = 1. \tag{33}$$

Using $a_0 = 0$, $a_1 = 0$ and $a_2 = 1$, we compute the Tribonacci numbers using both the recursive definition in (28) and the derived formula in (32). We verify that (28) and (32) give the same results up to $n = 300$ in Table 11.

n	$a_n = a_{n-3} + a_{n-2} + a_{n-1}, \;\; n \geq 3$	Equation (32)
0	Define $a_0 = 0$	0
1	Define $a_1 = 0$	0
2	Define $a_2 = 1$	1
3	1	
4	2	
5	4	
6	7	
7	13	
8	24	
9	44	
10	81	
20	35890	
30	15902591	
40	7046319384	
50	3122171529233	
100	53324762928098149064722658	
200	15555116989073938986569525465884451018665640926743832	
300	453751036586945692074245222930195738533...	
	...5193230814796219118584076403940718845682	

Table 11: Generating Tribonacci Numbers Using (28) and (32)

2.8 Generalized Tribonacci Numbers

This section describes how to generate the generalized Tribonacci numbers which use any three starting seeds, t_0, t_1 and t_2. The generalized Tribonacci sequence is given by,

$$t_n = t_{n-3} + t_{n-2} + t_{n-1} \text{ for } n \geq 3. \tag{34}$$

Let $T = \begin{bmatrix} -t_0 - t_1 + t_2 & t_1 - t_0 & t_0 \\ t_0 & t_2 - t_1 & t_1 \\ t_1 & t_0 + t_1 & t_2 \end{bmatrix}$.

$$TA^n = \begin{bmatrix} t_{n-1} & t_{n-2} + t_{n-1} & t_n \\ t_n & t_{n-1} + t_n & t_{n+1} \\ t_{n+1} & t_n + t_{n+1} & t_{n+2} \end{bmatrix} \text{ for } n \geq 2. \tag{35}$$

Proof of (35):

When $n = 2 : TA^2 = \begin{bmatrix} -t_0 - t_1 + t_2 & t_1 - t_0 & t_0 \\ t_0 & t_2 - t_1 & t_1 \\ t_1 & t_0 + t_1 & t_2 \end{bmatrix} \begin{bmatrix} 0\,0\,1 \\ 1\,1\,1 \\ 1\,2\,2 \end{bmatrix} = \begin{bmatrix} t_1 & t_0 + t_1 & t_2 \\ t_2 & t_1 + t_2 & t_3 \\ t_3 & t_2 + t_3 & t_4 \end{bmatrix}$

Assume (35) holds for all $k \leq n$.

Right multiplying by A, we have

$$TA^{n+1} = \begin{bmatrix} t_{n-1} & t_{n-2} + t_{n-1} & t_n \\ t_n & t_{n-1} + t_n & t_{n+1} \\ t_{n+1} & t_n + t_{n+1} & t_{n+2} \end{bmatrix} \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,1\,1 \end{bmatrix} = \begin{bmatrix} t_n & t_{n-1} + t_n & t_{n+1} \\ t_{n+1} & t_n + t_{n+1} & t_{n+2} \\ t_{n+2} & t_{n+1} + t_{n+2} & t_{n+3} \end{bmatrix}.$$

Therefore, (35) holds for $n+1$. By the Principle of Mathematical Induction, (35) holds for all $n \geq 2$. ■

Using the representation of A^n in (31) to calculate TA^n and setting corresponding entries equal to those in (35), we obtain the formula for the generalized Tribonacci numbers:

$$t_n = \frac{1}{(\lambda_1 - \lambda_2)\left(\lambda_2 - \overline{\lambda_2}\right)\left(\overline{\lambda_2} - \lambda_1\right)}[\lambda_1^n(\overline{\lambda_2} - \lambda_2)\left[(-t_0 - t_1 + t_2) + (t_1 - t_0)\lambda_1 + t_0\lambda_1^2\right]$$

$$-\lambda_2^n(\overline{\lambda_2} - \lambda_1)\left[(-t_0 - t_1 + t_2) + (t_1 - t_0)\lambda_2 + t_0\lambda_2^2\right]$$

$$+\overline{\lambda_2^n}(\lambda_2 - \lambda_1)\left[(-t_0 - t_1 + t_2) + (t_1 - t_0)\overline{\lambda_2} + t_0\overline{\lambda_2^2}\right]], \tag{36}$$

where t_0, t_1 and t_2 are the starting seeds and $n \geq 4$.

To illustrate the validity of the derived formula in (36), let's look at an example. Using the starting seeds $t_0 = 1$, $t_1 = 2$ and $t_2 = 3$, we compute the generalized Tribonacci numbers using both the recursive definition in (34) and the derived formula in (36). We verify that (34) and (36) give the same results up to $n = 300$ in Table 12.

n	$t_n = t_{n-3} + t_{n-2} + t_{n-1},\ \ n \geq 3$	Equation (36)
0	Let $t_0 = 1$	
1	Let $t_1 = 2$	
2	Let $t_2 = 3$	
3	6	
4	11	
5	20	
6	37	
7	68	
8	125	
9	230	
10	423	
20	187427	
30	83047505	
40	36797729645	
50	16304799367867	
100	278475910993686935750658203	
200	81232904494794246768797269284791756917949548058614301	
300	236960703319503932041285575095163188455...	
	...365601406997000607531688314676758448118 47	

Table 12: Generating Generalized Tribonacci Numbers Using (34) and (36)

2.9 The Limit of the Ratio of Consecutive Tribonacci Numbers

We observed in Section 2.5: More About the Golden Ratio that the limit of the ratio of two consecutive generalized Fibonacci numbers was the positive eigenvalue associated with the appropriate matrix. The same relationship occurs for the generalized Tribonacci numbers.

Define the sequence $x_n = \frac{t_{n+1}}{t_n}$ for $n \geq 0$ and let $\lim_{n\to\infty} x_n = L$ exist.
Assume $t_0, t_1, t_2 \geq 0$.

Using (34), we can rewrite $x_n = \frac{t_{n-2}+t_{n-1}+t_n}{t_n} = \frac{t_{n-2}}{t_n} + \frac{t_{n-1}}{t_n} + 1$

Then, $x_n = \frac{t_{n-1}}{t_{n-1}}\frac{t_{n-2}}{t_n} + \frac{1}{\frac{t_n}{t_{n-1}}} + 1.$

$x_n = \frac{1}{\frac{t_{n-1}}{t_{n-2}}}\frac{1}{\frac{t_n}{t_{n-1}}} + \frac{1}{\frac{t_n}{t_{n-1}}} + 1$

$x_n = \frac{1}{x_{n-2}}\frac{1}{x_{n-1}} + \frac{1}{x_{n-1}} + 1$

Taking the limit of both sides of the above sequence (assuming the limit exists), we obtain $L = \frac{1}{L^2} + \frac{1}{L} + 1$.

Then $L^3 - L^2 - L - 1 = 0$.

We know that sequence values of (t_n) are real-valued and positive. Therefore, from Table 10 we know that

$L = \frac{1}{3}\left[\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1\right] \approx 1.839286755$, which is the real-valued eigenvalue of the matrix A.

Therefore,

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = \frac{1}{3}\left[\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1\right] \approx 1.839286755... \qquad (37)$$

3 Conclusion

This thesis aims to provide the reader with an alternate way to derive Fibonacci numbers and other related sequences. The recursive definitions of Fibonacci numbers commonly appear in mathematics curriculum and studies done on these definitions have been extensive. This thesis, on the other hand, uses the theory of matrices to derive explicit formulas for the Fibonacci, Lucas, generalized Fibonacci, generalized weighted Fibonacci, Tribonacci and generalized Tribonacci numbers given any integer, n.

We began with a historical introduction which examined the works of mathematicians Fibonacci and Hemachandra. While Fibonacci gets credit for developing the "so called" Fibonacci numbers, we note that Indian mathematicians such as Hemachandra played a role in their development before Fibonacci's time. We also looked at some of the number theoretical properties that Fibonacci numbers posses and mentioned a few of the ways they appear in real life settings.

We then dove into the theory of matrices and developed formulas for the Fibonacci, Lucas, generalized Fibonacci, generalized weighted Fibonacci, Tribonacci and generalized Tribonacci numbers by diagonalizing associated matrices and examining their eigenvalues. We tabulated numerical results comparing both the recursive and derived formulas. It is known that these results can be extended further. In fact, generalizations can be made for the $n \times n$ case.

Fibonacci numbers are fascinating and their impact on the field of mathematics has been great. In 1963, *The Fibonacci Association* was established and is devoted to the study of Fibonacci numbers and other related sequences. *The Fibonacci Association* prints articles on these topics in their publication called *The Fibonacci Quarterly.* Countless other publications have been written about these numbers. Their appearance in different areas of mathematics and in other unusual places make Fibonacci numbers even more special. For this reason, their beauty can be appreciated by a wide range of mathematicians and even non-mathematicians. There is no doubt that Fibonacci numbers will continue to remain a curiosity to many for centuries to come.

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