

Journal of Computational and Applied Mathematics 76 (1996) 213-229

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Recurrence relations for the connection coefficients of orthogonal polynomials of a discrete variable

Stanisław Lewanowicz*

Institute of Computer Science, University of Wrocław, 51-151 Wrocław, Poland

Received 5 September 1995

Abstract

We give explicitly recurrence relations satisfied by the connection coefficients between two families of the classical orthogonal polynomials of a discrete variable (i.e., associated with the names of Charlier, Meixner, Krawtchouk and Hahn). Also, a recurrence relation is given for the coefficients in the formula expressing the *n*th associated polynomial in terms of the original polynomials.

Keywords: Classical orthogonal polynomials of a discrete variable; Associated polynomials; Connection coefficients; Recurrence relations

1. Introduction

Let $\{P_k(x)\}$ be any system of the classical orthogonal polynomials of a discrete variable, i.e., Charlier polynomials $C_k(x; a)$, Meixner polynomials $M_k(x; \beta, c)$, Krawtchouk polynomials $K_k(x; p, N)$ and Hahn polynomials $Q_n(x; \alpha, \beta, N)$:

$$\sum_{x=0}^{B-1} \varrho(x) P_k(x) P_l(x) = \delta_{kl} h_k \quad (k, l = 0, 1, ...),$$

where $h_k > 0$ (k = 0, 1, ...); the set of orthogonality is $\{0, 1, ..., B - 1\}$, where B equals $+\infty, +\infty$, N + 1 and N, respectively. Besides the three-term recurrence relation

$$xP_{k}(x) = \xi_{0}(k)P_{k-1}(x) + \xi_{1}(k)P_{k}(x) + \xi_{2}(k)P_{k+1}(x)$$

$$(k = 0, 1, \dots; P_{-1}(x) \equiv 0, P_{0}(x) \equiv 1)$$
(1.1)

these polynomials enjoy a number of similar properties [10, Ch. II; 11]. We shall need four of these properties.

^{*} E-mail: Stanislaw.Lewanowicz@ii.uni.wroc.pl.

^{0377-0427/96/\$15.00 © 1996} Elsevier Science B.V. All rights reserved *PII* \$0377-0427(96)00104-5

S. Lewanowicz/Journal of Computational and Applied Mathematics 76 (1996) 213–229

First, the weight function ρ satisfies a difference equation of the type

$$\Delta[\sigma(x)\varrho(x)] = \tau(x)\varrho(x), \tag{1.2}$$

where σ is a polynomial of degree at most 2, and τ is a first-degree polynomial.

Second, for arbitrary n, the polynomial P_n obeys the second-order difference equation

$$\boldsymbol{L}_{n}\boldsymbol{P}_{n}(\boldsymbol{x}) \equiv \left\{ \boldsymbol{\sigma}(\boldsymbol{x})\Delta\boldsymbol{\nabla} + \boldsymbol{\tau}(\boldsymbol{x})\Delta + \lambda_{n} \boldsymbol{I} \right\} \boldsymbol{P}_{n}(\boldsymbol{x}) = \boldsymbol{0}, \tag{1.3}$$

where $\Delta := E - I$, $\nabla := I - E^{-1}$, E^m $(m \in \mathbb{Z})$ is the *m*th shift operator, $E^m f(x) = f(x+m)$, I is the identity operator, If(x) = f(x), and λ_n is a constant given by

$$\lambda_n := -\frac{1}{2}n[(n-1)\sigma'' + 2\tau'] \quad (n \in \mathbb{N}).$$

$$(1.4)$$

(By convention, all the bold letter operators act on the variable x.)

Third, we have a pair of the so-called structure relations,

$$[\sigma(x) + \tau(x)]\Delta P_k(x) = d_0(k)P_{k-1}(x) + d_1(k)P_k(x) + d_2(k)P_{k+1}(x)$$
(1.5)

and

$$\sigma(x)\nabla P_k(x) = d_0(k)P_{k-1}(x) + [d_1(k) + \lambda_k]P_k(x) + d_2(k)P_{k+1}(x).$$
(1.6)

Fourth,

$$\sigma(x)\varrho(x)x^{k}\Big|_{x=0}^{x=B} = 0 \qquad (k = 0, 1, ...).$$
(1.7)

Let $\{P_k\}$ be any family of classical orthogonal polynomials of a discrete variable, and let $\{\bar{P}_k\}$ be a sequence of polynomials. We are looking for a formula of the type

$$\bar{P}_n = \sum_{k=0}^n c_{n,k} P_k.$$
(1.8)

The coefficients $c_{n,k}$ in (1.8) are called the *connection coefficients* between the polynomials $\{P_k\}$ and $\{\bar{P}_k\}$ (see [1, Lecture 7]).

Here are the interesting particular cases:

1. $\{\bar{P}_k\}$ is another sequence of classical orthogonal polynomials of a discrete variable;

2. $\{\overline{P}_k\}$ is a sequence of orthogonal polynomials associated with $\{P_k\}$ (see [2]).

Note that in each case, polynomial $\{\overline{P}_n\}$ satisfies a linear difference equation in x.

In a recent paper, Ronveaux et al. [11] have discussed the first case and proposed an algorithmic way of obtaining a recurrence relation (in k) of the form

$$\mathscr{L} c_{n,k} \equiv \sum_{i=0}^{r} A_i(k) c_{n,k+i} = 0.$$
(1.9)

In the present paper we propose an alternative technique of derivation of the recurrence relation (1.9), based on an idea introduced in [8, 9]. The difference operator \mathscr{L} is given in terms of σ ,

 τ and the difference operators \mathscr{X} and \mathscr{D} , defined implicitly by the right-hand sides of (1.1) and (1.5), respectively (see Theorems 3.1 and 3.4). Also, it should be stressed that in almost all the cases the order of the obtained recurrence relation is significantly lower than in [11] (for instance, an eighth-order relation, given in [11] for the Hahn-Hahn case, can be replaced by a second-order relation). Applications of the result to some pairs of the classical discrete orthogonal polynomials are given.

The case where \bar{P}_n is the *n*th associated polynomial is also discussed. A general result, given in Theorem 4.1, is applied to each classical family.

2. Identities involving the discrete Fourier coefficients

We shall need certain properties of the Fourier coefficients of an arbitrary polynomial f, deg f < B, defined by

$$a_k[f] := \frac{1}{h_k} b_k[f] \qquad (k = 0, 1, \dots, B - 1),$$
(2.1)

where

$$b_k[f] := \sum_{x=0}^{B-1} \varrho(x) P_k(x) f(x), \tag{2.2}$$

i.e., the coefficients in the expansion

$$f = \sum_{k=0}^{\deg f} a_k [f] P_k.$$

Let \mathscr{X}, \mathscr{D} and $\tilde{\mathscr{D}}$ be the difference operators (acting on k) defined by

$$\mathscr{X} := \xi_0(k)\mathscr{E}^{-1} + \xi_1(k)\mathscr{I} + \xi_2(k)\mathscr{E}, \tag{2.3}$$

$$\mathscr{D} := d_0(k)\mathscr{E}^{-1} + d_1(k)\mathscr{I} + d_2(k)\mathscr{E}, \tag{2.4}$$

$$\tilde{\mathscr{D}} := \mathscr{D} + \lambda_k \mathscr{I} \tag{2.5}$$

(cf. (1.1), (1.5) and (1.6), respectively), where \mathscr{I} is the identity operator, and \mathscr{E}^m the *m*th shift operator: $\mathscr{I} b_k[f] = b_k[f]$, $\mathscr{E}^m b_k[f] = b_{k+m}[f]$ ($m \in \mathbb{Z}$). For the sake of simplicity, we write \mathscr{E} in place of \mathscr{E}^1 . (We adopt the convention that all the script letter operators act on the variable k.)

Further, let us define the difference operators U and V (acting on x) by

$$\boldsymbol{U} := \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nabla} + \boldsymbol{\tau}(\boldsymbol{x})\boldsymbol{I}, \tag{2.6}$$

$$\boldsymbol{V} := [\boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{x})]\boldsymbol{\Delta} + \boldsymbol{\tau}(\boldsymbol{x})\boldsymbol{I}.$$
(2.7)

Notice that by virtue of $\Delta \nabla = \Delta - \nabla$, we can write

$$L_n = V - U + \lambda_n I. \tag{2.8}$$

We prove the following lemma.

Lemma 2.1. The coefficients (2.2) obey the identities:

$$b_k[xf(x)] = \mathscr{X} b_k[f], \tag{2.9}$$

$$\tilde{\mathscr{D}} b_k[\nabla f] = \lambda_k b_k[f], \qquad (2.10)$$

$$\mathscr{D}b_k[\Delta f] = \lambda_k \, b_k[f], \tag{2.11}$$

$$b_k[Uf] = -\mathscr{D}b_k[f], \tag{2.12}$$

$$b_k[Vf] = -\tilde{\mathscr{D}} b_k[f], \qquad (2.13)$$

$$b_k[\boldsymbol{L}_n f] = (\lambda_n - \lambda_k) b_k[f]. \tag{2.14}$$

Proof. In view of (1.1) and (2.3), identity (2.9) is obviously true.

We will prove the identity (2.10). Using (1.5), summing by parts, and then using (1.7) and the equation $\Delta[\sigma(x)\varrho(x)\nabla P_k(x)] = -\lambda_k \varrho(x)P_k(x)$ (cf. (1.2) and (1.3)), we get

$$\tilde{\mathscr{D}} b_k[\nabla f] = \tilde{\mathscr{D}} \sum_{x=0}^{B-1} \varrho(x) P_k(x) \Delta f(x-1) = \sum_{x=0}^{B-1} \varrho(x) \sigma(x) \nabla P_k(x) \Delta f(x-1)$$
$$= \varrho(x) \sigma(x) \nabla P_k(x) f(x-1) \Big|_{x=0}^{x=B} - \sum_{x=0}^{B-1} \Delta \left[\varrho(x) \sigma(x) \nabla P_k \right] f(x)$$
$$= \lambda_k \sum_{x=0}^{B-1} \varrho(x) P_k(x) f(x) = \lambda_k b_k[f].$$

The proof of (2.11) goes as follows.

$$\mathcal{D} b_k[\Delta f] = \mathcal{D} \sum_{x=0}^{B-1} \varrho(x) P_k(x) \Delta f(x) = \sum_{x=0}^{B-1} \varrho(x) [\sigma(x) + \tau(x)] \Delta P_k(x) \Delta f(x)$$
$$= \sum_{x=0}^{B-1} \varrho(x+1) \sigma(x+1) \nabla P_k(x+1) \Delta f(x)$$
$$= \sum_{y=1}^{B} \varrho(y) \sigma(y) \nabla P_k(y) \Delta f(y-1) = \sum_{x=0}^{B-1} \varrho(x) \sigma(x) \nabla P_k(x) \Delta f(x-1)$$
$$= -\sum_{x=0}^{B-1} \Delta [\sigma(x) \varrho(x) \nabla P_k(x)] f(x) = \lambda_k \sum_{x=0}^{B-1} \varrho(x) P_k(x) f(x) = \lambda_k b_k[f].$$

Here we used a.o. the equation $\sigma(x+1)\varrho(x+1) = [\sigma(x) + \tau(x)]\varrho(x)$ (cf. (1.2)). Similarly, we obtain

$$b_k[\sigma \nabla f] = \sum_{x=0}^{B-1} \varrho(x)\sigma(x)P_k(x)\nabla f = \sum_{x=0}^{B-1} \varrho(x)\sigma(x)P_k(x)\Delta f(x-1)$$
$$= -\sum_{x=0}^{B-1} \Delta \left[\sigma(x)\varrho(x)P_k(x)\right]f(x)$$

$$= -\sum_{x=0}^{B-1} [\tau(x)\varrho(x)P_k(x) + \sigma(x+1)\varrho(x+1)\Delta P_k(x)]f(x)$$

$$= -\sum_{x=0}^{B-1} \tau(x)\varrho(x)P_k(x)f(x) - \sum_{x=0}^{B-1} [\sigma(x) + \tau(x)]\varrho(x)\Delta P_k(x)f(x)$$

$$= -b_k[\tau f] - \sum_{x=0}^{B-1} \varrho(x)\mathscr{D}P_k(x)f(x)$$

$$= -b_k[\tau f] - \mathscr{D}b_k[f].$$

Hence follows the identity (2.12).

Identity (2.13) may be proved in an analogous way.

Using (2.8), (2.12) and (2.13), and remembering that $\tilde{\mathscr{D}} = \mathscr{D} + \lambda_k \mathscr{I}$ (cf. (2.5)), we have $b_k[\boldsymbol{L}_n f(x)] = b_k[\boldsymbol{V}f(x)] - b_k[\boldsymbol{U}f(x)] + \lambda_n b_k[f] = \{\mathscr{D} - \tilde{\mathscr{D}} + \lambda_n \mathscr{I}\} b_k[f] = (\lambda_n - \lambda_k) b_k[f].$ This proves the validity of (2.14). \Box

Remark 2.2. Identity (2.9) can be easily generalized to the form

$$b_k[qf] = q(\mathscr{X})b_k[f], \tag{2.15}$$

where q is any polynomial.

The next result refers to the case of the Hahn polynomials.

Lemma 2.3. Let $b_k[f]$ be defined by (2.2) with P_k being the Hahn polynomials, i.e., $P_k = Q_k(\cdot; \alpha, \beta, N)$. Let us define the difference operators G, H (acting on x) by

$$\boldsymbol{G} := (N-1-x)\boldsymbol{\Delta} - (\alpha+1)\boldsymbol{I}, \qquad (2.16)$$

$$\boldsymbol{H} := \boldsymbol{x} \boldsymbol{\nabla} + (\beta + 1) \boldsymbol{I}. \tag{2.17}$$

The following identities hold:

$$\mathscr{P}b_k[Gf(x)] = -(k+\beta+1)\tilde{\mathscr{P}}b_k[f], \qquad (2.18)$$

$$\mathcal{P}b_k[Hf(x)] = (k + \alpha + 1)\mathcal{P}b_k[f], \qquad (2.19)$$

where $\mathcal{P}, \tilde{\mathcal{P}}$ are the following first-order difference operators (acting on k):

$$\mathscr{P} := (k + \beta + 1)\pi(k)\mathscr{I} + \mathscr{E}, \tag{2.20}$$

$$\tilde{\mathscr{P}} := (k + \alpha + 1)\pi(k)\mathscr{I} - \mathscr{E}$$
(2.21)

with $\gamma = \alpha + \beta + 1$, and

$$\pi(k) := \frac{(k+\gamma)(k+\gamma+N)}{(2k+\gamma)_2}.$$
(2.22)

Proof. It suffices to observe that the operators (2.20), (2.21) and $\hat{\mathcal{Q}}$, $\tilde{\hat{\mathcal{Q}}}$ given by

$$egin{aligned} &\mathcal{Q} := -rac{(k+eta+1)\pi(k)}{k+\gamma}\mathscr{I} + rac{1}{k+1}\mathscr{E}, \ & ilde{\mathcal{Q}} := rac{(k+lpha+1)\pi(k)}{k+\gamma}\mathscr{I} + rac{1}{k+1}\mathscr{E} \end{aligned}$$

verify the equations

$$\mathscr{P}((N-1)\mathscr{I}-\mathscr{X}))=\mathscr{Q},\qquad \tilde{\mathscr{P}}\mathscr{X}=\tilde{\mathscr{Q}}\tilde{\mathscr{Q}},$$

so that we have

$$\mathscr{P}b_k[(N-1-x)\Delta f(x)] = \mathscr{Q}\mathscr{D}b_k[\Delta f(x)],$$

 $\tilde{\mathscr{P}}b_k[x\nabla f(x)] = \tilde{\mathscr{Q}}\tilde{\mathscr{D}}b_k[\nabla f(x)].$

Now use (2.11) and (2.10), respectively, and — noticing that $\lambda_k = k(k + \gamma)$ — check that

$$\begin{split} &\mathcal{Q}(\lambda_k\mathscr{I})-(lpha+1)\mathscr{P}=-(k+eta+1)\mathscr{P}, \ &\tilde{\mathcal{Q}}(\lambda_k\mathscr{I})+(eta+1)\tilde{\mathscr{P}}=(k+lpha+1)\mathscr{P}. \end{split}$$

Lemma 2.4. Let \mathcal{R} and $\tilde{\mathcal{R}}$ be the following first-order operators:

$$\mathscr{R} := (k+\alpha)\vartheta(k)\mathscr{E}^{-1} - \mathscr{I}, \tag{2.23}$$

$$\tilde{\mathscr{R}} := (k+\beta)\vartheta(k)\mathscr{E}^{-1} + \mathscr{I}$$
(2.24)

with

$$\vartheta(k) := \frac{(k+\gamma)(k+\gamma+N)}{(2k+\gamma-1)_2}.$$
(2.25)

Then we have the equality

$$\mathscr{RP} = \tilde{\mathscr{R}}\tilde{\mathscr{P}}.$$
 (2.26)

Proof. Eq. (2.26) can be verified by a straightforward calculation. \Box

3. Classical orthogonal polynomials of a discrete variable

Let $\{P_k\}$ and $\{\tilde{P}_k\}$ be any two families of the classical discrete orthogonal polynomials. We shall give a recurrence relation (in k) of the form

$$\mathscr{L} c_{n,k} \equiv \sum_{i=0}^{r} A_i(k) c_{n,k+i} = 0, \qquad (3.1)$$

obeyed by the connection coefficients $c_{n,k}$ in

$$\bar{P}_n = \sum_{k=0}^n c_{n,k} P_k.$$
(3.2)

Obviously, $c_{n,k}$ are the Fourier coefficients $a_k[\bar{P}_n]$. Let us write

$$b_{n,k} := b_k[\bar{P}_n] = h_k \ c_{n,k}. \tag{3.3}$$

Let P_n satisfy Eq. (1.3), and let

$$\bar{L}_{n}\bar{P}_{n}(x) \equiv \left\{\bar{\sigma}(x)\Delta \nabla + \bar{\tau}(x)\Delta + \bar{\lambda}_{n}I\right\}\bar{P}_{n}(x) = 0, \qquad (3.4)$$

where $\bar{\sigma}$, $\bar{\tau}$ are polynomials, deg $\bar{\sigma} \leq 2$, deg $\bar{\tau} = 1$, and $\bar{\lambda}_n := -\frac{1}{2}n[(n-1)\bar{\sigma}'' + 2\bar{\tau}']$. We shall use the notation

$$\varphi(x) := \bar{\sigma}(x) - \sigma(x), \tag{3.5}$$

$$\psi(x) := \overline{\tau}(x) - \tau(x). \tag{3.6}$$

We can write

$$\bar{L}_n = L_n + [\varphi(x) + \psi(x)]\Delta - \varphi(x)\nabla + (\bar{\lambda}_n - \lambda_n)I.$$
(3.7)

3.1. Connection between Hahn families

Assume that both sequences $\{P_k\}$, $\{\bar{P}_k\}$ belong to the Hahn family. We will prove the following.

Theorem 3.1. Let $\{P_k\}$, $\{\bar{P}_k\}$ be Hahn polynomials, $P_k = Q_k(\cdot; \alpha, \beta, N)$, and $\bar{P}_k = Q_k(\cdot; \eta, \zeta, N)$. The coefficients (3.3) satisfy the second-order recurrence relation

$$\tilde{\mathscr{L}} b_{n,k} = 0, \tag{3.8}$$

where the difference operator $\tilde{\mathscr{L}}$ is given by

$$\tilde{\mathscr{L}} := \mathscr{RP}(\nu_k\mathscr{I}) + (\varphi' + \psi')\mathscr{R}((k+\beta+1)\tilde{\mathscr{P}}) - \varphi'\tilde{\mathscr{R}}((k+\alpha+1)\mathscr{P})$$
(3.9)

with

$$v_k := \bar{\lambda}_n - \lambda_k + (\beta - \alpha)\varphi' - (\alpha + 1)\psi'. \tag{3.10}$$

Proof. In the considered case, we have (cf. Appendix, Tables 1 and 2)

$$\varphi(x) = \varphi' \cdot x, \qquad (\varphi + \psi)(x) = -(\varphi' + \psi') \cdot (N - 1 - x).$$

Eq. (3.7) implies

$$\bar{L}_n = L_n - (\varphi' + \psi')G - \varphi'H + \left[\bar{\lambda}_n - \lambda_n + (\beta - \alpha)\varphi' - (\alpha + 1)\psi'\right]I.$$

Using this result in the equation

$$b_k[\bar{L}_n f(x)] = 0, \tag{3.11}$$

where $f = \bar{P}_n$, applying the operator $\mathcal{T} = \mathcal{RP}$ (= $\tilde{\mathcal{RP}}$; cf. Lemma 2.4), and making use of Lemmas 2.1 and 2.3, we obtain formula (3.9).

The order of the recurrence can be deduced from (3.9) in view of the form of the difference operators \mathscr{P} , $\tilde{\mathscr{P}}$, \mathscr{R} and $\tilde{\mathscr{R}}$ (cf. (2.20)–(2.24)). \Box

Substituting (3.3), and the specific values of σ , τ , λ_k , h_k as well as the forms of the operators \mathscr{X} (see (2.3)) and \mathscr{D} (see (2.4)) given in the Appendix (see Table 1) in (3.8), we obtain the following corollary.

Corollary 3.2. The connection coefficients $c_{n,k}$ in

$$Q_n(x;\eta,\zeta,N) = \sum_{k=0}^n c_{n,k} Q_k(x;\alpha,\beta,N)$$
(3.12)

satisfy the second-order recurrence relation

$$\mathscr{L}c_{n,k} \equiv A_0(k)c_{n,k-1} + A_1(k)c_{n,k} + A_2(k)c_{n,k+1} = 0 \qquad (1 \le k \le n),$$
(3.13)

where

$$\begin{aligned} A_0(k) &= (k + n + \vartheta - 1)(n - k + 1)(k + \gamma + N)(2k + \gamma - 2)_4(2k + \gamma - 1), \\ A_1(k) &= k(2k + \gamma - 2)_3(N - k)(k + \gamma + N) \\ &\times \{(\alpha - \beta)n(n + \vartheta) + (\alpha - \beta + 2\zeta - 2\eta)k(k + \gamma) \\ &+ (\gamma + 1)[\alpha(\vartheta - 1) - \eta(\gamma - 1))]\}, \\ A_2(k) &= (k)_2(N - k - 1)_2(k + \beta + 1)(k + \alpha + 1) \\ &\times (k + n + \gamma + 1)(k - n + \gamma - \vartheta + 1)(k + \gamma + N)(2k + \gamma + 1), \end{aligned}$$

and where $\gamma := \alpha + \beta + 1$, and $\vartheta := \eta + \zeta + 1$, with the initial conditions $c_{n,n} = 1$, $c_{n,n+1} = 0$. The Pochhammer symbol $(a)_m$ has the following meaning:

$$(a)_0 := 1,$$
 $(a)_m := a(a+1)...(a+m-1)$ $(m = 1, 2, ...).$

Remark 3.3. Gasper [5] expressed $c_{n,k}$ in (3.12) explicitly as a multiple of

$$_{3}F_{2}\left(\left. \begin{array}{c} k-n, \, k+\alpha+1, \, n+k+\vartheta \\ k+\eta+1, \, 2k+\gamma+1 \end{array} \right| 1 \right)$$

He also gave certain conditions sufficient for non-negativity of $c_{n,k}$ for all n, k, N.

Thus, (3.13) is a new recurrence for the above hypergeometric functions. By the way, this recurrence may be also deduced from a general result given in [7].

3.2. Connection between Charlier, Meixner and Krawtchouk families

Now we consider the case where *none* of the families $\{P_k\}$, $\{\bar{P}_k\}$ is a Hahn family. We will prove the following.

Theorem 3.4. Let $\{P_k\}$, $\{\bar{P}_k\}$ be (independently chosen) Charlier, or Meixner, or Krawtchouk polynomials. The coefficients (3.3) satisfy the recurrence relation

$$\tilde{\mathscr{L}} b_{n,k} = 0$$
 $(1 \le k \le n; \ b_{n,n} = h_n, \ b_{n,n+1} = 0),$ (3.14)

where the difference operator $\tilde{\mathscr{L}}$ is given by

$$\tilde{\mathscr{L}} := \mathscr{D}\left(\mu_k \mathscr{I}\right) + \lambda_k \left(\psi' \mathscr{X} + \psi(-1) \mathscr{I}\right) \tag{3.15}$$

with

$$\mu_k := \bar{\lambda}_n - \lambda_k - \psi'. \tag{3.16}$$

The order of the recurrence relation (3.14) is not greater than 2.

Remark 3.5. Eq. (3.1) is obtained by substitution of (3.3) in (3.14).

Proof. Let us denote $f = \overline{P}_n$. As we have $\overline{\sigma} = \sigma$ (cf. Appendix, Tables 1 and 2), so Eq. (3.7) simplifies to

$$\bar{\boldsymbol{L}}_n f(\boldsymbol{x}) = \boldsymbol{L}_n f(\boldsymbol{x}) + \boldsymbol{\psi}(\boldsymbol{x}) \boldsymbol{\Delta} f(\boldsymbol{x}) + (\bar{\lambda}_n - \lambda_n) f(\boldsymbol{x}),$$

which can rewritten in the form

$$\bar{L}_n f(x) = L_n f(x) + \Delta(\psi(x-1)f(x)) + \kappa f(x)$$

with $\kappa = \overline{\lambda}_n - \lambda_n - \psi'$. Using this result in Eq. (3.11), we obtain

$$b_k[L_n f(x)] + b_k[\Delta(\psi(x-1)f(x))] + b_k[\kappa f(x)] = 0.$$

Applying the operator \mathscr{D} to both sides of the above equation, and making a repeated use of Lemma 2.1, we arrive at the recurrence relation (3.14) with the operator $\mathcal{\hat{L}}$ given in (3.15). \Box

Now, we can apply the general result given in Theorem 3.4 to all possible pairs of the families of non-Hahn classical discrete orthogonal polynomials. Computer algebra system MAPLE [3] was very helpful in obtaining the scalar form of the Eq. (3.14). The specific values of σ , τ , λ_k , h_k as well as the forms of the operators \mathscr{X} (see (2.3)) and \mathscr{D} (see (2.4)) are given in the Appendix for the *monic* Charlier polynomials (Table 1) and for the monic Meixner and Krawtchouk polynomials (Table 2).

3.2.1. Charlier–Charlier

For the connection coefficients $c_{n,k}$ in

$$C_n(x;b) = \sum_{k=0}^n c_{n,k}C_k(x;a),$$

we obtain the first-order recurrence relation

$$(n-k+1)c_{n,k-1}-k(a-b)c_{n,k}=0$$
 $(k=1,2,\ldots,n; c_{n,n}=1).$

Hence,

$$c_{n,k} = \binom{n}{k} (a-b)^{n-k} \qquad (k=0,1,\ldots,n)$$

which agrees with the result given in [11].

3.2.2. Meixner-Charlier

For the connection coefficients $c_{n,k}$ in

$$M_n(x;\beta,c)=\sum_{k=0}^n c_{n,k}C_k(x;a),$$

we obtain the second-order recurrence relation

$$q(k-n-1)c_{n,k-1} - k(k+\beta - aq - 1)c_{n,k} - a(k)_2c_{n,k+1} = 0$$

(k = 1,2,...,n; c_{n,n} = 1, c_{n,n+1} = 0),

where q := (1 - c)/c.

3.2.3. Krawtchouk–Charlier

For the connection coefficients $c_{n,k}$ in

$$K_n(x; p, N) = \sum_{k=0}^n c_{n,k} C_k(x; a),$$

we obtain the second-order recurrence relation

$$(k-1-n)c_{n,k-1} + pk(k-N+a/p-1)c_{n,k} + ap(k)_2c_{n,k+1} = 0$$

(k = 1, 2, ..., n; c_{n,n} = 1, c_{n,n+1} = 0).

3.2.4. Charlier–Meixner

For the connection coefficients $c_{n,k}$ in

$$C_n(x;a) = \sum_{k=0}^n c_{n,k} M_k(x;\beta,c),$$

we obtain the second-order recurrence relation

$$(1-c)^{2}(k-n-1)c_{n,k-1} + (1-c)k[c(2k+\beta-n+a-1)-a]c_{n,k}$$
$$+c^{2}(k)_{2}(k+\beta)c_{n,k+1} = 0 \qquad (k = 1, 2, ..., n)$$

with the initial conditions $c_{n,n} = 1$, $c_{n,n+1} = 0$.

3.2.5. Meixner–Meixner

For the connection coefficients $c_{n,k}$ in

$$M_n(x;\gamma,d) = \sum_{k=0}^n c_{n,k} M_k(x;\beta,c),$$

we obtain the second-order recurrence relation

$$(1-c)^{2}(1-d)(k-n-1)c_{n,k-1}$$

$$+(1-c)k \{c[2k+\beta-1-n-d(k-n+\beta-\gamma)] - d(k+\gamma-1)\}c_{n,k}$$

$$+c(c-d)(k)_{2}(k+\beta)c_{n,k+1} = 0 \qquad (k = 1, 2, ..., n)$$

with the initial conditions $c_{n,n} = 1$, $c_{n,n+1} = 0$.

For d = c, this equation simplifies to the first-order equation

$$(1-c)(n-k+1)c_{n,k-1}-ck(k-n+\beta-\gamma)c_{n,k}=0 \qquad (k=1,2,\ldots,n; \ c_{n,n}=1),$$

which implies the formula

$$M_n(x;\gamma,c) = \sum_{k=0}^n \binom{n}{k} \left(\frac{c}{c-1}\right)^k (\gamma-\beta)_k M_{n-k}(x;\beta,c).$$

Gasper [5] gave an explicit formula for $c_{n,k}$ in terms of

$$_{2}F_{1}\left(\left. \begin{array}{c} k-n,\,k+\beta \\ k+\gamma \end{array} \right| rac{c(1-d)}{d(1-c)}
ight).$$

He also showed that $c_{n,k} \ge 0$ iff $d \ge c$ and $\gamma \ge \beta$.

3.2.6. Krawtchouk-Meixner

For the connection coefficients $c_{n,k}$ in

$$K_n(x;N,p) = \sum_{k=0}^n c_{n,k} M_k(x;\beta,c),$$

we obtain the second-order recurrence relation

$$(1-c)^{2}(n-k+1)c_{n,k-1} + (c-1)k[(c-1)p(N-k+1) + c(2k-n+\beta-1)]c_{n,k} + c(cp-c-p)(k)_{2}(k+\beta)c_{n,k+1} = 0 \quad (k = 1, 2, ..., n),$$

with the initial conditions $c_{n,n} = 1$, $c_{n,n+1} = 0$.

Gasper [5] expressed $c_{n,k}$ in terms of

$$_{2}F_{1}\left(\left. \begin{array}{c} k-n,\,k+\beta \\ k-N \end{array} \right| \frac{c}{p(c-1)} \right).$$

3.2.7. Charlier-Krawtchouk

For the connection coefficients $c_{n,k}$ in

$$C_n(x;a) = \sum_{k=0}^n c_{n,k} K_k(x;p,N),$$

we obtain the second-order recurrence relation

$$(k - n - 1)c_{n,k-1}$$

-k [p(2k - N - n - 1) + a] $c_{n,k} - p^2(k)_2(N - k)c_{n,k+1} = 0$
(k = 1,2,...,n),

with the initial conditions $c_{n,n} = 1$, $c_{n,n+1} = 0$.

3.2.8. Meixner-Krawtchouk

For the connection coefficients $c_{n,k}$ in

$$M_n(x;\beta,c)=\sum_{k=0}^n c_{n,k}K_k(x;p,N),$$

we obtain the second-order recurrence relation

$$(1-c)(k-n-1)c_{n,k-1} - k [(1-c)p(2k-N-n-1) + c(k+\beta-1)]c_{n,k} - p(cp-p-c)(k)_2(N-k)c_{n,k+1} = 0 \quad (k = 1, 2, ..., n),$$

with the initial conditions $c_{n,n} = 1$, $c_{n,n+1} = 0$.

3.2.9. Krawtchouk-Krawtchouk

For the connection coefficients $c_{n,k}$ in

$$K_n(x;q,N) = \sum_{k=0}^n c_{n,k} K_k(x;p,N),$$

we obtain the second-order recurrence relation

$$(k - n - 1)c_{n,k-1} + k [(2p - q)k - (p - q)(N + 1) - np] c_{n,k} + p(p - q)(k)_2(N - k)c_{n,k+1} = 0$$

(k = 1, 2, ..., n),

with the initial conditions $c_{n,n} = 1$, $c_{n,n+1} = 1$. The explicit form is [11]

$$c_{n,k} = \binom{n}{k} (p-q)^{n-k} (N+1-n)_{n-k} \qquad (k=0,1,\ldots,n).$$

4. Associated polynomials

Given a system $\{P_k(x)\}$ of classical orthogonal polynomials of a discrete variable, the associated polynomials $\{P_k^{(1)}(x)\}$ are defined recursively by

$$xP_{k}^{(1)}(x) = \xi_{0}(k+1)P_{k-1}^{(1)}(x) + \xi_{1}(k+1)P_{k}^{(1)}(x) + \xi_{2}(k+1)P_{k+1}^{(1)}(x)$$

$$(k = 0, 1, \dots; P_{-1}^{(1)}(x) \equiv 0, P_{0}^{(1)}(x) \equiv 1),$$
(4.1)

notation being that of (1.1). Recently, Atakishiyev et al. [2] have shown that for any natural n, $f(x) = P_{n-1}^{(1)}(x)$ obeys the following second-order difference equation:

$$\boldsymbol{L}_{n}^{*}f(x) \equiv \boldsymbol{\nabla}\Delta[\boldsymbol{\sigma}(x)f(x)] - \boldsymbol{\nabla}[\boldsymbol{\tau}(x)f(x)] + \lambda_{n}f(x) = \kappa(\Delta + \boldsymbol{\nabla})\boldsymbol{P}_{n}(x), \qquad (4.2)$$

where we assume that $\{P_k\}$ are monic, and

$$\kappa := \frac{1}{2}\sigma'' - \tau'.$$

Let us look for a recurrence relation for the Fourier coefficients $a_k[P_{n-1}^{(1)}(x)]$, i.e., the connection coefficients $\{a_{n-1,k}^{(1)}\}$ in

$$P_{n-1}^{(1)}(x) = \sum_{k=0}^{n-1} a_{n-1,k}^{(1)} P_k(x).$$
(4.3)

Let us denote

$$b_{n-1,k}^{(1)} := b_k[P_{n-1}^{(1)}] = h_k a_{n-1,k}^{(1)}.$$
(4.4)

Theorem 4.1. The coefficients $b_{n-1,k}^{(1)}$ satisfy the recurrence relation

$$\tilde{\mathscr{L}}b_{n-1,k}^{(1)} = 0 \qquad (2 \leq k \leq n-r+1), \tag{4.5}$$

where

$$\tilde{\mathscr{L}} := \lambda_k \sigma(\mathscr{X}) - \mathscr{D}\tau(\mathscr{X}) + \lambda_n \mathscr{D}\lambda_k^{-1}\tilde{\mathscr{D}}.$$
(4.6)

The order r of the relation (4.5) equals 3 for Charlier, Meixner and Krawtchouk polynomials, and 4 for Hahn polynomials.

Proof. Let $f(x) = P_{n-1}^{(1)}(x)$. By virtue of (4.2), we have the identity

$$b_k[\boldsymbol{L}_n^*f(\boldsymbol{x})] = \kappa \, b_k[(\boldsymbol{\Delta} + \boldsymbol{\nabla})P_n(\boldsymbol{x})].$$

Applying the operator $\mathscr{T} := \mathscr{D}\lambda_k^{-1}\tilde{\mathscr{D}}$ to both sides of the above identity, and using (2.11), (2.10) and (2.15), we obtain

$$\tilde{\mathscr{L}}b_k[f] = \omega(k)$$

with the operator $\tilde{\mathscr{L}}$ given in (4.6) and

$$\omega(k) := \kappa \left(\mathscr{D} + \tilde{\mathscr{D}} \right) b_k[P_n].$$

Now, notice that $b_k[P_n] = h_n \delta_{kn}$, so that $\omega(k) = 0$ for k < n-1; hence the eq. (4.5). \Box

In the subsequent subsections we assume that the associated polynomials $\{P_k^{(1)}(x)\}$ are monic.

4.1. Associated Charlier polynomials

The coefficients $a_{n-1,k}^{(1)}$ in the formula

$$C_{n-1}^{(1)}(x;a) = \sum_{k=0}^{n-1} a_{n-1,k}^{(1)} C_k(x;a)$$

satisfy the third-order formula

$$(n+k)a_{n-1,k-1}^{(1)} + k(n+2k+1)a_{n-1,k}^{(1)} + (k)_2(k+2a+1)a_{n-1,k+1}^{(1)} + a(k)_3a_{n-1,k+2}^{(1)} = 0 \qquad (1 \le k \le n-1)$$

with the initial conditions $a_{n-1,n-1}^{(1)} = 1$, $a_{n-1,n}^{(1)} = a_{n-1,n+1}^{(1)} = 0$.

4.2. Associated Meixner polynomials

The coefficients $a_{n-1,k}^{(1)}$ in the formula

$$M_{n-1}^{(1)}(x;\beta,c) = \sum_{k=0}^{n-1} a_{n-1,k}^{(1)} M_k(x;\beta,c)$$

satisfy the third-order formula

$$(c-1)^{3}(n+k)a_{n-1,k-1}^{(1)} - (c-1)^{2}(c+1)k(n+2k+1)a_{n-1,k}^{(1)}$$
$$+(c-1)(k)_{2}[(c^{2}+3c+1)(k+1) + c(n+2\beta-1)]a_{n-1,k+1}^{(1)}$$
$$-(c)_{2}(k)_{3}(k+\beta+1)a_{n-1,k+2}^{(1)} = 0 \quad (1 \le k \le n-1)$$

with the initial conditions $a_{n-1,n-1}^{(1)} = 1$, $a_{n-1,n}^{(1)} = a_{n-1,n+1}^{(1)} = 0$.

4.3. Associated Krawtchouk polynomials

The coefficients $a_{n-1,k}^{(1)}$ in the formula

$$K_{n-1}^{(1)}(x; p, N) = \sum_{k=0}^{n-1} a_{n-1,k}^{(1)} K_k(x; p, N)$$

satisfy the third-order formula

$$(n+k)a_{n-1,k-1}^{(1)} - (2p-1)k(n+2k+1)a_{n-1,k}^{(1)}$$
$$-(k)_2[(p-1)_2(2N-n-5k-4) - (k+1)]a_{n-1,k+1}^{(1)}$$
$$+(p-1)_2(2p-1)(k)_3(N-k)a_{n-1,k+2}^{(1)} = 0 \quad (1 \le k \le n-1)$$

with the initial conditions $a_{n-1,n-1}^{(1)} = 1$, $a_{n-1,n}^{(1)} = a_{n-1,n+1}^{(1)} = 0$.

4.4. Associated symmetric Hahn polynomials

The coefficients $d_{n-1,k}^{(1)}$ in the formula

$$Q_{n-1}^{(1)}(x;\alpha,\alpha,N) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d_{n-1,k}^{(1)} Q_{n-2k-1}(x;\alpha,\alpha,N)$$

satisfy the second-order formula

$$B_0(k)d_{n-1,k-1}^{(1)} + B_1(k)d_{n-1,k}^{(1)} + B_2(k)d_{n-1,k+1}^{(1)} = 0 \qquad (0 \le k \le \lfloor (n-1)/2 \rfloor - 1),$$

where

$$B_{0}(k) = 2(n - 2k - 2)_{4}(4\kappa^{2} - 9)(\kappa - 1)_{3}^{2}(2k - 2\alpha - 1)(\kappa + k + 2)$$

$$\times (N - n + 2k - 1)_{2}(\kappa + N + \alpha + 1)_{2},$$

$$B_{1}(k) = 8(n - 2k - 2)_{2}\kappa^{4}(\kappa - 1)^{2}(2\kappa - 3)(2\kappa - 1)(2\kappa + 1)^{2}$$

$$\times \{(\alpha)_{2}[4(N + \alpha)^{2} - (2\alpha - 1)(2\alpha + 3) - 1] - (\kappa - \alpha)(\kappa + 1)$$

$$\times [(2\alpha + 1)(2k - N + \alpha + 2) + (2k + 1)(2n - 2k - 1) - N(N - 1)]$$

$$-n(n + 2\alpha + 1)[(N + \alpha)^{2} + (\alpha - 1)(\alpha + 2) - 1]\},$$

$$B_{2}(k) = 32(2\kappa + 3)(2\kappa + 1)^{3}(2\kappa - 1)^{2}\kappa^{6}(n - k - 1)(2k + 2\alpha + 3),$$

with $\kappa := n - 2k + \alpha - 1$, and the initial conditions $d_{n-1,0}^{(1)} = 1$, $d_{n-1,-1}^{(1)} = 0$.

Appendix

Data for the monic Charlier and Hahn polynomials				
	Charlier	Hahn		
	$C_k(x;a)$	$Q_k(x; \alpha, \beta, N)$		
	(a > 0)	$(\alpha, \beta > -1, N \in \mathbb{Z}^+)$		
σ	x	$x(N+\alpha-x)$		
τ	a-x	$(\beta+1)(N-1)-(\gamma+1)x$		
λ_k	k	$k(k+\gamma)$		
X	$ak\mathscr{E}^{-1} + (k+a)\mathscr{I} + \mathscr{E}$	$\frac{k(N-k)(k+\alpha)(k+\beta)(k+\gamma-1)(k+\gamma+N-1)}{(2k+\gamma-2)_2(2k+\gamma-1)_2} \mathscr{E}^{-1}$		
		$+\left\{\frac{\alpha-\beta+2N-2}{4}+\frac{(\beta^2-\alpha^2)(\gamma+2N-1)}{4(2k+\gamma-1)(2k+\gamma+1)}\right\}\mathscr{I}+\mathscr{E}$		
D	$ak \mathscr{E}^{-1}$	$\frac{k(k+\alpha)(k+\beta)(k+\gamma-1)_2(N-k)(k+\gamma+N-1)}{(2k+\gamma-2)_2(2k+\gamma-1)_2} \mathscr{E}^{-1}$		
		$-\frac{k(k+\gamma)[2k(k+\gamma)+(\gamma-\alpha)(\gamma-1)-N(\alpha-\beta)]}{(2k+\gamma-1)(2k+\gamma+1)}\mathscr{I}-k\mathscr{E}$		
h _k	$k!a^k$	$\frac{k!\Gamma(k+\alpha+1)\Gamma(k+\beta+1)(2k+\gamma+1)_{N-k-1}}{(k+\gamma)_k(N-k-1)!}$		

Table 1 Data for the monic Charlier and Hahn polynomials

Note: $\gamma := \alpha + \beta + 1$.

Data for the monic Meixner and Krawtchouk polynomials					
	Meixner	Krawtchouk			
	$M_k(x;\beta,c)$	$K_k(x; p, N)$			
	$(\beta > 0, c \in (0, 1))$	$(p \in (0,1), N \in \mathbb{Z}^+)$			
σ	x	x			
τ	$\beta c + (c-1)x$	$(1-p)^{-1}(Np-x)$			
λĸ	(1-c)k	$(1-p)^{-1}k$			
X	$\frac{ck(k+\beta-1)}{(1-c)^2}\mathscr{E}^{-1}$	$p(1-p)k(N-k+1)\mathscr{E}^{-1}$			
	$+\frac{[(c+1)k+\beta c]}{1-c}\mathscr{I}+\mathscr{E}$	$+[k+p(N-2k)]\mathcal{I}+\mathcal{E}$			
D	$\frac{ck(1-\beta-k)}{c-1}\mathscr{E}^{-1}+ck\mathscr{I}$	$pk(1+N-k)\mathscr{E}^{-1}-p(1-p)^{-1}k\mathscr{I}$			
h_k	$\frac{k!(\beta)_k c^k}{(1-c)^{\beta+2k}}$	$\frac{N!k!}{(N-k)!}p^k(1-p)^k$			

Table	e 2					
Data	for the	monio	Mairman	and	Vroutchouk	nolumorniala

References

- R. Askey, Orthogonal Polynomials and Special Functions, Regional Conf. Ser. Appl. Math. 21 (SIAM, Philadelphia, PA, 1975).
- [2] N.M. Atakishiyev, A. Ronveaux and K.B. Wolf, Difference equation for the associated polynomials on the linear lattice, Zh. Teoret. Mat. Fiz. 106 (1996) 76-83.
- [3] B.W. Char, K.O. Geddes, G.H. Gonnet, B.L. Leong, M.B. Monagan and S.M. Watt, Maple V Language Reference Manual (Springer, New York, 1991).
- [4] T.S. Chihara, An Introduction to Orthogonal Polynomials (Gordon and Breach, New York, 1978).
- [5] G. Gasper, Projection formulas for orthogonal polynomials of a discrete variable, J. Math. Anal. Appl. 45 (1974) 176–198.
- [6] R. Koekoek and R.F. Swarttouw, The Askey scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 94-05, Fac. Techn. Math. Informatics, Delft Univ. of Technology, Delft, 1994.
- [7] S. Lewanowicz, Recurrence relations for hypergeometric functions of unit argument, *Math. Comput.* **45** (1985) 521–535. Errata *ibid.* **48** (1987) 853.
- [8] S. Lewanowicz, Results on the associated classical orthogonal polynomials, in: Orthogonality, Moment Problems and Continued Fractions, Proc. Internat. Conf. in Honour of T.J. Stieltjes Jr., Delft, Netherlands, 31 October-4 November 1994; J. Comput. Appl. Math. 65 (1995) 215-231.
- [9] S. Lewanowicz, Second-order recurrence relation for the linearization coefficients of the classical orthogonal polynomials, J. Comput. Appl. Math. 69 (1996) 159-170.
- [10] A.F. Nikiforov, S.K. Suslov and V.B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable* (Springer, Berlin, 1991).
- [11] A. Ronveaux, S. Belmehdi, A. Zarzo and E. Godoy, Recurrence relations for connection coefficients: classical discrete orthogonal polynomials, Centre de Recherches Mathématiques Proc. and Lecture Notes, to appear.
- [12] A. Ronveaux, A. Zarzo and E. Godoy, Recurrence relations for connection coefficients between two families of orthogonal polynomials, J. Comput. Appl. Math. 62 (1995) 67-73.