International Journal of Mathematics and Mathematical Sciences
Volume 2012 (2012), Article ID 817157, 14 pages
http://dx.doi.org/10.1155/2012/817157

Abstract
Full-Text PDF
Full-Text HTML
Full-Text ePUB
Full-Text XML
Linked References
How to Cite this Article
Complete Special Issue

Research Article

Generalized (q, w)-Euler Numbers and Polynomials Associated with p-Adic q-Integral on $\mathbb{Z}_p$

H. Y. Lee, N. S. Jung, J. Y. Kang, and C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea

Received 18 June 2012; Accepted 4 October 2012

Academic Editor: A. Bayad

Abstract

We generalize the Euler numbers and polynomials by the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We observe an interesting phenomenon of "scattering" of the zeros of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ in complex plane.

1. Introduction

Recently, many mathematicians have studied in the area of the Euler numbers and polynomials (see [1–15]). The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [14], we introduced that Euler equation $E_n(x) = 0$ has symmetrical roots for $x = 1/2$ (see [14]). It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ in complex plane. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Compared with [1, 4, 5]. Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer, and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned} X &= \varprojlim_{N} \left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} \left(a + dp\mathbb{Z}_p\right), \\ a + dp^N\mathbb{Z}_p &= \left\{x \in X \mid x \equiv a \left(\bmod dp^N\right)\right\}, \end{aligned} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. For any positive integer N,

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{[dp^N]_q} \tag{1.3}$$

is known to be a distribution on X, compared with [1–10, 14]. For

$$g \in UD\left(\mathbb{Z}_p\right) = \left\{g \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\right\}. \tag{1.4}$$

Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{0\le x<p^N} g(x)(-q)^x. \tag{1.5}$$

From (1.5), we also obtain

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0), \tag{1.6}$$

where $g_1(x) = g(x+1)$ (see [1–3]).

From (1.6), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \tag{1.7}$$

where $g_n(x) = g(x+n)$.

As well-known definition, the Euler polynomials are defined by

$$\begin{aligned} F(t) &= \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \\ F(t,x) &= \frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \end{aligned} \tag{1.8}$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers (cf. [1–15]).

Our aim in this paper is to define the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. We investigate some properties which are related to the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. Especially, distribution of roots for $E_{n,q,w}(x:a) = 0$ is different from $E_n(x) = 0$'s. We also derive the existence of a specific interpolation function which interpolate the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$.

2. The Generalized (q,w)-Euler Numbers and Polynomials

Our primary goal of this section is to define the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. We also find generating functions of the generalized (q,w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x:a)$. Let a be strictly positive real number.

The generalized (q,w)-Euler numbers and polynomials $E_{n,q,w}(a)$, $E_{n,q,w}(x:a)$ are defined by

$$\sum_{n=0}^{\infty} E_{n,q,w}(a) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x), \tag{2.1}$$

$$\sum_{n=0}^{\infty} E_{n,q,w}(x:a) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ay} e^{(ay+x)t} d\mu_{-q}(y), \quad \text{for } t,\ w \in \mathbb{C}, \tag{2.2}$$

respectively.

From above definition, we obtain

$$\begin{aligned} E_{n,q,w}(a) &= \int_{\mathbb{Z}_p} w^{ax} (ax)^n d\mu_{-q}(x), \\ E_{n,q,w}(x:a) &= \int_{\mathbb{Z}_p} w^{ay} (x+ay)^n d\mu_{-q}(y). \end{aligned} \tag{2.3}$$

Let $g(x) = w^{ax} e^{axt}$. By (1.6) and using p-adic q-integral on $\mathbb{Z}_p$, we have

$$\begin{aligned} qI_{-q}(g_1) + I_{-q}(g) &= \int_{\mathbb{Z}_p} w^{a(x+1)} e^{a(x+1)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\ &= \left(qw^a e^{at} + 1\right) \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\ &= [2]_q. \end{aligned} \tag{2.4}$$

Hence, by (2.1), we obtain

$$\sum_{n=0}^{\infty} E_{n,q,w}(a) \frac{t^n}{n!} = \frac{[2]_q}{qw^a e^{at} + 1}. \tag{2.5}$$

By (1.6), (2.2) and $g(y)=w^{ay}e^{(ay+x)t}$, we have

$$\sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}=\frac{[2]_q}{qw^a e^{at}+1}e^{xt}. \tag{2.6}$$

After some elementary calculations, we obtain

$$\sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}=[2]_q\sum_{n=0}^{\infty}(-1)^n q^n w^{an}e^{ant}e^{xt}. \tag{2.7}$$

From (2.6), we have

$$\begin{aligned}E_{n,q,w}(x:a)&=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}E_{k,q,w}(a)\\&=\left(x+E_{q,w}(a)\right)^n,\end{aligned} \tag{2.8}$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

3. Basic Properties for the Generalized (q,w)-Euler Numbers and Polynomials

By (2.5), we have

$$\begin{aligned}\frac{\partial}{\partial x}\sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}&=\frac{\partial}{\partial x}\left(\frac{[2]_q}{qw^a e^{at}+1}e^{xt}\right)\\&=t\sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}nE_{n-1,q,w}(x:a)\frac{t^n}{n!}.\end{aligned} \tag{3.1}$$

By (3.1), we have the following differential relation.

Theorem 3.1. *For positive integers n, one has*

$$\frac{\partial}{\partial x}E_{n,q,w}(x:a)=nE_{n-1,q,w}(x:a). \tag{3.2}$$

By Theorem 3.1, we easily obtain the following corollary.

Corollary 3.2 (integral formula). *Consider that*

$$\int_p^q E_{n-1,q,w}(x:a)\,dx=\frac{1}{n}\left(E_{n,q,w}(q:a)-E_{n,q,w}(p:a)\right). \tag{3.3}$$

By (2.5), one obtains

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,q,w}(x+y:a)\frac{t^n}{n!}&=\frac{[2]_q}{qw^a e^{at}+1}e^{(x+y)t}\\&=\sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}\sum_{k=0}^{\infty}y^k\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}(x:a)\,y^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.4}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following addition theorem.

Theorem 3.3 (addition theorem). *For $n\in\mathbb{Z}_+$,*

$$E_{n,q,w}(x+y:a)=\sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}(x:a)\,y^{n-k}. \tag{3.5}$$

By (2.5), for $m\equiv 1 (\mathrm{mod}\ 2)$, one has

$$\sum_{n=0}^{\infty}\left(m^n\frac{[2]_q}{[2]_{q^m}}\sum_{k=0}^{m-1}(-1)^k q^k w^{ak}E_{n,q^m,w^m}\left(\frac{x+ak}{m}:a\right)\right)\frac{t^n}{n!}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-1}(-1)^k q^k w^{ak}\left(\sum_{n=0}^{\infty}E_{n,q^m,w^m}\left(\frac{x+ak}{m}:a\right)\right)\frac{(mt)^n}{n!}\\
&= \sum_{k=0}^{m-1}\left((-1)^k q^k w^{ak}\frac{[2]_q}{q^m w^{ma}e^{mat}+1}e^{(x+ak)t}\right)\\
&= \frac{[2]_q}{1+qw^a e^{at}}e^{xt}\\
&= \sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}.
\end{aligned}
\tag{3.6}
$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following multiplication theorem.

Theorem 3.4 (multiplication theorem). *For $m,n\in\mathbb{N}$*

$$
E_{n,q,w}(x:a) = m^n\frac{[2]_q}{[2]_{q^m}}\sum_{k=0}^{m-1}(-1)^k q^k w^{ak}E_{n,q^m,w^m}\left(\frac{x+ak}{m}:a\right).
\tag{3.7}
$$

From (1.6), one notes that

$$
\begin{aligned}
[2]_q &= \int_{\mathbb{Z}_p} qw^{ax+a}e^{(ax+a)t}d\mu_{-q}(x)+\int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-q}(x)\\
&= \sum_{n=0}^{\infty}\left(qw^a\int_{\mathbb{Z}_p} w^{ax}(ax+a)^n d\mu_{-q}(x)+\int_{\mathbb{Z}_p} w^{ax}(ax)^n d\mu_{-q}(x)\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(qw^aE_{n,q,w}(a:a)+E_{n,q,w}(a)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{3.8}
$$

From the above, we obtain the following theorem.

Theorem 3.5. *For $n\in\mathbb{Z}_+$, we have*

$$
qw^aE_{n,q,w}(a:a)+E_{n,q,w}(a) = \begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0.\end{cases}
\tag{3.9}
$$

By (2.8) in the above, we arrive at the following corollary.

Corollary 3.6. *For $n\in\mathbb{Z}_+$, one has*

$$
qw^a\left(a+E_{q,w}(a)\right)^n+E_{n,q,w}(a) = \begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases}
\tag{3.10}
$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

From (1.7), one notes that

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l w^{al}(al)^m\right)\frac{t^n}{m!}\\
&= q^n\int_{\mathbb{Z}_p} w^{ax+an}e^{(ax+an)t}d\mu_{-q}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-q}(x)\\
&= \sum_{m=0}^{\infty}\left(q^nw^{an}\int_{\mathbb{Z}_p} w^{ax}(ax+an)^m d\mu_{-q}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p} w^{ax}(ax)^m d\mu_{-q}(x)\right)\frac{t^m}{m!}\\
&= \sum_{m=0}^{\infty}\left(q^nw^{an}E_{m,w}(an:a)+(-1)^{n-1}E_{m,w}(a)\right)\frac{t^m}{m!}.
\end{aligned}
\tag{3.11}
$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following theorem.

Theorem 3.7. *For $n\in\mathbb{Z}_+$, one has*

$$
q^nw^{an}E_{m,w}(na:a)+(-1)^{n-1}E_{m,w}(a) = [2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}w^{al}q^l(al)^m.
\tag{3.12}
$$

4. The Analogue of the q-Euler Zeta Function

By using the generalized (q,w)-Euler numbers and polynomials, the generalized (q,w)-Euler zeta function and the generalized Hurwitz (q,w)-Euler zeta functions are defined. These functions interpolate the generalized (q,w)-Euler numbers and (q,w)-Euler polynomials, respectively. Let

$$F_{q,w}(x:a)(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} e^{ant} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}(x:a) \frac{t^n}{n!}. \tag{4.1}$$

By applying derivative operator, $d^k/dt^k|_{t=0}$ to the above equation, we have

$$\left.\frac{d^k}{dt^k} F_{q,w}(x:a)(t)\right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an+x)^k, \quad (k \in \mathbb{N}), \tag{4.2}$$

$$E_{k,q,w}(x:a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an+x)^k. \tag{4.3}$$

By using the above equation, we are now ready to define the generalized (q,w)-Euler zeta functions.

Definition 4.1. For $s \in \mathbb{C}$, one defines

$$\zeta_{q,w}^{(a)}(x:s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^{an}}{(an+x)^s}. \tag{4.4}$$

Note that $\zeta_w^{(a)}(x,s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $w \to 1, w \to 1$, and $a = 1$, then $\zeta_{q,w}^{(a)}(x:s) = \zeta(x:s)$ which is the Hurwitz Euler zeta functions. Relation between $\zeta_w^{(a)}(x:s)$ and $E_{k,w}(x:a)$ is given by the following theorem.

Theorem 4.2. *For $k \in \mathbb{N}$, one has*

$$\zeta_{q,w}^{(a)}(x:-k) = E_{k,w}(x:a). \tag{4.5}$$

By using (4.2), one notes that

$$\left.\frac{d^k}{dt^k} F_{q,w}(0:a)(t)\right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an)^k, \quad (k \in \mathbb{N}). \tag{4.6}$$

Hence, one obtains

$$E_{k,q,w}(a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an)^k. \tag{4.7}$$

By using the above equation, one is now ready to define the generalized Hurwitz (q,w)-Euler zeta functions.

Definition 4.3. Let $s \in \mathbb{C}$. One defines

$$\zeta_{q,w}^{(a)}(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^{an}}{(an)^s}. \tag{4.8}$$

Note that $\zeta_{q,w}^{(a)}(s)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $w \to 1, q \to 1$, and $a = 1$, then $\zeta_w^{(a)}(s) = \zeta(s)$ which is the Euler zeta functions. Relation between $\zeta_w^{(a)}(s)$ and $E_{k,w}(s)$ is given by the following theorem.

Theorem 4.4. *For $k \in \mathbb{N}$, one has*

$$\zeta_{q,w}^{(a)}(-k) = E_{k,q,w}(a). \tag{4.9}$$

5. Zeros of the Generalized (q,w)-Euler Polynomials $E_{n,q,w}(x:a)$

In this section, we investigate the reflection symmetry of the zeros of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$.

In the special case, $w = 1$ and $q \to 1$, $E_{n,q,w}(x:a)$ are called generalized Euler polynomials $E_n(x:a)$. Since

$$\begin{aligned} &\sum_{n=0}^{\infty} E_n(a-x:a) \frac{(-t)^n}{n!} \\ &= \frac{2}{e^{-at}+1} e^{(a-x)(-t)} \\ &= \frac{2}{e^{at}+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x:a) \frac{t^n}{n!}, \end{aligned} \tag{5.1}$$

we have

$$E_n(x:a) = (-1)^n E_n(a-x:a) \quad \text{for } n \in \mathbb{N}. \tag{5.2}$$

We observe that $E_n(x:a), x \in \mathbb{C}$ has $\operatorname{Re}(x) = a/2$ reflection symmetry in addition to the usual $\operatorname{Im}(x) = 0$ reflection symmetry analytic complex functions.

Let

$$F_{q,w}(x:t) = \frac{[2]_q}{qw^a e^{at}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!}. \tag{5.3}$$

Then, we have

$$\begin{aligned} F_{q^{-1},w^{-1}}(a-x:-t) &= \frac{[2]_{q^{-1}}}{q^{-1}w^{-a}e^{-at}+1} e^{(a-x)(-t)} \\ &= w^a \frac{[2]_q}{qw^a e^{at}+1} e^{xt} \\ &= w^a \sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!}. \end{aligned} \tag{5.4}$$

Hence, we arrive at the following complement theorem.

Theorem 5.1 (complement theorem). *For* $n \in \mathbb{N}$,

$$E_{n,q^{-1},w^{-1}}(a-x:a) = (-1)^n w^a E_{n,q,w}(x:a). \tag{5.5}$$

Throughout the numerical experiments, we can finally conclude that $E_{n,q,w}(x:a), x \in \mathbb{C}$ has not $\operatorname{Re}(x) = a/2$ reflection symmetry analytic complex functions. However, we observe that $E_{n,q,w}(x:a), x \in \mathbb{C}$ has $\operatorname{Im}(x) = 0$ reflection symmetry (see Figures 1, 2, and 3). The obvious corollary is that the zeros of $E_{n,q,w}(x:a)$ will also inherit these symmetries.

$$\text{If } E_{n,q,w}(x_0:a) = 0, \ \text{ then } E_{n,q,w}(x_0^*:a) = 0, \tag{5.6}$$

where $*$ denotes complex conjugation (see Figures 1, 2, and 3).

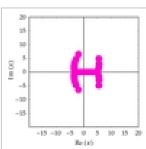

Figure 1: Zeros of $E_{n,q,w}(x:a)$ for $a = 1, 2, 3, 4$.

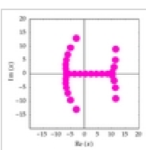

Figure 2: Zeros of $E_{n,q,w}(x:a)$ for $q = 1/10, 3/10, 7/10, 9/10$.

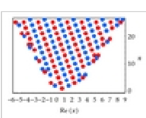

Figure 3: Real zeros of $E_{n,q,w}(x:a)$ for $1 \le n \le 25$.

We investigate the beautiful zeros of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ by using a computer. We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x:a)$ for $n = 30, a = 1, 2, 3, 4$, and $x \in \mathbb{C}$ (Figure 1). In Figure 1 (top-left), we choose $n = 30, q = 1/2, w = 1$, and $a = 1$. In Figure 1 (top-right), we choose $n = 30, q = 1/2, w = 2$, and $a = 2$. In Figure 1 (bottom-left), we choose $n = 30, q = 1/2, w = 3$, and $a = 3$. In Figure 1 (bottom-right), we choose $n = 30, q = 1/2, w = 4$, and $a = 4$.

We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x:a)$ for $n = 30, a = 2, w = 2$, and $x \in \mathbb{C}$ (Figure 2).

In Figure 2 (top-left), we choose $n = 30, q = 1/10, w = 2$, and $a = 2$. In Figure 2 (top-right), we choose $n = 30, q = 3/10, w = 2$, and $a = 2$. In Figure 2 (bottom-left), we choose $n = 30, q = 7/10, w = 2$, and $a = 2$. In Figure 2 (bottom-right), we choose $n = 30, q = 9/10, w = 2$ and $a = 2$.

Plots of real zeros of $E_{n,q,w}(x:a)$ for $1 \le n \le 25$ structure are presented (Figure 3).

In Figure 3 (top-left), we choose $q = 1/2, w = 1$, and $a = 2$. In Figure 3 (top-right), we choose $q = 1/2, w = 2$, and $a = 2$. In Figure 3 (bottom-left), we choose $q = 1/2, w = 3$, and $a = 2$. In Figure 3 (bottom-right), we choose $q = 1/2, w = 4$, and $a = 2$.

Stacks of zeros of $E_{n,q,w}(x:a)$ for $1 \le n \le 30, q = 1/2, w = 4$, and $a = 4$ from a 3-D structure are presented (Figure 4).

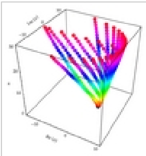

Figure 4: Stacks of zeros of $E_{n,q,w}(x:a)$ for $1 \le n \le 30$.

Our numerical results for approximate solutions of real zeros of the generalized $E_{n,q,w}(x:a)$ are displayed (Tables 1 and 2).

Table 1: Numbers of real and complex zeros of $E_{n,q,w}(x:a)$.

Table 2: Approximate solutions of $E_{n,q,w}(x:a)=0, x \in \mathbb{R}$.

We observe a remarkably regular structure of the complex roots of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$. We hope to verify a remarkably regular structure of the complex roots of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ (Table 1).

Next, we calculated an approximate solution satisfying $E_{n,q,w}(x:a), q=1/2, w=2, a=2, x \in \mathbb{R}$. The results are given in Table 2.

Figure 5 shows the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ for real $-9/10 \le q \le 9/10$ and $-5 \le x \le 5$, with the zero contour indicated in black (Figure 5). In Figure 5 (top-left), we choose $n=1$, $w=2$, and $a=2$. In Figure 5 (top-right), we choose $n=2$, $w=2$, and $a=2$. In Figure 5 (bottom-left), we choose $n=3$, $w=2$, and $a=2$. In Figure 5 (bottom-right), we choose $n=4$, $w=2$, and $a=2$.

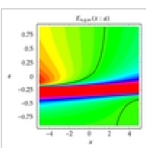

Figure 5: Zero contour of $E_{n,q,w}(x:a)$.

Finally, we will consider the more general problems. How many roots does $E_{n,q,w}(x:a)$ have? This is an open problem. Prove or disprove: $E_{n,q,w}(x:a)=0$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q,w}(x:a)}$ of $E_{n,q,w}(x:a), \mathrm{Im}(x:a) \ne 0$. Since n is the degree of the polynomial $E_{n,q,w}(x:a)$, the number of real zeros $R_{E_{n,q,w}(x:a)}$ lying on the real plane $\mathrm{Im}(x:a)=0$ is then $R_{E_{n,q,w}(x:a)} = n - C_{E_{n,q,w}(x:a)}$, where $C_{E_{n,q,w}(x:a)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q,w}(x:a)}$ and $C_{E_{n,q,w}(x:a)}$. We plot the zeros of $E_{n,q,w}(x:a)$, respectively (Figures 1–5). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $E_{n,q,w}(x:a)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ to appear in mathematics and physics.

References

1. M. Acikgoz and Y. Simsek, "On multiple interpolation functions of the Nörlund-type q-Euler polynomials," *Abstract and Applied Analysis*, vol. 2009, Article ID 382574, 14 pages, 2009. View at Publisher · View at Google Scholar
2. A. Bayad, "Modular properties of elliptic Bernoulli and Euler functions," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010. View at Google Scholar
3. M. Cenkci, M. Can, and V. Kurt, "p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 2, pp. 203–216, 2004. View at Google Scholar
4. L. Jang, "A note on Nörlund-type twisted q-Euler polynomials and numbers of higher order associated with fermionic invariant q-integrals," *Journal of Inequalities and Applications*, vol. 2010, Article ID 417452, 12 pages, 2010. View at Publisher · View at Google Scholar
5. T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007. View at Publisher · View at Google Scholar
6. T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002. View at Google Scholar · View at Zentralblatt MATH
7. T. Kim, "q-Euler numbers and polynomials associated with p-adic q-integrals," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007. View at Publisher · View at Google Scholar
8. T. Kim, J. Choi, Y-.H. Kim, and C. S. Ryoo, "A Note on the weighted p-adic q-Euler measure on $\mathbb{Z}$p," *Advanced Studies in Contemporary*

Mathematics, vol. 21, pp. 35–40, 2011. View at Google Scholar

9. B. A. Kupershmidt, “Reflection symmetries of q-Bernoulli polynomials,” *Journal of Nonlinear Mathematical Physics*, vol. 12, supplement 1, pp. 412–422, 2005. View at Publisher · View at Google Scholar
10. E.-J. Moon, S.-H. Rim, J.-H. Jin, and S.-J. Lee, “On the symmetric properties of higher-order twisted q-Euler numbers and polynomials,” *Advances in Difference Equations*, vol. 2010, Article ID 765259, 8 pages, 2010. View at Google Scholar
11. H. Ozden, Y. Simsek, and I. N. Cangul, “Euler polynomials associated with p-adic q-Euler measure,” *General Mathematics*, vol. 15, no. 2, pp. 24–37, 2007. View at Google Scholar
12. C. S. Ryoo, T. Kim, and L.-C. Jang, “Some relationships between the analogs of Euler numbers and polynomials,” *Journal of Inequalities and Applications*, vol. 2007, Article ID 86052, 22 pages, 2007. View at Publisher · View at Google Scholar · View at Zentralblatt MATH
13. C. S. Ryoo, “A note on the weighted q-Euler numbers and polynomials,” *Advanced Studies in Contemporary Mathematics*, vol. 21, pp. 47–54, 2011. View at Google Scholar
14. C. S. Ryoo and Y. S. Yoo, “A note on Euler numbers and polynomials,” *Journal of Concrete and Applicable Mathematics*, vol. 7, no. 4, pp. 341–348, 2009. View at Google Scholar · View at Zentralblatt MATH
15. Y. Simsek, O. Yurekli, and V. Kurt, “On interpolation functions of the twisted generalized Frobinuous-Euler numbers,” *Advanced Studies in Contemporary Mathematics*, vol. 14, pp. 49–68, 2007. View at Google Scholar