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On the Jacobsthal-Lucas Numbers by Matrix Method[1]

Fikri Köken and Durmuş Bozkurt

Selçuk University, Faculty of Art and Science
Department of Mathematics, 42075, Kampus, Konya, Turkey
kokenfikri@gmail.com, dbozkurt@selcuk.edu.tr

Abstract

In this study, we define the Jacobsthal Lucas E-matrix and R-matrix alike to the Fibonacci Q-matrix. Using this matrix represantation we have found some equalities and Binet-like formula for the Jacobsthal and Jacobsthal-Lucas numbers.

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1 Introduction

Fibonacci and Lucas numbers and their generalization have many interesting properties and applications in almost every field of science and art. For the prettiness and rich applications of these numbers and their relatives one can see science and the nature [3-7].

In 1960, Charles H. King studied on the following Q-matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

in his Ms thesis. He showed below

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \text{ and } \det(Q) = -1.$$

Moreover, it is clearly shown below

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \text{ (Cassini's formula)}.$$

[1]This study is the part of Ms Thesis of F. Köken.

The above equalities demonstrate that there is a very close link between the matrices and Fibonacci numbers [7].

We have defined Jacobsthal F-matrix given by

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \tag{1}$$

in our article [2] and have demonstrated a very close link between this matrix and Jacobsthal numbers.

More generally, there are some relations between the integer sequences and matrices [5-10].

In [1], the Jacobsthal and the Jacobsthal-Lucas sequences J_n and j_n are defined by the recurrence relations

$$J_0 = 0,\ J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2$$

$$j_0 = 2,\ j_1 = 1,\ j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2$$

respectively.

In this study, we have defined Jacobsthal-Lucas E-matrix by

$$E = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}. \tag{2}$$

It is feasible that, it can be written

$$\begin{bmatrix} j_{n+2} \\ j_{n+1} \end{bmatrix} = E \begin{bmatrix} J_{n+1} \\ J_n \end{bmatrix} \text{ and } 9 \begin{bmatrix} J_{n+2} \\ J_{n+1} \end{bmatrix} = \frac{1}{9}\ E\ \begin{bmatrix} j_{n+1} \\ j_n \end{bmatrix}$$

where J_n and j_n are the nth Jacobsthal and Jacobsthal Lucas numbers, respectively.

Furthermore, we have defined Jacobsthal-Lucas R-matrix by

$$R = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}. \tag{3}$$

Throughout this paper, J_n and j_n denote the nth Jacobsthal and Jacobsthal-Lucas numbers.

2 The Matrix Representation

In the section, we will get some properties of the Jacobsthal-Lucas E-matrix and R-matrix. Moreover, using these matrices, we have obtained the Cassini-like and the Binet-like formulas for the Jacobsthal and Jacobsthal Lucas numbers.

Theorem 1 *Let E be a matrix as in (2). Then, for all positive integers n, the following matrix power is held below*

$$E^n = \begin{cases} 3^n \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}, & \text{if } n \text{ even} \\ 3^{n-1} \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix}, & \text{if } n \text{ odd.} \end{cases} \quad (4)$$

Proof. We will use the Principle of Mathematical Induction (PMI) on odd and even n, seperately. (For odd n) When $n = 1$,

$$E^1 = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} j_2 & 2j_1 \\ j_1 & 2j_0 \end{bmatrix},$$

is correct. We assume that it is correct for odd $n = k$. Now, we show that it is correct for $n = k + 2$. We can write

$$E^{k+2} = E^k E^2 = 3^{k+1} \begin{bmatrix} j_{k+3} & 2j_{k+2} \\ j_{k+2} & 2j_{k+1} \end{bmatrix}$$

and the result is easily seen immediately.

(For even n) When $n = 2$,

$$E^2 = \begin{bmatrix} 27 & 18 \\ 9 & 18 \end{bmatrix} = 3^2 \begin{bmatrix} J_3 & 2J_2 \\ J_2 & 2J_1 \end{bmatrix}$$

is correct. We assume it is correct for even $n = k$. Finally, we show that it is correct for $n = k + 2$. We calculate

$$E^{k+2} = E^k E^2 = 3^{k+2} \begin{bmatrix} J_{k+3} & 2J_{k+2} \\ J_{k+2} & 2J_{k+1} \end{bmatrix}.$$

■

Corollary 2 *Let E^n be as in (4). For all positive integers n, the following determinantal equalities are held:*

i) $\det(E^n) = 3^{2n} \cdot 2^n$,

ii) $J_{n+1}J_{n-1} - J_n^2 = (-1)^n \cdot 2^{n-1}$,

iii) $j_{n+1}j_{n-1} - j_n^2 = (-1)^{n+1} \cdot 3^2 \cdot 2^{n-1}$.

Proof. We will use the PMI on n. When $n = 1$, it is seen to be true. We assume that it is true for $n = k$. We will show that it is true for $n = k + 1$.

$$\det(E^{k+1}) = \det(E^k)\det(E) = 3^{2k+2} \cdot 2^{k+1}.$$

and the proof of (i) is completed.

The proof of (ii) and (iii) are easily shown from (i) and (4). ■

Theorem 3 *Let n be an integer. The Binet-like formulas of the Jacobsthal and Jacobsthal Lucas numbers are*

$$J_n = \frac{2^n - (-1)^n}{3} \text{ and } j_n = 2^n + (-1)^n.$$

Proof. If we calculate the eigenvalues and eigenvectors of the *E-matrix*, they are allotted

$$\lambda_1 = 6,\ \lambda_2 = 3 \text{ and } v_1 = (-1,\ 1),\ v_2 = (2,\ 1),$$

respectively. We can diagonalize of the matrix E by

$$V = U^{-1}EU$$

where

$$U = (v_1^T, v_2^T) = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } V = diag(\lambda_1, \lambda_2) = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}.$$

Thus, applying from the properties of similar matrices, we can write that

$$E^n = UV^nU^{-1}. \tag{5}$$

By (4) and (5) matrix equations, desired results are obtained via

$$E^n = 3^{n-1} \begin{bmatrix} 2^{n+1}+1 & 2(2^n-1) \\ 2^n-1 & 2(2^{n-1}+1) \end{bmatrix}.$$

■

Theorem 4 *Let m and n be integers. Then, the following equalities are valid:*

i) $9J_{m+n} = j_m j_{n+1} + 2j_{m-1}j_n$,
ii) $J_{m+n} = J_m J_{n+1} + 2J_{m-1}J_n$,
iii) $j_{m+n} = j_n J_{m+1} + 2j_{n-1}J_m$,
iv) $9 \cdot (-1)^{n+1} \cdot 2^{n-1} J_{m-n} = j_{n-1}j_{m+1} - j_n j_m$,
v) $(-1)^n \cdot 2^{n-1} J_{m-n} = J_m J_{n-1} - J_{m-1}J_n$,
vi) $(-1)^{n+1} \cdot 2^{n-1} j_{m-n} = J_m j_{n-1} - J_{m-1}j_n$,
vii) $(-1)^{n+1} \cdot 2^n = J_n j_{n-1} - J_{n-1}j_n$.

Proof. Let the matrix E as in (2). Since

$$E^{m+n} = E^m E^n.$$

Thus, equalities *(i)*, *(ii)* and *(iii)* are easily seen.

If we compute E^{-n}, we obtain

$$E^{-n} = \begin{cases} \frac{(-1)^n}{6^n} \begin{bmatrix} 2J_{n-1} & -2J_n \\ -J_n & J_{n+1} \end{bmatrix}, & \text{for even } n, \\ \frac{(-1)^{n+1}}{3 \cdot 6^n} \begin{bmatrix} 2j_{n-1} & -2j_n \\ -j_n & j_{n+1} \end{bmatrix} & \text{for odd } n, \end{cases}$$

Since it is that

$$E^{m-n} = E^m E^{-n},$$

equalities *(iv)*, *(v)* and *(vi)* are clearly seen. For proof of (vii), it is obtained by taking $m = n$ in (i) of the Theorem 4, this completes the proof. ■

Theorem 5 *Let R be in (3). Then, there are following equalities,*

$$RF^n = \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix} \textit{ and } R \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix} = 9F^n$$

where F is the Jacobsthal F-matrix in (1).

Proof. We have recalled that $j_{n+1} = J_{n+1} + 4J_n$, $j_n = 2J_{n+1} - J_n$, $9J_{n+1} = j_{n+1} + 4j_n$ and $9J_n = 2j_{n+1} - j_n$ in [1]. Using these equations, we can easly see that this relation between R-matrix and E-matrix. ■

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