GENERALIZATIONS OF THE CLASSICAL LAGUERRE POLYNOMIALS AND SOME q-ANALOGUES

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GENERALIZATIONS OF THE CLASSICAL LAGUERRE POLYNOMIALS AND SOME q-ANALOGUES

PROEFSCHRIFT

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ROELOF KOEKOEK
doctorandus in de wiskunde
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Aan Caroline

Aan mijn ouders

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Introduction

In [22] and [23] H.L. Krall studied orthogonal polynomials satisfying fourth order differential equations. Moreover, he classified all sets of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with degree$[P_n(x)] = n$ which satisfy a fourth order differential equation of the form

$$p_4(x)y^{(4)}(x) + p_3(x)y^{(3)}(x) + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = 0$$

where $\{p_i(x)\}_{i=0}^{4}$ are polynomials with degree$[p_i(x)] \leq i$ and $\{p_i(x)\}_{i=1}^{4}$ are independent of the degree n. These sets of orthogonal polynomials include those which were called the classical orthogonal polynomials at that time, namely the Legendre, Laguerre, Hermite, Bessel and Jacobi polynomials. He also found three other sets of orthogonal polynomials satisfying a fourth order differential equation of this type. In [20] A.M. Krall studied these new sets of orthogonal polynomials in more details and named them the Legendre type, Laguerre type and Jacobi type polynomials. These polynomials are generalizations of the classical Legendre, Laguerre and Jacobi polynomials in the sense that the weight function for these orthogonal polynomials consists of the classical weight function together with a Dirac delta function at the endpoint(s) of the interval of orthogonality.

Later L.L. Littlejohn studied more generalizations of the classical Legendre polynomials in this way and named them after H.L. Krall : the Krall polynomials. These Krall polynomials are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$\frac{1}{A}\delta(x+1) + \frac{1}{B}\delta(x-1) + C,\ A > 0,\ B > 0 \ \text{ and } \ C > 0.$$

See [24] and [25]. These polynomials do not fit in the class of polynomials which satisfy a fourth order differential equation of the above type. The Krall polynomials satisfy a sixth order differential equation of a similar form.

A.M. Krall and L.L. Littlejohn did some work on the classification of higher order differential equations having orthogonal polynomial solutions. They tried to classify all differential equations of the form

$$\sum_{i=0}^{r} p_i(x)y^{(i)}(x) = 0,\ r \in \{2, 3, 4, \ldots\},$$

where $\{p_i(x)\}_{i=0}^{r}$ are polynomials with degree$[p_i(x)] \leq i$ and $\{p_i(x)\}_{i=1}^{r}$ are independent of n having orthogonal polynomial solutions $\{P_n(x)\}_{n=0}^{\infty}$ with degree$[P_n(x)] = n$. See [26] and [21].

In [19] T.H. Koornwinder found a general class of orthogonal polynomials which generalize the Legendre type, Jacobi type and Krall polynomials. These polynomials are orthogonal on the interval $[-1,1]$ with respect to the weight function

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1).$$

As a limit case he mentioned the polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ which are orthogonal on the interval $[0,\infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha}e^{-x}+M\delta(x).$$

These polynomials generalize the Laguerre type polynomials. Many important properties of these polynomials are listed in [13].

In section 3.5 we show that these generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ satisfy a differential equation of the form

$$M\sum_{i=0}^{\infty}a_i(x)y^{(i)}(x)+xy''(x)+(\alpha+1-x)y'(x)+ny(x)=0$$

which is of infinite order in general. For $M=0$ it reduces to the differential equation for the classical Laguerre polynomials. In the nonclassical case $M>0$ only for integer values of α this differential equation is of finite order $2\alpha+4$.

Most classical orthogonal polynomials mentioned before can be generalized by constructing an inner product in the following way. Let $w(x)$ denote the weight function for the classical orthogonal polynomials we want to generalize and let $[a,b]$ be the interval of orthogonality of these polynomials, where b might be infinity in the Laguerre case. Now we consider the inner product

$$<f,g>=\int_a^b w(x)f(x)g(x)dx+\sum_{i=0}^{\infty}M_i f^{(i)}(a)g^{(i)}(a)+\sum_{j=0}^{\infty}N_j f^{(j)}(b)g^{(j)}(b) \qquad (0.0.1)$$

where $M_i\geq 0$ and $N_j\geq 0$ for all $i,j=0,1,2,\ldots$. In the Laguerre case we only have one endpoint of the interval of orthogonality, so the last sum does not appear in that case.

In [3] and [4] H. Bavinck and H.G. Meijer studied generalizations of the Jacobi polynomials in this way in the symmetric case $\alpha=\beta$ and with only first derivatives in the inner product. In fact they computed the polynomials which are orthogonal with respect to

$$<f,g>=\frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\left\{\Gamma(\alpha+1)\right\}^2}\int_{-1}^{1}\left(1-x^2\right)^{\alpha}f(x)g(x)dx+ \\ +M\left[f(-1)g(-1)+f(1)g(1)\right]+N\left[f'(-1)g'(-1)+f'(1)g'(1)\right]$$

where $\alpha>-1$, $M\geq 0$ and $N\geq 0$.

F. Marcellan and A. Ronveaux similarly studied orthogonal polynomials in a general setting in [30]. They studied inner products of the form

$$< f, g > = \int_{-\infty}^{\infty} w(x)f(x)g(x)dx + \frac{1}{\lambda} f^{(r)}(c)g^{(r)}(c)$$

where $\lambda > 0$, c is real, r is a nonnegative integer and $w(x)$ is a positive weight function defined on the real line.

It is important to realize that the inner product (0.0.1) cannot be obtained from any weight function in general. Since many properties of the classical orthogonal polynomials depend on the existence of a positive weight function we cannot expect the new generalized orthogonal polynomials to have properties such as a three term recurrence relation. Further we cannot expect the polynomial with degree n to have exactly n real and simple zeros which are located in the interior of the interval of orthogonality. This interval of orthogonality is not even defined in the general case. So it is significant to study these new families of orthogonal polynomials.

Although one can derive some results in general, see for instance [30], we have chosen to explore an explicit example in detail. So we study generalizations of the classical Laguerre polynomials which are also further generalizations of the Laguerre type and Koornwinder's generalized Laguerre polynomials . In this case we only have one endpoint of the interval of orthogonality of the classical orthogonal polynomials.

In chapter 1 we give the definition of the classical Laguerre polynomials and some of their properties. Moreover, some notations and terminology are introduced in that chapter.

In chapter 2 we give the definition and some important properties of the new generalized Laguerre polynomials . Most of its contents was published before in [15].

In chapter 3 we list some properties of Koornwinder's generalized Laguerre polynomials . Most of these properties were given in [13] too. In section 3.5 we give a proof of the differential equation of infinite order mentioned before. This is a joint work with J. Koekoek published in [10].

Chapter 4 deals with another special case of the polynomials defined in chapter 2. This is the simplest case where the inner product cannot be derived from a weight function. We give some results concerning the zeros in this and some other special cases. Some results given in this chapter were published before in a joint work with H.G. Meijer in [18].

This is part one of this thesis.

In part two we consider a q-analogue of the classical Laguerre polynomials. This q-analogue was studied in detail by D.S. Moak in [31]. This is not the only q-analogue of the Laguerre polynomials. Similar results can be found for other q-analogues. We have chosen these q-Laguerre polynomials as an example. These particular q-Laguerre polynomials satisfy two different kinds of orthogonality relations. These results are given in chapter 5.

In chapter 6 we give the definition and some properties of the generalizations of these q-Laguerre polynomials. These polynomials can be considered as q-analogues of the polynomials given in chapter 2. The results given in this chapter were published before in [16] and [17].

In chapter 7 we consider a special case which was treated before in [14]. In this chapter we give some results concerning the zeros of these polynomials which contradict possible conjectures which may arise from the results in chapter 4.

PART ONE

Chapter 1

The classical Laguerre polynomials

1.1 Some notations and terminology

A (positive) weight function $w(x)$ on an interval $[a, b]$, which might be infinite, is a non-negative integrable function for which

$$\int_a^b w(x)dx > 0.$$

The polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with degree$[p_n(x)] = n$ are called orthogonal polynomials on the interval $[a, b]$ with respect to the weight function $w(x)$ if

$$\int_a^b w(x)p_m(x)p_n(x)dx = h_n\delta_{mn}, \; h_n > 0$$

where δ_{mn} denotes the Kronecker delta defined by

$$\delta_{mn} := \begin{cases} 1 & \text{if} \quad m = n \\ \\ 0 & \text{if} \quad m \neq n \end{cases} \qquad m, n = 0, 1, 2, \ldots.$$

The interval $[a, b]$ is called the interval of orthogonality.

From now on we always work in the space of all real polynomials, i.e. polynomials with real coefficients. An inner product $<,>$ is a positive definite symmetric bilinear form. This means that for all polynomials f, g and h and for each real number λ we have

- $< f, g > = < g, f >$
- $< f, g + h > = < f, g > + < f, h >$
- $< \lambda f, g > = < f, \lambda g > = \lambda < f, g >$
- $< f, f > \geq 0$ and $< f, f > = 0$ if and only if $f = 0$

The polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with degree$[p_n(x)] = n$ are called orthogonal polynomials with respect to the inner product $<,>$ if

$$< p_m, p_n > = h_n \delta_{mn}, \; h_n > 0.$$

Given an inner product $<,>$ the polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with degree$[p_n(x)] = n$ which are orthogonal with respect to that inner product are uniquely determined except for a multiplicative constant. One set of polynomials $\{r_n(x)\}_{n=0}^{\infty}$ orthogonal with respect to this inner product can be constructed by using the Gram-Schmidt orthogonalization process as follows.

We start with the monomials $\{x^n\}_{n=0}^{\infty}$ which span the space of all polynomials. Then we define

$$r_0(x) = x^0 = 1 \;\; ; \;\; r_n(x) = x^n - \sum_{i=0}^{n-1} \frac{< x^n, r_i(x) >}{< r_i(x), r_i(x) >} r_i(x), \; n = 1, 2, 3, \ldots.$$

An induction argument easily shows that these polynomials $\{r_n(x)\}_{n=0}^{\infty}$ are orthogonal with respect to the inner product $<,>$. Every other set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with degree$[p_n(x)] = n$ orthogonal with respect to this inner product $<,>$ satisfies the relation $p_n(x) = C_n r_n(x)$ for all n where $\{C_n\}_{n=0}^{\infty}$ are nonzero constants. These constants are often chosen such that the polynomials $\{p_n(x)\}_{n=0}^{\infty}$ all satisfy the same normalization condition.

1.2 The hypergeometric series

The hypergeometric series ${}_pF_q$ is defined by

$$ {}_pF_q\left(\left.\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_p)_k}{(b_1, b_2, \ldots, b_q)_k} \frac{z^k}{k!}, \tag{1.2.1}$$

where

$$(a_1, a_2, \ldots, a_p)_k := (a_1)_k (a_2)_k \cdots (a_p)_k$$

with

$$(a)_0 := 1 \;\text{ and }\; (a)_k := a(a+1)(a+2)\cdots(a+k-1), \; k = 1, 2, 3, \ldots.$$

When one of the numerator parameters a_i equals $-n$ where n is a nonnegative integer the hypergeometric series is a polynomial. Otherwise the radius of convergence ρ of the hypergeometric series is given by

$$\rho = \begin{cases} \infty & \text{if } \; p < q+1 \\ 1 & \text{if } \; p = q+1 \\ 0 & \text{if } \; p > q+1. \end{cases}$$

In the special case $p=2$, $q=1$ and $z=1$ we have a summation formula which is known as Vandermonde's summation formula

$$ {}_2F_1\left(\begin{array}{c} -n,b \\ c \end{array} \middle| 1\right) = \frac{(c-b)_n}{(c)_n},\ c \neq 0,-1,-2,\ldots,\ n=0,1,2,\ldots. \tag{1.2.2} $$

Later Gauss found another summation formula in this case :

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array} \middle| 1\right) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)},\ c-a-b>0,\ c \neq 0,-1,-2,\ldots. \tag{1.2.3} $$

1.3 The definition and properties of the classical Laguerre polynomials

For $\alpha > -1$ the classical Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ can be defined by

$$ \begin{aligned} L_n^{(\alpha)}(x) &= \binom{n+\alpha}{n} {}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array} \middle| x\right) \\ &= \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!} \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k,\ n=0,1,2,\ldots. \end{aligned} \tag{1.3.1} $$

From this definition we easily see that

$$ L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + \text{ lower order terms},\ n=0,1,2,\ldots. \tag{1.3.2} $$

These polynomials are orthogonal on the interval $[0,\infty)$ with respect to the weight function $\frac{1}{\Gamma(\alpha+1)} x^{\alpha} e^{-x}$. The orthogonality relation is

$$ \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \binom{n+\alpha}{n} \delta_{mn},\ m,n=0,1,2,\ldots. \tag{1.3.3} $$

An easy consequence of the definition (1.3.1) is

$$ L_n^{(\alpha)}(0) = \binom{n+\alpha}{n},\ n=0,1,2,\ldots. \tag{1.3.4} $$

The polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ satisfy a second order linear differential equation

$$ xy''(x) + (\alpha+1-x)\,y'(x) + ny(x) = 0, \tag{1.3.5} $$

which is often called the Laguerre equation.

The three term recurrence relation is

$$\begin{cases} (n+1)L_{n+1}^{(\alpha)}(x) + (x-2n-\alpha-1)\,L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 1,2,3,\ldots \\ L_0^{(\alpha)}(x) = 1 \ \text{ and } \ L_1^{(\alpha)}(x) = \alpha+1-x. \end{cases} \tag{1.3.6}$$

They satisfy the simple differentiation formula $\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)$ or more general

$$D^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x),\ k = 0,1,2,\ldots,n,\ n = 0,1,2,\ldots. \tag{1.3.7}$$

We will use another simple relation for the classical Laguerre polynomials

$$L_n^{(\alpha+1)}(x) = L_n^{(\alpha)}(x) - \frac{d}{dx}L_n^{(\alpha)}(x),\ n = 0,1,2,\ldots, \tag{1.3.8}$$

which is equivalent to formula (5.1.13) in [32] and which is easily shown to be correct by using the definition (1.3.1).

If we differentiate (1.3.8) once, multiply by x and apply the Laguerre equation (1.3.5) we find

$$nL_n^{(\alpha)}(x) + (\alpha+1)\frac{d}{dx}L_n^{(\alpha)}(x) = x\frac{d}{dx}L_n^{(\alpha+1)}(x),\ n = 0,1,2,\ldots. \tag{1.3.9}$$

This relation will be used in chapter 2 and chapter 3.

Finally, we have a Christoffel-Darboux formula

$$\begin{aligned} &(x-y)\binom{n+\alpha}{n}\sum_{k=0}^{n}\frac{L_k^{(\alpha)}(x)L_k^{(\alpha)}(y)}{\binom{k+\alpha}{k}} \\ &\quad = (n+1)\left[L_n^{(\alpha)}(x)L_{n+1}^{(\alpha)}(y) - L_{n+1}^{(\alpha)}(x)L_n^{(\alpha)}(y)\right],\ n = 0,1,2,\ldots. \end{aligned} \tag{1.3.10}$$

In the so-called confluent form the Christoffel-Darboux formula reads

$$\begin{aligned} &\binom{n+\alpha}{n}\sum_{k=0}^{n}\frac{\left\{L_k^{(\alpha)}(x)\right\}^2}{\binom{k+\alpha}{k}} \\ &\quad = (n+1)\left[L_{n+1}^{(\alpha)}(x)\frac{d}{dx}L_n^{(\alpha)}(x) - L_n^{(\alpha)}(x)\frac{d}{dx}L_{n+1}^{(\alpha)}(x)\right],\ n = 0,1,2,\ldots. \end{aligned} \tag{1.3.11}$$

Chapter 2

Generalizations of the classical Laguerre polynomials

2.1 The definition and the orthogonality relation

Consider the inner product

$$\left\{\begin{array}{l} < f,g > = \dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_0^\infty x^\alpha e^{-x} f(x)g(x)dx + \sum_{\nu=0}^{N} M_\nu f^{(\nu)}(0)g^{(\nu)}(0), \\ \alpha > -1,\ N \in \{0,1,2,\ldots\} \ \text{ and } \ M_\nu \geq 0 \text{ for all } \nu \in \{0,1,2,\ldots,N\}. \end{array}\right. \tag{2.1.1}$$

We will determine the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product (2.1.1). It is clear from the Gram-Schmidt orthogonalization process that there exists such a set of polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ with degree$\left[L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right] = n$. So we may write, by using (1.3.7)

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \sum_{k=0}^{n} A_k D^k L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k A_k L_{n-k}^{(\alpha+k)}(x),\ n = 0,1,2,\ldots, \tag{2.1.2}$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomial defined by (1.3.1) and the coefficients $\{A_k\}_{k=0}^{n}$ are real constants which may depend on $n, \alpha, M_0, M_1, \ldots, M_N$. Moreover, each polynomial $L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)$ is unique except for a multiplicative constant. We will choose this constant such that

$$L_n^{\alpha,0,0,\ldots,0}(x) = L_n^{(\alpha)}(x). \tag{2.1.3}$$

By using the representation (2.1.2) and (1.3.2) we easily see that the coefficient k_n of x^n in the polynomial $L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)$ equals

$$k_n = \frac{(-1)^n}{n!} A_0. \tag{2.1.4}$$

This implies that $A_0 \neq 0$.

Let $p(x) = x^m$. We choose $L_0^{\alpha,M_0,M_1,\ldots,M_N}(x) = 1$ for the moment and we will determine the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=1}^{\infty}$ in such a way that $< p, L_n^{\alpha,M_0,M_1,\ldots,M_N} > = 0$ for all $m \in \{0,1,2,\ldots,n-1\}$.

By using the definition (1.3.1) and Vandermonde's summation formula (1.2.2) we find

$$\begin{aligned}
&\int_0^\infty x^{\alpha+m}e^{-x}L_{n-k}^{(\alpha+k)}(x)dx \\
&= \binom{n+\alpha}{n-k}\sum_{j=0}^{n-k}\frac{(-n+k)_j\Gamma(\alpha+m+j+1)}{(\alpha+k+1)_j j!} = \\
&= \binom{n+\alpha}{n-k}\Gamma(m+\alpha+1)\,{}_2F_1\left(\begin{array}{c}-n+k, m+\alpha+1\\ k+\alpha+1\end{array}\middle|\,1\right) = \\
&= \binom{n-m-1}{n-k}\Gamma(m+\alpha+1),\ k=0,1,2,\ldots,n,\ m,n=0,1,2,\ldots. \qquad (2.1.5)
\end{aligned}$$

First we consider the case that $n \geq N+2$ and $N+1 \leq m \leq n-1$. Then it is clear that

$$p^{(\nu)}(0) = 0 \text{ for all } \nu \in \{0,1,2,\ldots,N\}.$$

Since

$$\binom{n-m-1}{n-k} = \frac{(n-m-1)!}{(n-k)!\Gamma(k-m)} = 0 \text{ for } k=0,1,2,\ldots,m \text{ and } m<n$$

we see, by using (2.1.5), that $< p, L_n^{\alpha,M_0,M_1,\ldots,M_N} > = 0$ is equivalent to

$$\begin{aligned}
0 &= \sum_{k=m+1}^{n}\frac{(-1)^k A_k}{\Gamma(\alpha+1)}\int_0^\infty x^{\alpha+m}e^{-x}L_{n-k}^{(\alpha+k)}(x)dx \\
&= \frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)}\sum_{k=m+1}^{n}(-1)^k\binom{n-m-1}{n-k}A_k,\ m=N+1,N+2,\ldots,n-1.
\end{aligned}$$

If we substitute $m = n-1, n-2, \ldots, N+1$ respectively we easily obtain

$$A_{N+2} = A_{N+3} = \cdots = A_n = 0 \text{ for } n \geq N+2.$$

Hence, the expression (2.1.2) reduces to

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \sum_{k=0}^{N+1} A_k D^k L_n^{(\alpha)}(x) \qquad (2.1.6)$$

for $n \geq N+2$. For $n \leq N+1$ (2.1.6) is trivial. In that case the coefficients $\{A_k\}_{k=n+1}^{N+1}$ can be chosen arbitrarily. This proves that the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ can be defined by (2.1.6) for all $n \in \{0,1,2,\ldots\}$.

In order to define the coefficients $\{A_k\}_{k=0}^{N+1}$ we now have to consider for $n = 1, 2, 3, \ldots$

$$< p, L_n^{\alpha,M_0,M_1,\ldots,M_N} > = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots, \min(n-1, N). \tag{2.1.7}$$

Since $p(x) = x^m$ we easily see that

$$p^{(\nu)}(0) = m!\delta_{m\nu}, \ \nu = 0, 1, 2, \ldots, N.$$

Hence, (2.1.7) implies, by using (2.1.1), (2.1.5), (2.1.6), (1.3.4) and (1.3.7), that

$$\frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \sum_{k=m+1}^{\min(n,N+1)} (-1)^k \binom{n-m-1}{n-k} A_k + \\ + (-1)^m m! M_m \sum_{k=0}^{\min(n,N+1)} (-1)^k \binom{n+\alpha}{n-m-k} A_k = 0, \ m = 0, 1, 2, \ldots, \min(n-1, N),$$

where

$$\binom{u}{v} = 0 \quad \text{for} \quad u \neq -1, -2, -3, \ldots \quad \text{and} \quad v = -1, -2, -3, \ldots.$$

This condition is necessary and sufficient for the orthogonality. For $n \geq N+1$ this is a system of $N+1$ equations for the $N+2$ coefficients $\{A_k\}_{k=0}^{N+1}$. If $1 \leq n \leq N+1$ we have n equations for the $n+1$ coefficients $\{A_k\}_{k=0}^{n}$. Since

$$\binom{n-m-1}{n-k} = \binom{n+\alpha}{n-m-k} = 0 \quad \text{for} \quad k \geq n+1 \quad \text{and} \quad m = 0, 1, 2, \ldots, \min(n-1, N)$$

and

$$\binom{n-m-1}{n-k} = \binom{n-m-1}{k-m-1}$$

we simply write

$$\frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)} \sum_{k=m+1}^{N+1} (-1)^k \binom{n-m-1}{k-m-1} A_k + \\ + (-1)^m m! M_m \sum_{k=0}^{N+1} (-1)^k \binom{n+\alpha}{n-m-k} A_k = 0, \ m = 0, 1, 2, \ldots, \min(n-1, N).$$

However, we will define the coefficients $\{A_k\}_{k=0}^{N+1}$ in such a way that

$$\binom{m+\alpha}{m} \sum_{k=m+1}^{N+1} (-1)^k \binom{n-m-1}{k-m-1} A_k + \\ + (-1)^m M_m \sum_{k=0}^{N+1} (-1)^k \binom{n+\alpha}{n-m-k} A_k = 0, \ m = 0, 1, 2, \ldots, N \tag{2.1.8}$$

is valid for all $n \in \{0,1,2,\ldots\}$. For $n \geq N+1$ this is the same system of equations. For $n \leq N$ we have added the following conditions on the arbitrary coefficients $\{A_k\}_{k=n+1}^{N+1}$:

$$\binom{m+\alpha}{m} \sum_{k=m+1}^{N+1} \frac{(m+1-n)_{k-m-1}}{(k-m-1)!} A_k = M_m \sum_{k=0}^{N+1} (-1)^k \binom{n+\alpha}{n-m-k} A_k$$

for $m = n, n+1, n+2, \ldots, N$, since $\binom{a}{k} = (-1)^k \frac{(-a)_k}{k!}$. Hence

$$\begin{cases} \displaystyle\binom{n+\alpha}{n} \sum_{k=n+1}^{N+1} A_k = M_n A_0 \\ \displaystyle\binom{n+\alpha+i}{n+i} \sum_{k=n+i+1}^{N+1} \frac{(i+1)_{k-n-i-1}}{(k-n-i-1)!} A_k = 0, \ i = 1,2,3,\ldots,N-n. \end{cases}$$

This implies for $n \leq N$ that $A_{n+2} = A_{n+3} = \cdots = A_{N+1} = 0$ and $\binom{n+\alpha}{n} A_{n+1} = M_n A_0$. However, in the sequel we only need

$$\binom{n+\alpha}{n} (A_{n+1} + A_{n+2} + \cdots + A_{N+1}) = M_n A_0 \ \text{ for } \ n \leq N. \tag{2.1.9}$$

With (2.1.6) and (2.1.8) we have found the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$. We changed the choice of $L_0^{\alpha,M_0,M_1,\ldots,M_N}(x) = 1$ such that (2.1.6) also holds for $n = 0$. This can be done since each polynomial $L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)$ is unique except for a multiplicative constant. In view of the chosen normalization (2.1.3) these polynomials clearly are generalizations of the classical Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ defined by (1.3.1).

Finally, we note that (2.1.8) for $m = N$ leads to

$$\binom{N+\alpha}{N} A_{N+1} = M_N \sum_{k=0}^{N+1} (-1)^k \binom{n+\alpha}{n-N-k} A_k. \tag{2.1.10}$$

This implies that $A_{N+1} = 0$ for $M_N = 0$.

Now we will show that

$$< L_n^{\alpha,M_0,M_1,\ldots,M_N}, L_n^{\alpha,M_0,M_1,\ldots,M_N} > = \binom{n+\alpha}{n} A_0 \left(A_0 + A_1 + \cdots + A_{N+1}\right). \tag{2.1.11}$$

This implies that

$$A_0 \left(A_0 + A_1 + \cdots + A_{N+1}\right) > 0. \tag{2.1.12}$$

In order to show that (2.1.11) is true we note, by using (2.1.4) and the orthogonality, that

$$< L_n^{\alpha,M_0,M_1,\ldots,M_N}, L_n^{\alpha,M_0,M_1,\ldots,M_N} > = \frac{(-1)^n}{n!} A_0 < x^n, L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) > .$$

Now we use (2.1.6), (1.3.7) and (2.1.5) for $m = n$ to find for $n \geq N+1$

$$\begin{aligned} < x^n, L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) > &= \sum_{k=0}^{N+1} \frac{(-1)^k}{\Gamma(\alpha+1)} A_k \int_0^\infty x^{n+\alpha} e^{-x} L_{n-k}^{(\alpha+k)}(x) dx \\ &= (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \sum_{k=0}^{N+1} A_k. \end{aligned}$$

This proves (2.1.11) in the case $n \geq N+1$.

For $n \leq N$ we find

$$< x^n, L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) > = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \sum_{k=0}^{n} A_k + (-1)^n n! M_n A_0.$$

Now we use (2.1.9) to conclude that (2.1.11) is also true for $n \leq N$.

Hence, we have obtained the orthogonality relation

$$\begin{aligned} &< L_m^{\alpha,M_0,M_1,\ldots,M_N}, L_n^{\alpha,M_0,M_1,\ldots,M_N} > \\ &\quad = \binom{n+\alpha}{n} A_0 \left(A_0 + A_1 + \cdots + A_{N+1}\right) \delta_{mn},\ m,n = 0,1,2,\ldots. \end{aligned} \tag{2.1.13}$$

If $M_0 = M_1 = \cdots = M_N = 0$ (2.1.8) and (2.1.3) lead to $A_0 = 1$ and $A_1 = A_2 = \cdots = A_{N+1} = 0$. This implies that (2.1.13) is a generalization of (1.3.3), since $L_n^{\alpha,0,0,\ldots,0}(x) = L_n^{(\alpha)}(x)$.

2.2 Another representation

In this section we will prove the following three relations involving the classical Laguerre polynomials :

$$xD^{k+1} L_n^{(\alpha+1)}(x) = (n-k) D^k L_n^{(\alpha)}(x) + (\alpha+k+1) D^{k+1} L_n^{(\alpha)}(x),\ k = 0,1,2,\ldots, \tag{2.2.1}$$

$$x^k D^k L_n^{(\alpha+k)}(x) = \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} (i-n)_{k-i} (\alpha+k)_i D^i L_n^{(\alpha)}(x),\ k = 0,1,2,\ldots \tag{2.2.2}$$

and

$$D^k L_n^{(\alpha)}(x) = \sum_{i=0}^{k} \binom{k}{i} \frac{(i-n)_{k-i}}{(\alpha+i)_i (\alpha+2i+1)_{k-i}} x^i D^i L_n^{(\alpha+i)}(x),\ k = 0,1,2,\ldots. \tag{2.2.3}$$

Relation (2.2.1) which is a generalization of relation (1.3.9) is used to prove relations (2.2.2) and (2.2.3). The relations (2.2.2) and (2.2.3) can be used to show that definition (2.1.6) is equivalent to

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \sum_{k=0}^{N+1} B_k x^k D^k L_n^{(\alpha+k)}(x), \tag{2.2.4}$$

for some coefficients $\{B_k\}_{k=0}^{N+1}$. These coefficients are related to the coefficients $\{A_k\}_{k=0}^{N+1}$.

By using relation (2.2.2) we find

$$A_i = \sum_{k=i}^{N+1} (-1)^{i+k} \binom{k}{i} (i-n)_{k-i} (\alpha+k)_i B_k,\ i = 0, 1, 2, \ldots, N+1.$$

With (2.2.3) we find

$$B_k = \frac{1}{(\alpha+k)_k} \sum_{j=k}^{N+1} \binom{j}{k} \frac{(k-n)_{j-k}}{(\alpha+2k+1)_{j-k}} A_j,\ k = 0, 1, 2, \ldots, N+1.$$

To prove (2.2.1) we use induction on k. For $k = 0$ relation (2.2.1) reduces to relation (1.3.9). This shows that it is a generalization of (1.3.9).

Now we assume that (2.2.1) holds for $k = m-1$. Then we find by using (1.3.8) :

$$\begin{aligned} & xD^{m+1} L_n^{(\alpha+1)}(x) = x\frac{d}{dx} D^m L_n^{(\alpha+1)}(x) \\ = & \frac{d}{dx}\left[xD^m L_n^{(\alpha+1)}(x)\right] - D^m L_n^{(\alpha+1)}(x) \\ = & \frac{d}{dx}\left[(n+1-m)D^{m-1} L_n^{(\alpha)}(x) + (\alpha+m)D^m L_n^{(\alpha)}(x)\right] - D^m L_n^{(\alpha+1)}(x) \\ = & (n-m)D^m L_n^{(\alpha)}(x) + (\alpha+m+1)D^{m+1} L_n^{(\alpha)}(x). \end{aligned}$$

This proves relation (2.2.1).

Now we prove relation (2.2.2) which is a second generalization of (1.3.9). Again we use induction on k. For $k = 0$ relation (2.2.2) is trivial. For $k = 1$ it reduces to (1.3.9). This shows that it is another generalization of (1.3.9).

Now we assume that (2.2.2) holds for $k = m-1$. Hence

$$\begin{aligned} & x^{m-1} D^{m-1} L_n^{(\alpha+m-1)}(x) \\ = & \sum_{i=0}^{m-1} (-1)^{i+m-1} \binom{m-1}{i} (i-n)_{m-1-i} (\alpha+m-1)_i D^i L_n^{(\alpha)}(x). \end{aligned} \tag{2.2.5}$$

If we take the derivative of (2.2.5) and multiply by x we obtain

$$\begin{aligned} & x^m D^m L_n^{(\alpha+m-1)}(x) + (m-1)x^{m-1} D^{m-1} L_n^{(\alpha+m-1)}(x) \\ = & x \sum_{i=0}^{m-1} (-1)^{i+m-1} \binom{m-1}{i} (i-n)_{m-1-i} (\alpha+m-1)_i D^{i+1} L_n^{(\alpha)}(x). \end{aligned} \tag{2.2.6}$$

Now we multiply equation (2.2.5) by $m-1$ and subtract it from equation (2.2.6) obtaining

$$\begin{aligned} x^m D^m L_n^{(\alpha+m-1)}(x) &= \sum_{i=0}^{m-1} (-1)^{i+m-1} \binom{m-1}{i} (i-n)_{m-1-i} (\alpha+m-1)_i \times \\ & \times \left[xD^{i+1} L_n^{(\alpha)}(x) - (m-1)D^i L_n^{(\alpha)}(x)\right]. \end{aligned}$$

We replace α by $\alpha+1$ and use (2.2.1) and (1.3.8) to find

$$\begin{aligned} x^m D^m L_n^{(\alpha+m)}(x) &= \sum_{i=0}^{m-1}(-1)^{i+m-1}\binom{m-1}{i}(i-n)_{m-1-i}(\alpha+m)_i \times \\ &\qquad \times\left[(n-m-i+1)D^i L_n^{(\alpha)}(x)+(\alpha+m+i)D^{i+1}L_n^{(\alpha)}(x)\right]. \end{aligned}$$

Hence

$$\begin{aligned} & x^m D^m L_n^{(\alpha+m)}(x) \\ =\; & \sum_{i=0}^{m-1}(-1)^{i+m-1}\binom{m-1}{i}(n-m-i+1)(i-n)_{m-1-i}(\alpha+m)_i D^i L_n^{(\alpha)}(x) + \\ & \qquad + \sum_{i=0}^{m-1}(-1)^{i+m-1}\binom{m-1}{i}(i-n)_{m-1-i}(\alpha+m)_{i+1}D^{i+1}L_n^{(\alpha)}(x) \\ =\; & \sum_{i=1}^{m-1}(-1)^{i+m}(i-n)_{m-1-i}(\alpha+m)_i \times \\ & \qquad \times\left[\binom{m-1}{i-1}(i-1-n)-\binom{m-1}{i}(n-m-i+1)\right]D^i L_n^{(\alpha)}(x) + \\ & \qquad + (-1)^{m-1}(n-m+1)(-n)_{m-1}L_n^{(\alpha)}(x)+(\alpha+m)_m D^m L_n^{(\alpha)}(x) \\ =\; & \sum_{i=1}^{m-1}(-1)^{i+m}\binom{m}{i}(i-n)_{m-i}(\alpha+m)_i D^i L_n^{(\alpha)}(x) + \\ & \qquad + (-1)^m(-n)_m L_n^{(\alpha)}(x)+(\alpha+m)_m D^m L_n^{(\alpha)}(x) \\ =\; & \sum_{i=0}^{m}(-1)^{i+m}\binom{m}{i}(i-n)_{m-i}(\alpha+m)_i D^i L_n^{(\alpha)}(x). \end{aligned}$$

This proves relation (2.2.2).

To prove relation (2.2.3) we start with relation (2.2.2) and write it in the following way :

$$C_k(x)=\sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i}(i-n)_{k-i}(\alpha+k)_i D_i(x),\ k=0,1,2,\ldots. \tag{2.2.7}$$

It is not difficult to see that the system defined by (2.2.7) has a unique solution for $\{D_i(x)\}_{i=0}^{\infty}$. We will show that this solution is given by

$$D_i(x)=\sum_{j=0}^{i}\binom{i}{j}\frac{(j-n)_{i-j}}{(\alpha+j)_j(\alpha+2j+1)_{i-j}}C_j(x),\ i=0,1,2,\ldots. \tag{2.2.8}$$

To prove this we substitute (2.2.8) in the right-hand side of (2.2.7)

$$S_k(x):=\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+k}\binom{k}{i}\binom{i}{j}(i-n)_{k-i}(j-n)_{i-j}\frac{(\alpha+k)_i}{(\alpha+j)_j(\alpha+2j+1)_{i-j}}C_j(x)$$

and show that this equals $C_k(x)$. Changing the order of summation and using

$$(j-n)_{i-j}(i-n)_{k-i} = (j-n)_{k-j}$$

and

$$(-1)^{i+j}\binom{k}{i}\binom{i}{j} = \binom{k}{j}\frac{(j-k)_{i-j}}{(i-j)!}$$

we find

$$S_k(x) = \sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\frac{(j-n)_{k-j}}{(\alpha+j)_j}C_j(x)\sum_{i=j}^{k}\frac{(j-k)_{i-j}(\alpha+k)_i}{(\alpha+2j+1)_{i-j}(i-j)!}.$$

Now we use the fact that

$$(\alpha+k)_{i+j} = (\alpha+k)_j(\alpha+k+j)_i$$

to see that the last sum equals

$$\begin{aligned} T(j,k) &:= \sum_{i=j}^{k}\frac{(j-k)_{i-j}(\alpha+k)_i}{(\alpha+2j+1)_{i-j}(i-j)!} = \sum_{i=0}^{k-j}\frac{(j-k)_i(\alpha+k)_{i+j}}{(\alpha+2j+1)_i i!} \\ &= (\alpha+k)_j\sum_{i=0}^{k-j}\frac{(j-k)_i(\alpha+k+j)_i}{(\alpha+2j+1)_i i!} = (\alpha+k)_j \cdot {}_2F_1\left(\left.\begin{array}{c} j-k, \alpha+k+j \\ \alpha+2j+1 \end{array}\right| 1\right). \end{aligned}$$

By using Vandermonde's summation formula (1.2.2) we obtain

$$T(j,k) = (\alpha+k)_j\frac{(j-k+1)_{k-j}}{(\alpha+2j+1)_{k-j}} = \begin{cases} 0, & j<k \\ (\alpha+k)_k, & j=k. \end{cases}$$

Hence, since

$$S_k(x) = \sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\frac{(j-n)_{k-j}}{(\alpha+j)_j}C_j(x)T(j,k)$$

we have

$$S_k(x) = \frac{(\alpha+k)_k}{(\alpha+k)_k}C_k(x) = C_k(x).$$

This proves (2.2.8) and therefore (2.2.3).

2.3 Representation as hypergeometric series

From (2.1.6) and (1.3.1) we obtain

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \binom{n+\alpha}{n}\sum_{k=0}^{N+1}A_k D^k {}_1F_1\left(\left.\begin{array}{c} -n \\ \alpha+1 \end{array}\right| x\right) = \binom{n+\alpha}{n}\sum_{m=0}^{n}C_m\frac{x^m}{m!}$$

where

$$C_m = \sum_{k=0}^{N+1} \frac{(-n)_{m+k}}{(\alpha+1)_{m+k}} A_k = \frac{(-n)_m}{(\alpha+1)_{m+N+1}} \sum_{k=0}^{N+1} (m-n)_k (m+\alpha+k+1)_{N+1-k} A_k.$$

From (2.1.12) it follows that $A_0 + A_1 + \cdots + A_{N+1} \neq 0$. So we may write

$$C_m = (A_0 + A_1 + \cdots + A_{N+1}) \frac{(-n)_m}{(\alpha+N+2)_m} \frac{(m+\beta_0)(m+\beta_1)\cdots(m+\beta_N)}{(\alpha+1)_{N+1}}$$

for some complex numbers $\beta_j,\ j = 0,1,2,\ldots,N$. Since

$$m + \beta_j = \beta_j \frac{(\beta_j+1)_m}{(\beta_j)_m} \quad \text{for} \quad \beta_j \neq 0,-1,-2,\ldots$$

we find in that case

$$\begin{aligned} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) &= \frac{\beta_0\beta_1\cdots\beta_N}{(\alpha+1)_{N+1}} \binom{n+\alpha}{n} (A_0 + A_1 + \cdots + A_{N+1}) \times \\ &\qquad \times {}_{N+2}F_{N+2}\left(\begin{matrix} -n, \beta_0+1, \beta_1+1, \ldots, \beta_N+1 \\ \alpha+N+2, \beta_0, \beta_1, \ldots, \beta_N \end{matrix} \middle| \, x \right). \quad (2.3.1) \end{aligned}$$

In the case that β_j is a nonpositive integer for some j we have to take the analytic continuation of (2.3.1).

For $M_0 = M_1 = \cdots = M_N = 0$ we have $A_0 = 1$ and $A_1 = A_2 = \cdots = A_{N+1} = 0$. In that case we find

$$\begin{aligned} C_m &= \frac{(-n)_m}{(\alpha+1)_{m+N+1}} (m+\alpha+1)_{N+1} = \frac{(-n)_m}{(\alpha+N+2)_m} \frac{(m+\alpha+1)_{N+1}}{(\alpha+1)_{N+1}} \\ &= \frac{(-n)_m}{(\alpha+N+2)_m} \frac{(m+\beta_0)(m+\beta_1)\cdots(m+\beta_N)}{(\alpha+1)_{N+1}} \end{aligned}$$

where

$$\beta_j = \alpha + j + 1,\ j = 0,1,2,\ldots,N.$$

We then find for (2.3.1) :

$$\begin{aligned} L_n^{\alpha,0,0,\ldots,0}(x) &= \binom{n+\alpha}{n} {}_{N+2}F_{N+2}\left(\begin{matrix} -n, \alpha+2, \alpha+3, \ldots, \alpha+N+2 \\ \alpha+N+2, \alpha+1, \alpha+2, \ldots, \alpha+N+1 \end{matrix} \middle| \, x \right) \\ &= \binom{n+\alpha}{n} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| \, x \right). \end{aligned}$$

So (2.3.1) can be considered as a generalization of the hypergeometric series representation of the classical Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ given in (1.3.1).

2.4 A second order differential equation

In [18] we found a second order linear differential equation satisfied by the polynomials $\left\{L_n^{\alpha,M,N}(x)\right\}_{n=0}^{\infty}$. The method used there can be applied in the general case, but in [9] we found a more simple and elegant proof of the second order linear differential equation for the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$. We will give this latter proof here.

Theorem 2.1. *The polynomials* $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ *satisfy a second order linear differential equation of the form*

$$xp_2(x)y''(x) - p_1(x)y'(x) + np_0(x)y(x) = 0 \tag{2.4.1}$$

where $p_0(x)$, $p_1(x)$ *and* $p_2(x)$ *are polynomials with*

$$\begin{cases} p_0(x) = A_0\left(A_0 + A_1 + \cdots + A_{N+1}\right)x^{N+1} + \text{ lower order terms} \\ p_1(x) = A_0\left(A_0 + A_1 + \cdots + A_{N+1}\right)x^{N+2} + \text{ lower order terms} \\ p_2(x) = A_0\left(A_0 + A_1 + \cdots + A_{N+1}\right)x^{N+1} + \text{ lower order terms} \end{cases} \tag{2.4.2}$$

and

$$p_1(x) = xp_2'(x) + (x - \alpha - N - 2)\,p_2(x). \tag{2.4.3}$$

Proof. We start with the differential equation (1.3.5) for the classical Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$:

$$x\frac{d^2}{dx^2}L_n^{(\alpha)}(x) + (\alpha + 1 - x)\frac{d}{dx}L_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0. \tag{2.4.4}$$

Differentiation of (2.4.4) leads to

$$xD^{k+2}L_n^{(\alpha)}(x) + (\alpha + k + 1 - x)D^{k+1}L_n^{(\alpha)}(x) + (n-k)D^kL_n^{(\alpha)}(x) = 0,\ k = 0, 1, 2, \ldots. \tag{2.4.5}$$

By using $k = N - 1$ in (2.4.5) we find from (2.1.6)

$$xL_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \sum_{k=0}^{N} b_k(x)D^kL_n^{(\alpha)}(x)$$

where

$$\begin{cases} b_k(x) = A_kx,\ k = 0, 1, 2, \ldots, N-2 \\ b_{N-1}(x) = A_{N-1}x - (n - N + 1)A_{N+1} \\ b_N(x) = A_Nx - (\alpha + N - x)A_{N+1}. \end{cases}$$

Then we take $k = N - 2$ in (2.4.5) to obtain

$$x^2 L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \sum_{k=0}^{N-1} \tilde{b}_k(x) D^k L_n^{(\alpha)}(x)$$

where

$$\begin{cases} \tilde{b}_k(x) = x b_k(x), \; k = 0,1,2,\ldots,N-3 \\ \\ \tilde{b}_{N-2}(x) = x b_{N-2}(x) - (n-N+2) b_N(x) \\ \\ \tilde{b}_{N-1}(x) = x b_{N-1}(x) - (\alpha+N-1-x) b_N(x). \end{cases}$$

Repeating this process we finally find by taking $k = 0$ in (2.4.5)

$$x^N L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = q_0(x) L_n^{(\alpha)}(x) + q_1(x) \frac{d}{dx} L_n^{(\alpha)}(x) \tag{2.4.6}$$

for some polynomials q_0 and q_1 satisfying

$$\begin{cases} q_0(x) = A_0 x^N + \text{ lower order terms} \\ \\ q_1(x) = (A_1 + A_2 + \cdots + A_{N+1})\, x^N + \text{ lower order terms.} \end{cases} \tag{2.4.7}$$

Differentiation of (2.4.6) gives us

$$\begin{aligned} & x^N \frac{d}{dx} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) + N x^{N-1} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) \\ = \;& q_0'(x) L_n^{(\alpha)}(x) + \left[q_0(x) + q_1'(x)\right] \frac{d}{dx} L_n^{(\alpha)}(x) + q_1(x) \frac{d^2}{dx^2} L_n^{(\alpha)}(x). \end{aligned}$$

Now we multiply this by x and use (2.4.4) and (2.4.6) to find

$$x^{N+1} \frac{d}{dx} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = r_0(x) L_n^{(\alpha)}(x) + r_1(x) \frac{d}{dx} L_n^{(\alpha)}(x) \tag{2.4.8}$$

where

$$\begin{cases} r_0(x) = x q_0'(x) - N q_0(x) - n q_1(x) \\ \\ r_1(x) = x q_0(x) + x q_1'(x) + (x - \alpha - N - 1) q_1(x). \end{cases} \tag{2.4.9}$$

It follows from (2.4.7) and (2.4.9) that

$$\begin{cases} r_0(x) = -n\,(A_1 + A_2 + \cdots + A_{N+1})\, x^N + \text{ lower order terms} \\ \\ r_1(x) = (A_0 + A_1 + \cdots + A_{N+1})\, x^{N+1} + \text{ lower order terms.} \end{cases} \tag{2.4.10}$$

In the same way we obtain from (2.4.8) by using (2.4.4)

$$x^{N+2} \frac{d^2}{dx^2} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = s_0(x) L_n^{(\alpha)}(x) + s_1(x) \frac{d}{dx} L_n^{(\alpha)}(x) \tag{2.4.11}$$

where

$$\begin{cases} s_0(x) = xr_0'(x) - (N+1)r_0(x) - nr_1(x) \\ \\ s_1(x) = xr_0(x) + xr_1'(x) + (x - \alpha - N - 2)r_1(x). \end{cases} \tag{2.4.12}$$

And applying (2.4.10) to (2.4.12) we find

$$\begin{cases} s_0(x) = -n\left(A_0 + A_1 + \cdots + A_{N+1}\right)x^{N+1} + \text{ lower order terms} \\ \\ s_1(x) = \left(A_0 + A_1 + \cdots + A_{N+1}\right)x^{N+2} + \text{ lower order terms.} \end{cases} \tag{2.4.13}$$

Now we eliminate the derivative of the classical Laguerre polynomial from (2.4.6) and (2.4.8) and find

$$\begin{aligned} & \left[q_0(x)r_1(x) - q_1(x)r_0(x)\right] L_n^{(\alpha)}(x) \\ = \;& x^N \left[r_1(x)L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) - xq_1(x)\frac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right]. \end{aligned}$$

Since $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} \neq 0$ we conclude that

$$q_0(x)r_1(x) - q_1(x)r_0(x) = x^N p_2(x) \tag{2.4.14}$$

for some polynomial p_2. In the same way we obtain from (2.4.6) and (2.4.11)

$$q_0(x)s_1(x) - q_1(x)s_0(x) = x^N p_1(x) \tag{2.4.15}$$

for some polynomial p_1. And from (2.4.8) and (2.4.11) it follows that

$$r_0(x)s_1(x) - r_1(x)s_0(x) = nx^{N+1}p_0(x) \tag{2.4.16}$$

for some polynomial p_0. Here we used the fact that

$$q_0(x) = A_0x^N \;\text{ and }\; r_0(x) = s_0(x) = 0 \;\text{ for }\; n = 0,$$

which follows from (2.4.6), (2.4.9) and (2.4.12).

In view of (2.4.6), (2.4.8) and (2.4.11) it is clear that the determinant

$$\begin{vmatrix} x^N L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & q_0(x) & q_1(x) \\ \\ x^{N+1}\dfrac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & r_0(x) & r_1(x) \\ \\ x^{N+2}\dfrac{d^2}{dx^2}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & s_0(x) & s_1(x) \end{vmatrix}$$

must be zero. The first column of this determinant can be divided by x^N. Hence, by using (2.4.14), (2.4.15) and (2.4.16), we find

$$\begin{aligned}
0 &= \begin{vmatrix} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & q_0(x) & q_1(x) \\ \\ x\dfrac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & r_0(x) & r_1(x) \\ \\ x^2\dfrac{d^2}{dx^2}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) & s_0(x) & s_1(x) \end{vmatrix} \\
&= x^{N+2}p_2(x)\frac{d^2}{dx^2}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) - x^{N+1}p_1(x)\frac{d}{dx}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) + \\
&\qquad + x^{N+1}np_0(x)L_n^{\alpha,M_0,M_1,\ldots,M_N}(x).
\end{aligned}$$

This proves (2.4.1).

Now (2.4.2) easily follows from (2.4.14), (2.4.15) and (2.4.16), by using (2.4.7), (2.4.10) and (2.4.13).

Finally we prove (2.4.3). Differentiation of (2.4.14) gives us

$$x^Np_2'(x) + Nx^{N-1}p_2(x) = q_0'(x)r_1(x) + q_0(x)r_1'(x) - q_1'(x)r_0(x) - q_1(x)r_0'(x).$$

Now it follows by using (2.4.14), (2.4.12), (2.4.9) and (2.4.15) that

$$\begin{aligned}
& x^N\left[xp_2'(x) + (x-\alpha-N-2)p_2(x)\right] \\
={}& x\left[q_0'(x)r_1(x) + q_0(x)r_1'(x) - q_1'(x)r_0(x) - q_1(x)r_0'(x)\right] + \\
&\qquad + (x-\alpha-2N-2)\left[q_0(x)r_1(x) - q_1(x)r_0(x)\right] \\
={}& q_0(x)\left[xr_0(x) + xr_1'(x) + (x-\alpha-N-2)r_1(x)\right] + \\
&\qquad - q_1(x)\left[xr_0'(x) - (N+1)r_0(x) - nr_1(x)\right] + \\
&\qquad - xq_0(x)r_0(x) - Nq_0(x)r_1(x) - nq_1(x)r_1(x) + \\
&\qquad + xq_0'(x)r_1(x) - xq_1'(x)r_0(x) - (x-\alpha-N-1)q_1(x)r_0(x) \\
={}& q_0(x)s_1(x) - q_1(x)s_0(x) + r_1(x)\left[xq_0'(x) - Nq_0(x) - nq_1(x)\right] + \\
&\qquad - r_0(x)\left[xq_0(x) + xq_1'(x) + (x-\alpha-N-1)q_1(x)\right] \\
={}& q_0(x)s_1(x) - q_1(x)s_0(x) + r_1(x)r_0(x) - r_0(x)r_1(x) = x^Np_1(x).
\end{aligned}$$

We simply divide by x^N to obtain (2.4.3). This proves the theorem.

If $M_0 = M_1 = \cdots = M_N = 0$ we have $A_0 = 1$ and $A_1 = A_2 = \cdots = A_{N+1} = 0$. Moreover, we have $L_n^{\alpha,0,0,\ldots,0}(x) = L_n^{(\alpha)}(x)$. This implies that (2.4.6) is valid with $q_0(x) = x^N$ and $q_1(x) = 0$. Then it follows from (2.4.9) that $r_0(x) = 0$ and $r_1(x) = x^{N+1}$. And from (2.4.12) we then obtain $s_0(x) = -nx^{N+1}$ and $s_1(x) = (x-\alpha-1)x^{N+1}$. So we have, by using (2.4.14), (2.4.15) and (2.4.16) that $p_2(x) = x^{N+1}$, $p_1(x) = (x-\alpha-1)x^{N+1}$ and $p_0(x) = x^{N+1}$. Hence, for $M_0 = M_1 = \cdots = M_N = 0$ the differential equation (2.4.1) can be divided by x^{N+1} to yield the Laguerre equation (1.3.5). This shows that (2.4.1) can be considered as a generalization of (1.3.5).

2.5 Recurrence relation

All sets of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with degree$[P_n(x)] = n$ which are orthogonal on an interval with respect to a positive weight function satisfy a three term recurrence relation. The classical Laguerre polynomials for instance, satisfy (1.3.6). The polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ in general fail to have this property, but we will prove the following result.

Theorem 2.2. *The polynomials* $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ *satisfy a* $(2N+3)$*-term recurrence relation of the form*

$$x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \sum_{k=\max(0,n-N-1)}^{n+N+1} E_k^{(n)} L_k^{\alpha,M_0,M_1,\ldots,M_N}(x),\ n = 0,1,2,\ldots. \tag{2.5.1}$$

Proof. Since $x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)$ is a polynomial of degree $n+N+1$ we have

$$x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) = \sum_{k=0}^{n+N+1} E_k^{(n)} L_k^{\alpha,M_0,M_1,\ldots,M_N}(x),\ n = 0,1,2,\ldots \tag{2.5.2}$$

for some real coefficients $E_k^{(n)},\ k = 0,1,2,\ldots,n+N+1$.

Taking the inner product with $L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)$ on both sides of (2.5.2) we find by using (2.1.1) for $n = 0,1,2,\ldots$ and $m = 0,1,2,\ldots,n+N+1$:

$$\begin{aligned} &< L_m^{\alpha,M_0,M_1,\ldots,M_N}, L_m^{\alpha,M_0,M_1,\ldots,M_N} > \cdot E_m^{(n)} \\ &\quad = < x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x), L_m^{\alpha,M_0,M_1,\ldots,M_N}(x) > \\ &\quad = < x^{N+1}L_m^{\alpha,M_0,M_1,\ldots,M_N}(x), L_n^{\alpha,M_0,M_1,\ldots,M_N}(x) > . \end{aligned} \tag{2.5.3}$$

In view of the orthogonality property of the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ we conclude that $E_m^{(n)} = 0$ for $m+N+1 < n$. This proves (2.5.1).

The coefficients $\{A_k\}_{k=0}^{N+1}$ in the definition (2.1.6) depend on n. To distinguish two coefficients with the same index, but depending on a different value of n we will write $A_k(n)$ instead of A_k. Comparing the leading coefficients on both sides of (2.5.2) we obtain by using this notation and (2.1.4)

$$E_{n+N+1}^{(n)} = \frac{k_n}{k_{n+N+1}} = (-1)^{N+1}\frac{(n+N+1)!}{n!}\frac{A_0(n)}{A_0(n+N+1)} \neq 0,\ n = 0,1,2,\ldots.$$

If we define (compare with (2.1.11))

$$\Lambda_n := < L_n^{\alpha,M_0,M_1,\ldots,M_N}, L_n^{\alpha,M_0,M_1,\ldots,M_N} > = \binom{n+\alpha}{n} A_0\left(A_0 + A_1 + \cdots + A_{N+1}\right)$$

then we find by using (2.5.3), (2.1.4) and the orthogonality that

$$E_{n-N-1}^{(n)} = \frac{k_{n-N-1}\Lambda_n}{k_n\Lambda_{n-N-1}} \neq 0,\ n = N+1, N+2,\ldots.$$

2.6 A Christoffel-Darboux type formula

From the recurrence relation (2.5.1) we easily obtain

$$\begin{aligned}&\left(x^{N+1}-y^{N+1}\right)L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y)\\ =&\sum_{m=\max(0,k-N-1)}^{k+N+1}E_m^{(k)}\left[L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y)+\right.\\ &\left.-L_m^{\alpha,M_0,M_1,\ldots,M_N}(y)L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)\right],\ k=0,1,2,\ldots.\end{aligned}\tag{2.6.1}$$

We divide by Λ_k and sum over $k=0,1,2,\ldots,n$:

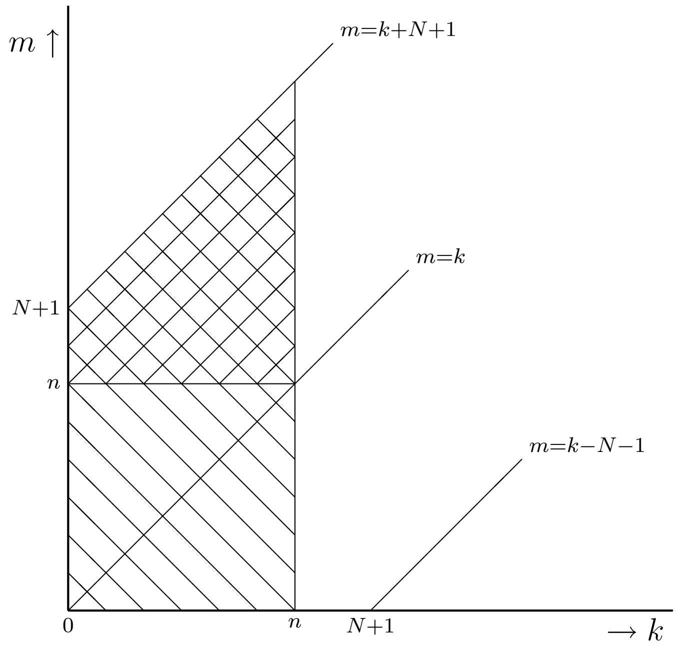
$$\begin{aligned}&\left(x^{N+1}-y^{N+1}\right)\sum_{k=0}^{n}\frac{L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y)}{\Lambda_k}\\ =&\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}\left[L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y)+\right.\\ &\left.-L_m^{\alpha,M_0,M_1,\ldots,M_N}(y)L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)\right],\ n=0,1,2,\ldots.\end{aligned}$$

Now we use (2.5.3) to see that

$$\frac{E_m^{(k)}}{\Lambda_k}=\frac{E_k^{(m)}}{\Lambda_m},\ k-N-1\le m\le k+N+1,\ k,m=0,1,2,\ldots.$$

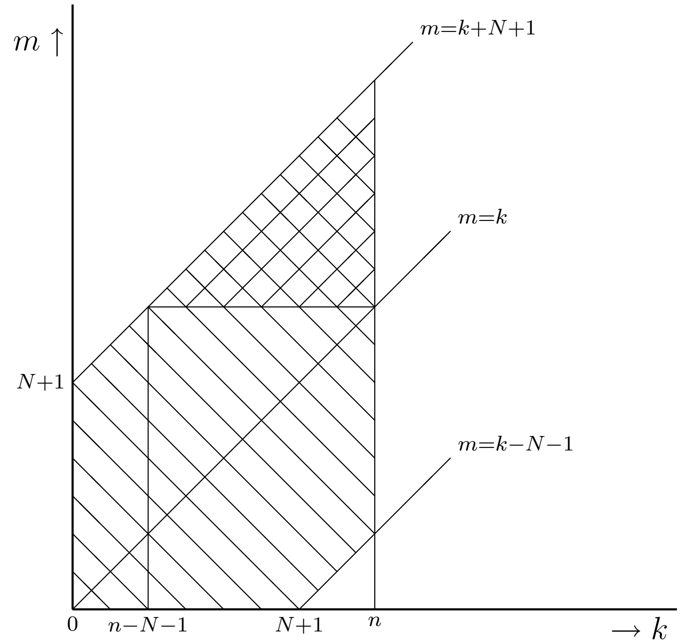
Now we have the following situations :

For $n\le N$ we have



$$\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{k+N+1} = \sum_{k=0}^{n}\sum_{m=0}^{n} + \sum_{k=0}^{n}\sum_{m=n+1}^{k+N+1} = \sum_{k=0}^{n}\sum_{m=n+1}^{k+N+1}$$

and for $n \geq N+1$ we have



$$\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{k+N+1} = \sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{n} + \sum_{k=n-N}^{n}\sum_{m=n+1}^{k+N+1} = \sum_{k=n-N}^{n}\sum_{m=n+1}^{k+N+1}.$$

So it follows from (2.6.1) by using this observation that

$$\begin{aligned}
&\left(x^{N+1} - y^{N+1}\right)\sum_{k=0}^{n}\frac{L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y)}{\Lambda_k}\\
&= \sum_{k=\max(0,n-N)}^{n}\sum_{m=n+1}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}\left[L_m^{\alpha,M_0,M_1,\ldots,M_N}(x)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y) + \right.\\
&\qquad\left. - L_m^{\alpha,M_0,M_1,\ldots,M_N}(y)L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)\right] \qquad (2.6.2)
\end{aligned}$$

for $n = 0, 1, 2, \ldots$. This can be considered as a generalization of the Christoffel-Darboux formula (1.3.10) for the classical Laguerre polynomials.

If we divide the Christoffel-Darboux type formula (2.6.2) by $x - y$ and let y tend to x then we find the confluent form

$$(N+1)x^N\sum_{k=0}^{n}\frac{\left\{L_k^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}^2}{\Lambda_k}$$

$$= \sum_{k=\max(0,n-N)}^{n} \sum_{m=n+1}^{k+N+1} \frac{E_m^{(k)}}{\Lambda_k} \left[L_k^{\alpha,M_0,M_1,\ldots,M_N}(x) \frac{d}{dx} L_m^{\alpha,M_0,M_1,\ldots,M_N}(x) + \right.$$
$$\left. - L_m^{\alpha,M_0,M_1,\ldots,M_N}(x) \frac{d}{dx} L_k^{\alpha,M_0,M_1,\ldots,M_N}(x) \right]$$

for $n = 0, 1, 2, \ldots$. This formula can be considered as a generalization of (1.3.11).

Chapter 3

Koornwinder's generalized Laguerre polynomials

3.1 The definition, the orthogonality relation and some elementary properties

Koornwinder's generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ are orthogonal on the interval $[0,\infty)$ with respect to the positive weight function

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha}e^{-x}+M\delta(x). \tag{3.1.1}$$

The generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ form a special case of the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ described in the preceding chapter. In this case the inner product (2.1.1) reduces to

$$< f,g > = \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} x^{\alpha}e^{-x}f(x)g(x)dx + Mf(0)g(0),\ \alpha>-1,\ M\geq 0. \tag{3.1.2}$$

For simplicity we always write M instead of M_0 in this chapter.

The polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ are defined by

$$L_n^{\alpha,M}(x)=\left[1+M\binom{n+\alpha}{n-1}\right]L_n^{(\alpha)}(x)+M\binom{n+\alpha}{n}\frac{d}{dx}L_n^{(\alpha)}(x),\ n=0,1,2,\ldots. \tag{3.1.3}$$

In this case the system of equations (2.1.8) for the coefficients $\{A_k\}_{k=0}^{N+1}$ simply yields

$$-A_1+M\left[\binom{n+\alpha}{n}A_0-\binom{n+\alpha}{n-1}A_1\right]=0.$$

A solution which also admits the normalization $L_n^{\alpha,0}(x) = L_n^{(\alpha)}(x)$ is

$$A_0 = 1 + M\binom{n+\alpha}{n-1} \quad \text{and} \quad A_1 = M\binom{n+\alpha}{n}.$$

By using (1.3.7) and (1.3.4) we easily find for $n \geq 1$

$$\begin{aligned} L_n^{\alpha,M}(0) &= \left[1 + M\binom{n+\alpha}{n-1}\right]\binom{n+\alpha}{n} - M\binom{n+\alpha}{n}\binom{n+\alpha}{n-1} \\ &= \binom{n+\alpha}{n} = L_n^{(\alpha)}(0). \end{aligned} \tag{3.1.4}$$

This formula remains valid for $n = 0$. More general we have

$$\begin{aligned} &\left\{\frac{d^k}{dx^k} L_n^{\alpha,M}(x)\right\}_{x=0} \\ &= (-1)^k \left[1 + M\binom{n+\alpha}{n-1}\right]\binom{n+\alpha}{n-k} + (-1)^{k+1} M\binom{n+\alpha}{n}\binom{n+\alpha}{n-k-1} \\ &= (-1)^k \left[\binom{n+\alpha}{n-k} + \frac{kM}{(\alpha+1)}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-k}\right], \ k = 0, 1, 2, \ldots. \end{aligned}$$

From the definition (3.1.3) it follows that

$$L_n^{\alpha,M}(x) = L_n^{(\alpha)}(x) + \frac{M}{(\alpha+1)}\binom{n+\alpha}{n}\left[nL_n^{(\alpha)}(x) + (\alpha+1)\frac{d}{dx}L_n^{(\alpha)}(x)\right].$$

By using the relation (1.3.9) this implies that

$$L_n^{\alpha,M}(x) = L_n^{(\alpha)}(x) + \frac{M}{(\alpha+1)}\binom{n+\alpha}{n} x\frac{d}{dx}L_n^{(\alpha+1)}(x), \ n = 0, 1, 2, \ldots.$$

This is the representation (2.2.4) with

$$B_0 = 1 \quad \text{and} \quad B_1 = \frac{M}{(\alpha+1)}\binom{n+\alpha}{n}.$$

To find the representation as hypergeometric series we note that

$$L_n^{\alpha,M}(x) = \binom{n+\alpha}{n}\sum_{m=0}^{n} C_m \frac{x^m}{m!}$$

where

$$\begin{aligned} C_m &= \left[1 + M\binom{n+\alpha}{n-1}\right]\frac{(-n)_m}{(\alpha+1)_m} - M\binom{n+\alpha}{n-1}\frac{(-n+1)_m}{(\alpha+2)_m} \\ &= \frac{(-n)_m}{(\alpha+1)_{m+1}}\left[(m+\alpha+1) + (m+\alpha+1)M\binom{n+\alpha}{n-1} + (m-n)M\binom{n+\alpha}{n}\right] \\ &= \frac{(-n)_m}{(\alpha+1)_{m+1}}\left\{\left[1 + M\binom{n+\alpha+1}{n}\right]m + (\alpha+1)\right\}. \end{aligned}$$

Hence

$$L_n^{\alpha,M}(x) = \binom{n+\alpha}{n}\left[1 + M\binom{n+\alpha+1}{n}\right]\sum_{m=0}^{n}\frac{(-n)_m}{(\alpha+1)_{m+1}}(m+\gamma)\frac{x^m}{m!}, \tag{3.1.5}$$

where

$$\gamma = \frac{\alpha+1}{1 + M\binom{n+\alpha+1}{n}} > 0. \tag{3.1.6}$$

Since

$$m + \gamma = \gamma\frac{(\gamma+1)_m}{(\gamma)_m}$$

we have from (3.1.5)

$$L_n^{\alpha,M}(x) = \binom{n+\alpha}{n}{}_2F_2\left(\begin{array}{c} -n, \gamma+1 \\ \alpha+2, \gamma \end{array}\middle| x\right), \tag{3.1.7}$$

where γ is defined by (3.1.6).

By using (2.1.11) we find that the orthogonality relation is

$$\begin{aligned} < L_m^{\alpha,M}, L_n^{\alpha,M} > &= \binom{n+\alpha}{n} \times \\ &\times \left[1 + M\binom{n+\alpha}{n-1}\right]\left[1 + M\binom{n+\alpha+1}{n}\right]\delta_{mn},\ m,n = 0,1,2,\ldots, \end{aligned} \tag{3.1.8}$$

where the inner product $<,>$ is defined by (3.1.2).

3.2 A second order differential equation

In order to find the second order differential equation (2.4.1) in this case we note that (2.4.6) equals the definition (3.1.3). Differentiation of (3.1.3) gives us

$$\frac{d}{dx}L_n^{\alpha,M}(x) = \left[1 + M\binom{n+\alpha}{n-1}\right]\frac{d}{dx}L_n^{(\alpha)}(x) + M\binom{n+\alpha}{n}\frac{d^2}{dx^2}L_n^{(\alpha)}(x).$$

Now we use the Laguerre equation (1.3.5) to obtain

$$\begin{aligned} x\frac{d}{dx}L_n^{\alpha,M}(x) &= \left[1 + M\binom{n+\alpha+1}{n}\right]x\frac{d}{dx}L_n^{(\alpha)}(x) + \\ &\quad - (\alpha+1)M\binom{n+\alpha}{n}\frac{d}{dx}L_n^{(\alpha)}(x) - nM\binom{n+\alpha}{n}L_n^{(\alpha)}(x). \end{aligned} \tag{3.2.1}$$

This equals (2.4.8) so we can proceed in the same way. Another way leading to the same result is the following.

Elimination of the derivative of the classical Laguerre polynomial from (3.1.3) and (3.2.1) leads to

$$r(x)L_n^{(\alpha)}(x) = q(x)L_n^{\alpha,M}(x) - M\binom{n+\alpha}{n}x\frac{d}{dx}L_n^{\alpha,M}(x) \tag{3.2.2}$$

where

$$\begin{cases} q(x) = \left[1 + M\binom{n+\alpha+1}{n}\right]x - (\alpha+1)M\binom{n+\alpha}{n} \\ r(x) = \left[1 + M\binom{n+\alpha}{n-1}\right]q(x) + nM^2\binom{n+\alpha}{n}^2. \end{cases}$$

By using (3.1.3) and (3.2.2) we find a second order linear differential equation for Koornwinder's generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ of the form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0 \tag{3.2.3}$$

where

$$\begin{cases} a_2(x) = M^2\binom{n+\alpha}{n}^2 xr(x) \\ a_1(x) = M\binom{n+\alpha}{n}\left[1 + M\binom{n+\alpha}{n-1}\right]xr(x) + \\ \qquad - M\binom{n+\alpha}{n}q(x)r(x) + M^2\binom{n+\alpha}{n}^2[r(x) - xr'(x)] \\ a_0(x) = \{r(x)\}^2 - \left[1 + M\binom{n+\alpha}{n-1}\right]q(x)r(x) + \\ \qquad - M\binom{n+\alpha}{n}\left[q'(x)r(x) - q(x)r'(x)\right]. \end{cases} \tag{3.2.4}$$

Some tedious computations show that

$$r(x) = \left[1 + M\binom{n+\alpha}{n-1}\right]\left[1 + M\binom{n+\alpha+1}{n}\right]x - (\alpha+1)M\binom{n+\alpha}{n} \tag{3.2.5}$$

and that the differential equation given by (3.2.3) and (3.2.4) can be divided by $M^2\binom{n+\alpha}{n}^2$ so that we obtain a differential equation of the form

$$xp_2(x)y''(x) - p_1(x)y'(x) + np_0(x)y(x) = 0 \tag{3.2.6}$$

where

$$\begin{cases} p_2(x) = r(x) \\ \\ p_1(x) = (\alpha+1)M\binom{n+\alpha}{n} - (\alpha+1-x)r(x) \\ \\ p_0(x) = r(x) - M\binom{n+\alpha}{n}\left[1+M\binom{n+\alpha+1}{n}\right] \end{cases} \tag{3.2.7}$$

with $r(x)$ defined by (3.2.5). Note that the polynomials $p_2(x)$, $p_1(x)$ and $p_0(x)$ all have the same leading coefficient

$$\left[1+M\binom{n+\alpha}{n-1}\right]\left[1+M\binom{n+\alpha+1}{n}\right].$$

For $M=0$ the differential equation given by (3.2.6) and (3.2.7) can be divided by $r(x)$ in order to find the Laguerre equation (1.3.5).

3.3 The three term recurrence relation

The polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ are orthogonal on the interval $[0,\infty)$ with respect to the positive weight function (3.1.1). Hence, they satisfy a three term recurrence relation of the form

$$xL_n^{\alpha,M}(x) = A_nL_{n+1}^{\alpha,M}(x) + B_nL_n^{\alpha,M}(x) + C_nL_{n-1}^{\alpha,M}(x),\ n=1,2,3,\ldots \tag{3.3.1}$$

for some real coefficients A_n, B_n and C_n.

In order to find these coefficients we compare the coefficients of x^{n+1} on both sides of (3.3.1) to obtain by using (3.1.3) and (1.3.2)

$$A_n = -(n+1)\frac{1+M\binom{n+\alpha}{n-1}}{1+M\binom{n+\alpha+1}{n}},\ n=1,2,3,\ldots. \tag{3.3.2}$$

By taking the inner product with $L_{n-1}^{\alpha,M}(x)$ on both sides of (3.3.1) we find by using the orthogonality relation (3.1.8)

$$C_n = \frac{< L_n^{\alpha,M}(x), xL_{n-1}^{\alpha,M}(x) >}{< L_{n-1}^{\alpha,M}(x), L_{n-1}^{\alpha,M}(x) >} = -(n+\alpha)\frac{1+M\binom{n+\alpha+1}{n}}{1+M\binom{n+\alpha}{n-1}},\ n=1,2,3,\ldots. \tag{3.3.3}$$

If we substitute $x = 0$ in (3.3.1) and use (3.1.4) we find

$$\binom{n+\alpha+1}{n+1}A_n + \binom{n+\alpha}{n}B_n + \binom{n+\alpha-1}{n-1}C_n = 0,\ n = 1,2,3,\ldots$$

or simplified

$$(n+\alpha)(n+\alpha+1)A_n + (n+1)(n+\alpha)B_n + n(n+1)C_n = 0,\ n = 1,2,3,\ldots. \tag{3.3.4}$$

Now it easily follows from (3.3.2), (3.3.3) and (3.3.4) that

$$B_n = (n+\alpha+1)\frac{1+M\binom{n+\alpha}{n-1}}{1+M\binom{n+\alpha+1}{n}} + n\frac{1+M\binom{n+\alpha+1}{n}}{1+M\binom{n+\alpha}{n-1}},\ n = 1,2,3,\ldots. \tag{3.3.5}$$

With (3.3.1), (3.3.2), (3.3.3) and (3.3.5) we have found the three term recurrence relation for Koornwinder's generalized Laguerre polynomials .

3.4 The Christoffel-Darboux formula

In this section we will derive the Christoffel-Darboux formula for Koornwinder's generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$.

In this special case the Christoffel-Darboux type formula (2.6.2) reduces to the common Christoffel-Darboux formula

$$(x-y)\sum_{k=0}^{n}\frac{L_k^{\alpha,M}(x)L_k^{\alpha,M}(y)}{\lambda_k} = \frac{A_n}{\lambda_n}\left[L_{n+1}^{\alpha,M}(x)L_n^{\alpha,M}(y) - L_{n+1}^{\alpha,M}(y)L_n^{\alpha,M}(x)\right] \tag{3.4.1}$$

where (compare with (3.1.8))

$$\lambda_n := < L_n^{\alpha,M}, L_n^{\alpha,M} > = \binom{n+\alpha}{n}\left[1+M\binom{n+\alpha}{n-1}\right]\left[1+M\binom{n+\alpha+1}{n}\right] \tag{3.4.2}$$

and A_n is defined by (3.3.2) for $n = 1,2,3,\ldots$ and $A_0 := -(1+M)^{-1}$.

If we divide (3.4.1) by $x-y$ and let y tend to x we find the confluent form

$$\sum_{k=0}^{n}\frac{\left\{L_k^{\alpha,M}(x)\right\}^2}{\lambda_k} = \frac{A_n}{\lambda_n}\left[L_n^{\alpha,M}(x)\frac{d}{dx}L_{n+1}^{\alpha,M}(x) - L_{n+1}^{\alpha,M}(x)\frac{d}{dx}L_n^{\alpha,M}(x)\right]. \tag{3.4.3}$$

Hence, with (3.4.1), (3.4.3), (3.3.2) and (3.4.2) we have found the Christoffel-Darboux formula for Koornwinder's generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$:

$$\begin{aligned}&(x-y)\binom{n+\alpha}{n}\left[1+M\binom{n+\alpha+1}{n}\right]^2\sum_{k=0}^{n}\frac{L_k^{\alpha,M}(x)L_k^{\alpha,M}(y)}{\lambda_k}\\ = \;&(n+1)\left[L_n^{\alpha,M}(x)L_{n+1}^{\alpha,M}(y) - L_{n+1}^{\alpha,M}(x)L_n^{\alpha,M}(y)\right],\ n = 0,1,2,\ldots\end{aligned} \tag{3.4.4}$$

and its confluent form

$$\begin{aligned} & \binom{n+\alpha}{n}\left[1+M\binom{n+\alpha+1}{n}\right]^2 \sum_{k=0}^{n} \frac{\left\{L_k^{\alpha,M}(x)\right\}^2}{\lambda_k} \\ = \; & (n+1)\left[L_{n+1}^{\alpha,M}(x)\frac{d}{dx}L_n^{\alpha,M}(x) - L_n^{\alpha,M}(x)\frac{d}{dx}L_{n+1}^{\alpha,M}(x)\right], n = 0, 1, 2, \ldots . \end{aligned} \tag{3.4.5}$$

It is clear that for $M = 0$ the formulas (3.4.4) and (3.4.5) respectively reduce to the Christoffel-Darboux formulas (1.3.10) and (1.3.11) for the classical Laguerre polynomials.

3.5 A higher order differential equation

In this section we will prove the following theorem.

Theorem 3.1. *For $M > 0$ the polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ satisfy a unique differential equation of the form*

$$M\sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + xy''(x) + (\alpha+1-x)y'(x) + ny(x) = 0, \tag{3.5.1}$$

where $\{a_i(x)\}_{i=0}^{\infty}$ are continuous functions on the real line and $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n.

Moreover, the functions $\{a_i(x)\}_{i=0}^{\infty}$ are polynomials given by

$$\begin{cases} a_0(x) = \binom{n+\alpha+1}{n-1} \\ \\ a_i(x) = \dfrac{1}{i!}\displaystyle\sum_{j=1}^{i}(-1)^{i+j+1}\binom{\alpha+1}{j-1}\binom{\alpha+2}{i-j}(\alpha+3)_{i-j}x^j, \; i = 1, 2, 3, \ldots . \end{cases} \tag{3.5.2}$$

It is clear that for $M = 0$ (3.5.1) reduces to the Laguerre equation (1.3.5).

Note that for $\alpha \neq 0, 1, 2, \ldots$ we have $\text{degree}[a_i(x)] = i, \; i = 1, 2, 3, \ldots$. This implies that if $M > 0$ the differential equation (3.5.1) is of infinite order in that case.

For nonnegative integer values of α we have

$$\begin{cases} \text{degree}[a_i(x)] = i, & i = 1, 2, 3, \ldots, \alpha+2 \\ \\ \text{degree}[a_i(x)] = \alpha+2, & i = \alpha+3, \alpha+4, \alpha+5, \ldots, 2\alpha+4 \\ \\ a_i(x) = 0, & i = 2\alpha+5, 2\alpha+6, 2\alpha+7, \ldots . \end{cases}$$

This implies that for nonnegative integer values of α and $M > 0$ the differential equation (3.5.1) is of order $2\alpha + 4$.

Proof. We will use the notation $a_0(x) := a_0(n,\alpha,x)$ and $a_i(x) := a_i(\alpha,x)$ for $i = 1,2,3,\ldots$. In order to show that the polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ for $M>0$ satisfy a unique differential equation of the form (3.5.1), we set $y(x) = L_n^{\alpha,M}(x)$ in (3.5.1) and use the definition (3.1.3) and the Laguerre equation (2.4.4) to find

$$\begin{aligned} &M\left[1+M\binom{n+\alpha}{n-1}\right]\sum_{i=0}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x)+M^2\binom{n+\alpha}{n}\sum_{i=0}^{\infty}a_i(x)D^{i+1}L_n^{(\alpha)}(x)+\\ &\quad+M\binom{n+\alpha}{n}\left[x\frac{d^3}{dx^3}L_n^{(\alpha)}(x)+(\alpha+1-x)\frac{d^2}{dx^2}L_n^{(\alpha)}(x)+n\frac{d}{dx}L_n^{(\alpha)}(x)\right]=0.\end{aligned}$$

Now we use (2.4.5) for $k=1$ to obtain

$$\begin{aligned} &M\left[\sum_{i=0}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x)+\binom{n+\alpha}{n}\frac{d}{dx}L_n^{(\alpha)}(x)-\binom{n+\alpha}{n}\frac{d^2}{dx^2}L_n^{(\alpha)}(x)\right]+\\ &\quad+M^2\left[\binom{n+\alpha}{n-1}\sum_{i=0}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x)+\binom{n+\alpha}{n}\sum_{i=0}^{\infty}a_i(x)D^{i+1}L_n^{(\alpha)}(x)\right]=0\end{aligned}$$

for all real x, $\alpha>-1$, $M>0$ and $n=0,1,2,\ldots$. Since the expressions between square brackets are independent of M this requires that

$$\sum_{i=0}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x)+\binom{n+\alpha}{n}\frac{d}{dx}L_n^{(\alpha)}(x)-\binom{n+\alpha}{n}\frac{d^2}{dx^2}L_n^{(\alpha)}(x)=0 \tag{3.5.3}$$

and

$$n\sum_{i=0}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x)+(\alpha+1)\sum_{i=0}^{\infty}a_i(x)D^{i+1}L_n^{(\alpha)}(x)=0 \tag{3.5.4}$$

for all real x and $n=0,1,2,\ldots$.

First we show that (3.5.3) and (3.5.4) have at most one solution for $\{a_i(x)\}_{i=0}^{\infty}$. This means that we have to show that

$$\sum_{i=0}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x)=0,\ n=0,1,2,\ldots \tag{3.5.5}$$

and

$$n\sum_{i=0}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x)+(\alpha+1)\sum_{i=0}^{\infty}a_i(x)D^{i+1}L_n^{(\alpha)}(x)=0,\ n=0,1,2,\ldots \tag{3.5.6}$$

only have the trivial solution. Note that (3.5.5) and (3.5.6) imply for all real x

$$a_0(n,\alpha,x)L_n^{(\alpha)}(x)+\sum_{i=1}^{\infty}a_i(\alpha,x)D^iL_n^{(\alpha)}(x)=0,\ n=0,1,2,\ldots \tag{3.5.7}$$

and

$$a_0(n,\alpha,x)\frac{d}{dx}L_n^{(\alpha)}(x)+\sum_{i=1}^{\infty}a_i(\alpha,x)D^{i+1}L_n^{(\alpha)}(x)=0,\ n=0,1,2,\ldots. \tag{3.5.8}$$

Substitution of $n = 0$ and $n = 1$ in (3.5.7) and (3.5.8) gives us

$$a_0(0,\alpha,x) = 0,\ a_0(1,\alpha,x) = 0 \ \text{ and } \ a_1(\alpha,x) = 0 \ \text{ for all real } x.$$

Now we set $n = 2$ in (3.5.8) to obtain

$$a_0(2,\alpha,x) = 0 \ \text{ for all real } x \text{ except possibly for } x = \alpha + 2,$$

being the zero of $\frac{d}{dx}L_2^{(\alpha)}(x)$. Now we use the continuity of $a_0(x)$ to conclude that $a_0(2,\alpha,x) = 0$ for all real x. Then we obtain from (3.5.7) by setting $n = 2$

$$a_2(\alpha,x) = 0 \ \text{ for all real } x.$$

Repeating this process we finally find

$$\begin{cases} a_0(x) = a_0(n,\alpha,x) = 0 \ \text{ for all real } x \ \text{ and } n = 0,1,2,\ldots \\ \\ a_i(x) = a_i(\alpha,x) = 0 \ \text{ for all real } x \ \text{ and } i = 1,2,3,\ldots. \end{cases}$$

This proves that (3.5.5) and (3.5.6) only have the trivial solution. Hence, (3.5.3) and (3.5.4) have at most one solution.

Now we will show that (3.5.2) is a solution for (3.5.3) and (3.5.4).

We start with

$$\sum_{i=1}^{\infty} a_i(x)D^iL_n^{(\alpha)}(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \frac{(-1)^{i+j+1}}{i!}\binom{\alpha+1}{j-1}\binom{\alpha+2}{i-j}(\alpha+3)_{i-j}x^jD^iL_n^{(\alpha)}(x),$$

where x is real and n is a nonnegative integer. Changing the order of summation twice we obtain

$$\begin{aligned}
&\sum_{i=1}^{\infty} a_i(x)D^iL_n^{(\alpha)}(x) \\
&= \sum_{j=1}^{\infty}(-1)^{j+1}\binom{\alpha+1}{j-1}x^j\sum_{i=j}^{\infty}\frac{(-1)^i}{i!}\binom{\alpha+2}{i-j}(\alpha+3)_{i-j}D^iL_n^{(\alpha)}(x) \\
&= \sum_{j=1}^{\infty}(-1)^{j+1}\binom{\alpha+1}{j-1}x^j\sum_{i=0}^{\infty}\frac{(-1)^{i+j}}{(i+j)!}\binom{\alpha+2}{i}(\alpha+3)_iD^{i+j}L_n^{(\alpha)}(x) \\
&= -\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+2}{i}(\alpha+3)_i\sum_{j=1}^{\infty}\binom{\alpha+1}{j-1}\frac{x^j}{(i+j)!}D^{i+j}L_n^{(\alpha)}(x). \qquad (3.5.9)
\end{aligned}$$

Now we use the definition of the classical Laguerre polynomials (1.3.1) to find

$$\begin{aligned}
\sum_{j=1}^{\infty}\binom{\alpha+1}{j-1}\frac{x^j}{(i+j)!}D^{i+j}L_n^{(\alpha)}(x) &= \binom{n+\alpha}{n}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha+1}{j-1}\frac{(-n)_{i+j+k}x^{j+k}}{(\alpha+1)_{i+j+k}(i+j)!k!} \\
&= \binom{n+\alpha}{n}\sum_{m=1}^{\infty}C_mx^m,\ i = 0,1,2,\ldots, \qquad (3.5.10)
\end{aligned}$$

where

$$\begin{aligned}
C_m &= \sum_{j=1}^{m} \binom{\alpha+1}{j-1} \frac{(-n)_{i+m}}{(\alpha+1)_{i+m}(i+j)!(m-j)!} \\
&= \frac{(-n)_{i+m}}{(\alpha+1)_{i+m}} \sum_{j=0}^{m-1} \binom{\alpha+1}{j} \frac{1}{(i+j+1)!(m-1-j)!}, \; m = 1, 2, 3, \ldots.
\end{aligned}$$

Since $(i+j+1)! = (i+1)!(i+2)_j$,

$$\binom{\alpha+1}{j} = (-1)^j \frac{(-\alpha-1)_j}{j!} \quad \text{and} \quad \frac{1}{(m-1-j)!} = (-1)^j \frac{(-m+1)_j}{(m-1)!}$$

we obtain

$$C_m = \frac{(-n)_{i+m}}{(\alpha+1)_{i+m}} \frac{1}{(m-1)!(i+1)!} {}_2F_1\left(\begin{matrix} -m+1, -\alpha-1 \\ i+2 \end{matrix} \middle| 1 \right), \; m = 1, 2, 3, \ldots.$$

Now we use the summation formula (1.2.3) to find for $m = 1, 2, 3, \ldots$

$$\begin{aligned}
C_m &= \frac{(-n)_{i+m}}{(\alpha+1)_{i+m}} \frac{1}{(m-1)!(i+1)!} \frac{\Gamma(m+\alpha+i+2)\Gamma(i+2)}{\Gamma(m+i+1)\Gamma(\alpha+i+3)} \\
&= \frac{(-n)_{i+m}(\alpha+i+3)_{m-1}}{(\alpha+1)_{i+m}(m-1)!(i+m)!}.
\end{aligned}$$

Hence, with (3.5.9) and (3.5.10) we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} a_i(x) D^i L_n^{(\alpha)}(x) \\
&= -\binom{n+\alpha}{n} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+2}{i} (\alpha+3)_i \sum_{m=1}^{\infty} \frac{(-n)_{i+m}}{(\alpha+1)_{i+m}} \frac{(\alpha+i+3)_{m-1}}{(m-1)!(i+m)!} x^m \\
&= -\binom{n+\alpha}{n} \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} \frac{(-\alpha-2)_i(-n)_{i+m}(\alpha+3)_{i+m-1}}{i!(\alpha+1)_{i+m}(m-1)!(i+m)!} x^m.
\end{aligned}$$

Now we use the facts that $(-n)_{i+m} = (-n)_m(-n+m)_i$, $(i+m)! = m!(m+1)_i$ and

$$\frac{(\alpha+3)_{i+m-1}}{(\alpha+1)_{i+m}} = \frac{\alpha+i+m+1}{(\alpha+1)(\alpha+2)}$$

to find

$$\begin{aligned}
\sum_{i=1}^{\infty} a_i(x) D^i L_n^{(\alpha)}(x) &= -\frac{\binom{n+\alpha}{n}}{(\alpha+1)(\alpha+2)} \times \\
&\quad \times \sum_{m=1}^{\infty} \frac{(-n)_m}{(m-1)!} \frac{x^m}{m!} \sum_{i=0}^{\infty} \frac{(-\alpha-2)_i(-n+m)_i}{(m+1)_i i!} (\alpha+i+m+1).
\end{aligned}$$

We split the last sum into two parts and use the summation formula (1.2.3) obtaining

$$\begin{aligned}
&\sum_{i=0}^{\infty}\frac{(-\alpha-2)_i(-n+m)_i}{(m+1)_i i!}(\alpha+i+m+1)\\
= \; &(\alpha+m+1){}_2F_1\left(\left.\begin{array}{c}-\alpha-2,-n+m\\ m+1\end{array}\right|1\right)+\\
&\quad+\frac{(\alpha+2)(n-m)}{(m+1)}{}_2F_1\left(\left.\begin{array}{c}-\alpha-1,-n+m+1\\ m+2\end{array}\right|1\right)\\
= \; &(\alpha+m+1)\frac{\Gamma(n+\alpha+3)\Gamma(m+1)}{\Gamma(m+\alpha+3)\Gamma(n+1)}+\frac{(\alpha+2)(n-m)}{(m+1)}\frac{\Gamma(n+\alpha+2)\Gamma(m+2)}{\Gamma(m+\alpha+3)\Gamma(n+1)}\\
= \; &(\alpha+1)\binom{n+\alpha+1}{n}\frac{m!}{(\alpha+1)_{m+2}}\left[(m+2\alpha+3)n+(\alpha+1)(\alpha+2)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^{\infty}a_i(x)D^iL_n^{(\alpha)}(x) = &-\frac{1}{(\alpha+2)}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n}\times\\
&\times\sum_{m=1}^{\infty}\frac{(-n)_m}{(\alpha+1)_{m+2}}\left[(m+2\alpha+3)n+(\alpha+1)(\alpha+2)\right]\frac{x^m}{(m-1)!} \qquad (3.5.11)
\end{aligned}$$

for all real x and $n=0,1,2,\ldots$.

To complete the proof of (3.5.3) we use (3.5.2) and (1.3.1) to find

$$\begin{aligned}
&a_0(x)L_n^{(\alpha)}(x)+\binom{n+\alpha}{n}\frac{d}{dx}L_n^{(\alpha)}(x)-\binom{n+\alpha}{n}\frac{d^2}{dx^2}L_n^{(\alpha)}(x)\\
= \; &\binom{n+\alpha}{n}\sum_{k=0}^{\infty}\left[\binom{n+\alpha+1}{n-1}\frac{(-n)_k}{(\alpha+1)_k}+\binom{n+\alpha}{n}\left\{\frac{(-n)_{k+1}}{(\alpha+1)_{k+1}}-\frac{(-n)_{k+2}}{(\alpha+1)_{k+2}}\right\}\right]\frac{x^k}{k!}\\
= \; &\binom{n+\alpha}{n}\sum_{k=0}^{\infty}\left[\binom{n+\alpha+1}{n-1}\frac{(-n)_k}{(\alpha+1)_k}+(n+\alpha+1)\binom{n+\alpha}{n}\frac{(-n)_{k+1}}{(\alpha+1)_{k+2}}\right]\frac{x^k}{k!}\\
= \; &\frac{1}{(\alpha+2)}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n}\times\\
&\quad\times\sum_{k=1}^{\infty}\frac{(-n)_k}{(\alpha+1)_{k+2}}\left[n(k+2\alpha+3)+(\alpha+1)(\alpha+2)\right]\frac{x^k}{(k-1)!} \qquad (3.5.12)
\end{aligned}$$

for all real x and $n=0,1,2,\ldots$. With (3.5.11) and (3.5.12) we have proved (3.5.3).

To prove (3.5.4) we observe (compare with (3.5.9)) that

$$\sum_{i=1}^{\infty}a_i(x)D^{i+1}L_n^{(\alpha)}(x)=-\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+2}{i}(\alpha+3)_i\sum_{j=1}^{\infty}\binom{\alpha+1}{j-1}\frac{x^j}{(i+j)!}D^{i+j+1}L_n^{(\alpha)}(x)$$

for all real x and $n=0,1,2,\ldots$. In the same way as before we find

$$\sum_{j=1}^{\infty}\binom{\alpha+1}{j-1}\frac{x^j}{(i+j)!}D^{i+j+1}L_n^{(\alpha)}(x)=\binom{n+\alpha}{n}\sum_{m=1}^{\infty}D_m x^m,\ i=0,1,2,\ldots,$$

where for $m = 1, 2, 3, \ldots$

$$D_m = \sum_{j=1}^{m} \binom{\alpha+1}{j-1} \frac{(-n)_{i+m+1}}{(\alpha+1)_{i+m+1}(i+j)!(m-j)!} = \frac{(-n)_{i+m+1}(\alpha+i+3)_{m-1}}{(\alpha+1)_{i+m+1}(m-1)!(i+m)!}.$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{\infty} a_i(x) D^{i+1} L_n^{(\alpha)}(x) \\
&= -\binom{n+\alpha}{n} \sum_{m=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-\alpha-2)_i(-n)_{i+m+1}(\alpha+3)_{i+m-1}}{i!(\alpha+1)_{i+m+1}(m-1)!(i+m)!} x^m \\
&= -\frac{\binom{n+\alpha}{n}}{(\alpha+1)(\alpha+2)} \sum_{m=1}^{\infty} \frac{(-n)_{m+1}}{(m-1)!}\frac{x^m}{m!}\, {}_2F_1\left(\left.\begin{array}{c} -\alpha-2, -n+m+1 \\ m+1 \end{array}\right| 1\right) \\
&= -\frac{\binom{n+\alpha}{n}}{(\alpha+1)(\alpha+2)} \sum_{m=1}^{\infty} \frac{(-n)_{m+1}}{(m-1)!}\frac{x^m}{m!} \frac{\Gamma(n+\alpha+2)\Gamma(m+1)}{\Gamma(m+\alpha+3)\Gamma(n)} \\
&= -\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1} \sum_{m=1}^{\infty} \frac{(-n)_{m+1}}{(\alpha+1)_{m+2}} \frac{x^m}{(m-1)!}.
\end{aligned} \tag{3.5.13}$$

With (3.5.11) and (3.5.13) we have found that

$$\begin{aligned}
& n\sum_{i=1}^{\infty} a_i(x) D^i L_n^{(\alpha)}(x) + (\alpha+1)\sum_{i=1}^{\infty} a_i(x) D^{i+1} L_n^{(\alpha)}(x) \\
&= -\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1} \times \\
&\quad \times \sum_{m=1}^{\infty} \frac{(-n)_m}{(\alpha+1)_{m+2}} \left[(m+2\alpha+3)n + (\alpha+1)(\alpha+2) + (\alpha+1)(m-n)\right] \frac{x^m}{(m-1)!} \\
&= -(n+\alpha+1)\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1} \sum_{m=1}^{\infty} \frac{(-n)_m}{(\alpha+1)_{m+1}} \frac{x^m}{(m-1)!}
\end{aligned} \tag{3.5.14}$$

for all real x and $n = 0, 1, 2, \ldots$.

To complete the proof of (3.5.4) we use (3.5.2) and (1.3.1) to see that for all real x and $n = 0, 1, 2, \ldots$

$$\begin{aligned}
& n a_0(x) L_n^{(\alpha)}(x) + (\alpha+1) a_0(x) \frac{d}{dx} L_n^{(\alpha)}(x) \\
&= \binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1} \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_{k+1}} \left[n(\alpha+k+1) + (-n+k)(\alpha+1)\right] \frac{x^k}{k!} \\
&= (n+\alpha+1)\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1} \sum_{k=1}^{\infty} \frac{(-n)_k}{(\alpha+1)_{k+1}} \frac{x^k}{(k-1)!}.
\end{aligned} \tag{3.5.15}$$

With (3.5.14) and (3.5.15) we have proved (3.5.4).

This shows that Koornwinder's generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ satisfy the differential equation defined by (3.5.1) and (3.5.2).

Chapter 4

Other special cases and miscellaneous results

4.1 Another special case

In this section we consider the inner product

$$\begin{cases} < f,g > = \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty x^\alpha e^{-x} f(x)g(x)dx + Mf(0)g(0) + Nf'(0)g'(0), \\ \\ \alpha > -1,\ M \geq 0,\ N \geq 0. \end{cases} \tag{4.1.1}$$

This is a special case of the inner product (2.1.1). In this chapter we mostly write M and N instead of M_0 and M_1 respectively.

For $N > 0$ the inner product (4.1.1) cannot be obtained from any weight function, since then $< 1, x^2 > \neq < x, x >$.

Since many of the well-known properties of orthogonal polynomials depend on the existence of a positive weight function, we may not expect the polynomials which are orthogonal with respect to the inner product (4.1.1) to satisfy a three term recurrence relation and to have real and simple zeros which are located in the interior of the interval of orthogonality. Moreover, since we are dealing with an inner product which cannot be obtained from a weight function we cannot speak of an interval of orthogonality.

In this chapter we will investigate the properties of the polynomials $\left\{L_n^{\alpha,M,N}(x)\right\}_{n=0}^\infty$ which are orthogonal with respect to the inner product (4.1.1). These polynomials were found in [11] and later described in more details in [18]. In this case it is quite easy yet to establish the definition (2.1.6) since the system of equations (2.1.8) for the coefficients $\{A_k\}_{k=0}^{N+1}$ is still manageable.

It turns out that the representation (2.3.1) as hypergeometric series is quite controllable : one can quite easily establish the behaviour of the coefficients $\{\beta_i\}_{i=0}^N$ in this case.

Finally, we have some results concerning the zeros of these orthogonal polynomials in this yet simple case. We state and prove these results in this and some other special cases in this chapter.

4.2 The definition, the orthogonality relation and some elementary properties

The polynomials $\left\{L_n^{\alpha,M,N}(x)\right\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product (4.1.1) are defined by

$$L_n^{\alpha,M,N}(x) = A_0 L_n^{(\alpha)}(x) + A_1 \frac{d}{dx} L_n^{(\alpha)}(x) + A_2 \frac{d^2}{dx^2} L_n^{(\alpha)}(x), \tag{4.2.1}$$

where

$$\begin{cases} A_0 = 1 + M\binom{n+\alpha}{n-1} + \dfrac{n(\alpha+2)-(\alpha+1)}{(\alpha+1)(\alpha+3)} N \binom{n+\alpha}{n-2} + \\ \qquad\qquad + \dfrac{MN}{(\alpha+1)(\alpha+2)}\binom{n+\alpha}{n-1}\binom{n+\alpha+1}{n-2} \\ A_1 = M\binom{n+\alpha}{n} + \dfrac{(n-1)}{(\alpha+1)} N \binom{n+\alpha}{n-1} + \dfrac{2MN}{(\alpha+1)^2}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-2} \\ A_2 = \dfrac{N}{(\alpha+1)}\binom{n+\alpha}{n-1} + \dfrac{MN}{(\alpha+1)^2}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1}. \end{cases} \tag{4.2.2}$$

It is clear that for $N = 0$ we have

$$L_n^{\alpha,M,0}(x) = \left[1 + M\binom{n+\alpha}{n-1}\right] L_n^{(\alpha)}(x) + M\binom{n+\alpha}{n}\frac{d}{dx}L_n^{(\alpha)}(x) = L_n^{\alpha,M}(x).$$

These are Koornwinder's generalized Laguerre polynomials described in the preceding chapter. Of course, for $M = N = 0$ we have the normalization $L_n^{\alpha,0,0}(x) = L_n^{(\alpha)}(x)$.

Note that

$$A_0 \geq 1, \; A_1 \geq 0 \;\text{ and }\; A_2 \geq 0. \tag{4.2.3}$$

In this case the system of equations (2.1.8) for the coefficients $\{A_k\}_{k=0}^{N+1}$ reduces to a system of two equations

$$\begin{cases} -A_1 + (n-1)A_2 + M\left[\binom{n+\alpha}{n}A_0 - \binom{n+\alpha}{n-1}A_1 + \binom{n+\alpha}{n-2}A_2\right] = 0 \\ (\alpha+1)A_2 - N\left[\binom{n+\alpha}{n-1}A_0 - \binom{n+\alpha}{n-2}A_1 + \binom{n+\alpha}{n-3}A_2\right] = 0. \end{cases} \tag{4.2.4}$$

A straightforward tedious computation shows that (4.2.2) is a solution for (4.2.4) which admits the normalization $L_n^{\alpha,0,0}(x) = L_n^{(\alpha)}(x)$.

The orthogonality relation is (compare with (2.1.13))

$$< L_m^{\alpha,M,N}, L_n^{\alpha,M,N} > = \binom{n+\alpha}{n} A_0 \left(A_0 + A_1 + A_2\right) \delta_{mn}, \; m,n = 0,1,2,\ldots,$$

where the inner product $<,>$ is given by (4.1.1).

By using (4.2.1), (4.2.2), (1.3.4) and (4.2.4) we obtain

$$\begin{aligned}
& M L_n^{\alpha,M,N}(0) \\
= \; & M \left[\binom{n+\alpha}{n} A_0 - \binom{n+\alpha}{n-1} A_1 + \binom{n+\alpha}{n-2} A_2\right] \\
= \; & A_1 - (n-1) A_2 \\
= \; & M \binom{n+\alpha}{n} + \frac{MN}{(\alpha+1)^2} \binom{n+\alpha}{n} \left[2\binom{n+\alpha+1}{n-2} - (n-1)\binom{n+\alpha+1}{n-1}\right] \\
= \; & M \binom{n+\alpha}{n} \left[1 - \frac{N}{(\alpha+1)} \binom{n+\alpha+1}{n-2}\right].
\end{aligned}$$

Hence, for $M > 0$ we have

$$L_n^{\alpha,M,N}(0) = \binom{n+\alpha}{n} \left[1 - \frac{N}{(\alpha+1)} \binom{n+\alpha+1}{n-2}\right]. \tag{4.2.5}$$

For $M = 0$ we find the same formula by direct computation. Note that $L_n^{\alpha,M,N}(0)$ does not depend on M.

In the same way we obtain from (4.2.1), (4.2.2), (1.3.4), (1.3.7) and (4.2.4) :

$$\begin{aligned}
-N \left\{\frac{d}{dx} L_n^{\alpha,M,N}(x)\right\}_{x=0} &= N \left[\binom{n+\alpha}{n-1} A_0 - \binom{n+\alpha}{n-2} A_1 + \binom{n+\alpha}{n-3} A_2\right] \\
&= (\alpha+1) A_2 \\
&= N \binom{n+\alpha}{n-1} + \frac{MN}{(\alpha+1)} \binom{n+\alpha}{n} \binom{n+\alpha+1}{n-1}.
\end{aligned}$$

Hence, for $N > 0$ we have

$$\left\{\frac{d}{dx} L_n^{\alpha,M,N}(x)\right\}_{x=0} = -\binom{n+\alpha}{n-1} - \frac{M}{(\alpha+1)} \binom{n+\alpha}{n} \binom{n+\alpha+1}{n-1}. \tag{4.2.6}$$

For $N = 0$ we find the same by direct computation. Note that $\left\{\frac{d}{dx} L_n^{\alpha,M,N}(x)\right\}_{x=0}$ does not depend on N and that $\left\{\frac{d}{dx} L_n^{\alpha,M,N}(x)\right\}_{x=0} < 0$ for $n = 1,2,3,\ldots$.

The second representation (2.2.4) in this case reads

$$L_n^{\alpha,M,N}(x) = B_0 L_n^{(\alpha)}(x) + B_1 x \frac{d}{dx} L_n^{(\alpha+1)}(x) + B_2 x^2 \frac{d^2}{dx^2} L_n^{(\alpha+2)}(x), \tag{4.2.7}$$

where

$$\begin{cases} B_0 = 1 - \dfrac{N}{(\alpha+1)}\dbinom{n+\alpha+1}{n-2} \\[2ex] B_1 = \dfrac{M}{(\alpha+1)}\dbinom{n+\alpha}{n} + \dfrac{N(\alpha+2)}{(\alpha+1)(\alpha+3)}\dbinom{n+\alpha}{n-2} \\[2ex] B_2 = \dfrac{N}{(\alpha+1)(\alpha+2)(\alpha+3)}\dbinom{n+\alpha}{n-1} + \\ \qquad\qquad + \dfrac{MN}{(\alpha+1)^2(\alpha+2)(\alpha+3)}\dbinom{n+\alpha}{n}\dbinom{n+\alpha+1}{n-1}. \end{cases} \tag{4.2.8}$$

This representation easily follows from the definition (4.2.1) and (4.2.2) by using (2.2.3).

Note that (4.2.5) and (4.2.6) easily follow from this definition (4.2.7) and (4.2.8). From (4.2.8) we easily see that

$$B_1 \geq 0 \quad \text{and} \quad B_2 \geq 0.$$

Later we will show that $B_0 < 0$ if $N > 0$ and n is sufficiently large.

4.3 Representation as hypergeometric series

If we write

$$L_n^{\alpha,M,N}(x) = \binom{n+\alpha}{n}\sum_{m=0}^{n} C_m \frac{x^m}{m!},$$

then we have in this case

$$\begin{aligned} C_m &= \left[\frac{(-n)_m}{(\alpha+1)_m}A_0 + \frac{(-n)_{m+1}}{(\alpha+1)_{m+1}}A_1 + \frac{(-n)_{m+2}}{(\alpha+1)_{m+2}}A_2\right] \\ &= \frac{(-n)_m}{(\alpha+1)_{m+2}}\left[A_0(m+\alpha+1)(m+\alpha+2) + \right. \\ &\qquad\qquad \left. + A_1(m-n)(m+\alpha+2) + A_2(m-n)(m-n+1)\right] \\ &= (A_0+A_1+A_2)\,\frac{(-n)_m}{(\alpha+3)_m}\,\frac{(m+\beta_0)(m+\beta_1)}{(\alpha+1)(\alpha+2)}, \end{aligned}$$

where

$$\begin{cases} (A_0+A_1+A_2)\,(\beta_0+\beta_1) = (2\alpha+3)A_0 + (\alpha+2-n)A_1 - (2n-1)A_2 \\[2ex] (A_0+A_1+A_2)\,\beta_0\beta_1 = (\alpha+1)(\alpha+2)A_0 - n(\alpha+2)A_1 + n(n-1)A_2. \end{cases} \tag{4.3.1}$$

Hence, for $\beta_0 \neq 0, -1, -2, \ldots$ and $\beta_1 \neq 0, -1, -2, \ldots$ (2.3.1) reduces to

$$L_n^{\alpha,M,N}(x) = \binom{n+\alpha}{n}\frac{\beta_0\beta_1}{(\alpha+1)(\alpha+2)}\,(A_0+A_1+A_2)\,{}_3F_3\left(\left.\begin{array}{c} -n, \beta_0+1, \beta_1+1 \\ \alpha+3, \beta_0, \beta_1 \end{array}\right| x\right).$$

By using (4.2.5) we conclude that

$$\frac{\beta_0\beta_1}{(\alpha+1)(\alpha+2)}\left(A_0+A_1+A_2\right)=1-\frac{N}{(\alpha+1)}\binom{n+\alpha+1}{n-2} \tag{4.3.2}$$

and therefore

$$L_n^{\alpha,M,N}(x)=\binom{n+\alpha}{n}\left[1-\frac{N}{(\alpha+1)}\binom{n+\alpha+1}{n-2}\right]{}_3F_3\left(\begin{matrix}-n,\beta_0+1,\beta_1+1\\ \alpha+3,\beta_0,\beta_1\end{matrix}\,\middle|\,x\right). \tag{4.3.3}$$

Since $\binom{n+\alpha}{n}C_n=(-1)^nA_0\neq 0$ we have $\beta_0\neq -n\neq\beta_1$. If $\beta_0\in\{0,-1,-2,\ldots,-n+1\}$ or $\beta_1\in\{0,-1,-2,\ldots,-n+1\}$ we simply have to take the analytic continuation of (4.3.3). For $\beta_0<-n$ and $\beta_1<-n$ formula (4.3.3) remains valid.

The following example shows that β_0 and β_1 need not to be real. If we take $\alpha=0$, $M=0$, $N=1$ and $n=1$, then it follows from (4.2.2) that $A_0=1$, $A_1=0$ and $A_2=1$. So we have in that case by using (4.3.1)

$$\beta_0+\beta_1=1 \text{ and } \beta_0\beta_1=1.$$

Hence

$$(\beta_0-\beta_1)^2=(\beta_0+\beta_1)^2-4\beta_0\beta_1=-3<0.$$

Now we will examine β_0 and β_1 in somewhat greater detail. First, we take $N>0$. With (4.2.3) we have $A_0+A_1+A_2\geq 1>0$. Since

$$\binom{n+\alpha}{n-i}=\frac{\Gamma(n+\alpha+1)}{\Gamma(n-i+1)\Gamma(\alpha+i+1)}\sim\frac{n^{\alpha+i}}{\Gamma(\alpha+i+1)} \text{ for } n\to\infty \tag{4.3.4}$$

we conclude that the right-hand side of (4.3.2) is negative for $N>0$ and n sufficiently large. Without loss of generality this implies that

$$\beta_0<0 \text{ and } \beta_1>0$$

for n sufficiently large. Further we have with (4.2.6)

$$\begin{aligned}&-\binom{n+\alpha}{n}\left(A_0+A_1+A_2\right)\frac{n(\beta_0+1)(\beta_1+1)}{(\alpha+1)(\alpha+2)(\alpha+3)}\\ =&\binom{n+\alpha}{n}C_1=\left\{\frac{d}{dx}L_n^{\alpha,M,N}(x)\right\}_{x=0}<0,\ n=1,2,3,\ldots.\end{aligned}$$

Hence

$$(\beta_0+1)(\beta_1+1)>0 \text{ for } n=1,2,3,\ldots.$$

We conclude that

$$-1<\beta_0<0 \text{ and } \beta_1>0$$

if n is large enough.

By using (4.3.4) we find from (4.2.2) for $n \to \infty$:

$$A_0 \sim \begin{cases} \dfrac{N(\alpha+2)}{(\alpha+1)(\alpha+3)} \dfrac{n^{\alpha+3}}{\Gamma(\alpha+3)} & \text{if } M=0 \\ \\ \dfrac{MN}{(\alpha+1)(\alpha+2)} \dfrac{n^{2\alpha+4}}{\Gamma(\alpha+2)\Gamma(\alpha+4)} & \text{if } M>0, \end{cases} \tag{4.3.5}$$

$$A_1 \sim \begin{cases} \dfrac{N}{(\alpha+1)} \dfrac{n^{\alpha+2}}{\Gamma(\alpha+2)} & \text{if } M=0 \\ \\ \dfrac{2MN}{(\alpha+1)^2} \dfrac{n^{2\alpha+3}}{\Gamma(\alpha+1)\Gamma(\alpha+4)} & \text{if } M>0 \end{cases} \tag{4.3.6}$$

and

$$A_2 \sim \begin{cases} \dfrac{N}{(\alpha+1)} \dfrac{n^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if } M=0 \\ \\ \dfrac{MN}{(\alpha+1)^2} \dfrac{n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)} & \text{if } M>0. \end{cases} \tag{4.3.7}$$

Hence, for $n \to \infty$ we have

$$\begin{aligned} &(2\alpha+3)A_0 + (\alpha+2-n)A_1 - (2n-1)A_2 \sim \\ &\qquad \sim \begin{cases} \dfrac{N\alpha}{(\alpha+1)(\alpha+3)} \dfrac{n^{\alpha+3}}{\Gamma(\alpha+2)} & \text{if } M=0 \\ \\ -\dfrac{MN}{(\alpha+1)^2(\alpha+2)} \dfrac{n^{2\alpha+4}}{\Gamma(\alpha+1)\Gamma(\alpha+4)} & \text{if } M>0. \end{cases} \end{aligned} \tag{4.3.8}$$

Now it follows from (4.3.1), (4.3.2), (4.3.5), (4.3.6), (4.3.7) and (4.3.8) for $n \to \infty$:

$$\beta_0 + \beta_1 \sim \begin{cases} \alpha & \text{if } M=0 \\ \\ -1 & \text{if } M>0 \end{cases}$$

and

$$\beta_0\beta_1 \sim \begin{cases} -(\alpha+1) & \text{if } M=0 \\ \\ -\dfrac{(\alpha+1)(\alpha+2)\Gamma(\alpha+3)}{Mn^{\alpha+1}} & \text{if } M>0. \end{cases}$$

Hence for $M=0$ we have for $n \to \infty$:

$$\beta_0 \to -1 \quad \text{and} \quad \beta_1 \to \alpha+1$$

and for $M>0$ we have for $n \to \infty$:

$$\beta_0 \to -1 \quad \text{and} \quad \beta_1 \to 0.$$

If $N = 0$ we have Koornwinder's generalized Laguerre polynomials . In that case we have (compare with (3.1.7) and (3.1.6)) :

$$L_n^{\alpha,M,0}(x) = \binom{n+\alpha}{n} {}_2F_2\left(\begin{array}{c} -n, \gamma+1 \\ \alpha+2, \gamma \end{array} \middle| x \right),$$

where

$$\gamma = \frac{\alpha+1}{1+M\binom{n+\alpha+1}{n}} > 0.$$

Note that we have in this case

$$\begin{cases} \gamma = \alpha+1 & \text{if } M = 0 \\ \gamma \to 0 & \text{if } M > 0 \text{ and } n \to \infty. \end{cases}$$

4.4 The zeros

All polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with degree$[P_n(x)] = n$ which are orthogonal on an interval with respect to a positive weight function have the nice property that the polynomial $P_n(x)$ has n real and simple zeros which are located in the interior of the interval of orthogonality. Our polynomials $\left\{L_n^{\alpha,M,N}(x)\right\}_{n=0}^{\infty}$ fail to have this property. However, we will prove the following theorem.

Theorem 4.1. *The polynomial $L_n^{\alpha,M,N}(x)$ has n real and simple zeros. At least $n-1$ of these zeros lie in $(0,\infty)$.*

In other words : at most one zero of $L_n^{\alpha,M,N}(x)$ lies in $(-\infty, 0]$.

Proof. For $n \geq 1$ we have $< 1, L_n^{\alpha,M,N}(x) > = 0$. Hence

$$\frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} L_n^{\alpha,M,N}(x) dx + M L_n^{\alpha,M,N}(0) = 0.$$

This implies that the polynomial $L_n^{\alpha,M,N}(x)$ changes sign on $(0,\infty)$ at least once. Suppose that $x_1, x_2, \ldots, x_k$ are those zeros of $L_n^{\alpha,M,N}(x)$ which are positive and have odd multiplicity. Define

$$p(x) := k_n(x-x_1)(x-x_2)\cdots(x-x_k),$$

where $k_n = \frac{(-1)^n}{n!} A_0$ denotes the leading coefficient in the polynomial $L_n^{\alpha,M,N}(x)$. This implies that

$$p(x) L_n^{\alpha,M,N}(x) \geq 0 \quad \text{for all} \quad x \geq 0.$$

Now we define

$$h(x) := (x+d)p(x)$$

in such a way that $h'(0) = 0$. Hence

$$0 = h'(0) = dp'(0) + p(0).$$

Since

$$\frac{p'(0)}{p(0)} = \left\{\frac{d}{dx} \ln |p(x)|\right\}_{x=0} = -\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k}\right) < 0$$

we have

$$d = -\frac{p(0)}{p'(0)} > 0.$$

Hence

$$h(x) L_n^{\alpha,M,N}(x) \geq 0 \text{ for all } x \geq 0.$$

This implies that

$$< h, L_n^{\alpha,M,N} > = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha e^{-x} h(x) L_n^{\alpha,M,N}(x) dx + M h(0) L_n^{\alpha,M,N}(0) > 0.$$

Hence, degree$[h] \geq n$ which implies that $k \geq n-1$. This implies that all positive zeros must be simple.

So we have : at most one zero of $L_n^{\alpha,M,N}(x)$ is located outside the interval $(0,\infty)$. This immediately implies that all zeros of $L_n^{\alpha,M,N}(x)$ are real. This proves the theorem.

Now we will examine the nonpositive zero of $L_n^{\alpha,M,N}(x)$ in somewhat greater detail. First we will prove the following result.

Theorem 4.2. *If $N > 0$ and n is sufficiently large the polynomial $L_n^{\alpha,M,N}(x)$ has a zero x_n in $(-\infty, 0]$.*

For $M > 0$ this nonpositive zero is bounded :

$$-\frac{1}{2}\sqrt{\frac{N}{M}} \leq x_n \leq 0. \tag{4.4.1}$$

For all $M \geq 0$ we have

$$x_n \to 0 \text{ for } n \to \infty. \tag{4.4.2}$$

Proof. From (2.1.4) and (4.2.3) we obtain that $L_n^{\alpha,M,N}(x) > 0$ for all $x < -B$ if $B > 0$ is sufficiently large. This implies that the polynomial $L_n^{\alpha,M,N}(x)$ has a zero in $(-\infty, 0]$ if and only if $L_n^{\alpha,M,N}(0) \leq 0$. By using (4.2.5) and (4.3.4) we conclude that $L_n^{\alpha,M,N}(0) \leq 0$ for $N > 0$ and n sufficiently large.

Now we take $N > 0$ and n large enough such that $L_n^{\alpha,M,N}(x)$ has a zero x_n in $(-\infty, 0]$. Let $x_1, x_2, \ldots, x_{n-1}$ denote the positive zeros of $L_n^{\alpha,M,N}(x)$ and define

$$r(x) := (x - x_1)(x - x_2) \cdots (x - x_{n-1}).$$

Then we have

$$L_n^{\alpha,M,N}(x)=\frac{(-1)^n}{n!}A_0r(x)(x-x_n),\ x_n\le 0. \tag{4.4.3}$$

Since degree$[r]=n-1$ we have

$$\begin{aligned}0 &= <r,L_n^{\alpha,M,N}>=\frac{(-1)^n}{n!}A_0\frac{1}{\Gamma(\alpha+1)}\int_0^\infty x^\alpha e^{-x}r^2(x)(x-x_n)dx+\\ &\quad -\frac{(-1)^n}{n!}A_0Mr^2(0)x_n+\frac{(-1)^n}{n!}A_0Nr'(0)\left[r(0)-x_nr'(0)\right].\end{aligned} \tag{4.4.4}$$

Since the integral in (4.4.4) is positive we must have

$$-Mr^2(0)x_n+Nr'(0)\left[r(0)-x_nr'(0)\right]\le 0.$$

Hence

$$0\le-\left[M\{r(0)\}^2+N\{r'(0)\}^2\right]x_n\le -Nr(0)r'(0)=N\left|r(0)r'(0)\right|,$$

since $r(0)$ and $r'(0)$ have opposite signs. It follows that

$$-2\sqrt{MN}\left|r(0)r'(0)\right|x_n\le-\left[M\{r(0)\}^2+N\{r'(0)\}^2\right]x_n\le N\left|r(0)r'(0)\right|.$$

Hence

$$-2\sqrt{MN}x_n\le N.$$

This implies that for $M>0$ the zero x_n is bounded :

$$-\frac{1}{2}\sqrt{\frac{N}{M}}\le x_n\le 0.$$

This proves (4.4.1).

It remains to show that (4.4.2) is true. From Taylor's theorem we have for $x<0$:

$$\begin{aligned}L_n^{\alpha,M,N}(x) &= L_n^{\alpha,M,N}(0)+x\left\{\frac{d}{dx}L_n^{\alpha,M,N}(x)\right\}_{x=0}+\frac{x^2}{2}\left\{\frac{d^2}{dx^2}L_n^{\alpha,M,N}(x)\right\}_{x=0}+\\ &\quad +\frac{x^3}{6}\left\{\frac{d^3}{dx^3}L_n^{\alpha,M,N}(x)\right\}_{x=\xi},\ x<\xi<0.\end{aligned} \tag{4.4.5}$$

In view of Rolle's theorem every zero of $\frac{d}{dx}L_n^{\alpha,M,N}(x)$ lies between two consecutive zeros of $L_n^{\alpha,M,N}(x)$. In (4.2.6) we have seen that $\left\{\frac{d}{dx}L_n^{\alpha,M,N}(x)\right\}_{x=0}<0$. Hence, all zeros of $\frac{d}{dx}L_n^{\alpha,M,N}(x)$ are positive. This implies that $\frac{d}{dx}L_n^{\alpha,M,N}(x)$ is negative and increasing for $x<0$. In the same way we conclude that $\frac{d^2}{dx^2}L_n^{\alpha,M,N}(x)$ must be positive and decreasing for $x<0$. Hence

$$\left\{\frac{d^3}{dx^3}L_n^{\alpha,M,N}(x)\right\}_{x=\xi}<0\ \text{ for }\ x<\xi<0.$$

From (4.4.5) we conclude

$$L_n^{\alpha,M,N}(x) > ax^2+bx+c,\ x<0, \tag{4.4.6}$$

where, by using (4.2.5) and (4.2.6)

$$c = L_n^{\alpha,M,N}(0) = \binom{n+\alpha}{n}\left[1-\frac{N}{(\alpha+1)}\binom{n+\alpha+1}{n-2}\right]$$

and

$$b = \left\{\frac{d}{dx}L_n^{\alpha,M,N}(x)\right\}_{x=0} = -\binom{n+\alpha}{n-1} - \frac{M}{(\alpha+1)}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1}.$$

And by using the representation (4.2.7), (4.2.8), (1.3.4) and (1.3.7) we find

$$\begin{aligned} 2a &= \left\{\frac{d^2}{dx^2}L_n^{\alpha,M,N}(x)\right\}_{x=0} \\ &= \binom{n+\alpha}{n-2} + \frac{2M}{(\alpha+1)}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-2} + \\ &\quad + \frac{\{n(\alpha+2)-\alpha\}N}{(\alpha+1)(\alpha+2)(\alpha+4)}\binom{n+\alpha}{n-1}\binom{n+\alpha+1}{n-2} + \\ &\quad + \frac{2MN}{(\alpha+1)^2(\alpha+2)(\alpha+3)}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1}\binom{n+\alpha+2}{n-2}. \end{aligned}$$

By using (4.3.4) we find for $n\to\infty$:

$$c \sim -\frac{N}{(\alpha+1)}\frac{n^{2\alpha+3}}{\Gamma(\alpha+1)\Gamma(\alpha+4)} \quad \text{for all } \ M \geq 0,$$

$$b \sim \begin{cases} -\dfrac{n^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if } \ M=0 \\[2ex] -\dfrac{M}{(\alpha+1)}\dfrac{n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)} & \text{if } \ M>0 \end{cases}$$

and

$$a \sim \begin{cases} \dfrac{N}{2(\alpha+1)(\alpha+4)}\dfrac{n^{2\alpha+5}}{\Gamma(\alpha+2)\Gamma(\alpha+4)} & \text{if } \ M=0 \\[2ex] \dfrac{MN}{(\alpha+1)^2(\alpha+2)(\alpha+3)}\dfrac{n^{3\alpha+6}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+5)} & \text{if } \ M>0. \end{cases}$$

This implies for the sum of the roots of $ax^2+bx+c=0$ if $n\to\infty$:

$$-\frac{b}{a} \sim \begin{cases} \dfrac{2(\alpha+1)(\alpha+4)\Gamma(\alpha+4)}{Nn^{\alpha+4}} & \text{if } \ M=0 \\[2ex] \dfrac{(\alpha+1)(\alpha+2)(\alpha+3)\Gamma(\alpha+5)}{Nn^{\alpha+4}} & \text{if } \ M>0 \end{cases}$$

and for the product of the roots

$$\frac{c}{a} \sim \begin{cases} -\dfrac{2(\alpha+1)(\alpha+4)}{n^2} & \text{if } M=0 \\[2ex] -\dfrac{(\alpha+1)(\alpha+2)\Gamma(\alpha+5)}{Mn^{\alpha+3}} & \text{if } M>0. \end{cases}$$

Hence

$$-\frac{b}{a} \to 0 \text{ and } \frac{c}{a} \to 0 \text{ for } n \to \infty.$$

In view of (4.4.6) the nonpositive zero x_n of $L_n^{\alpha,M,N}(x)$ lies between the two roots of $ax^2+bx+c=0$. Hence

$$x_n \to 0 \text{ for } n \to \infty.$$

This proves (4.4.2) and therefore the theorem.

Finally we will prove the following theorem.

Theorem 4.3. *Let $N>0$ and let n be sufficiently large such that the polynomial $L_n^{\alpha,M,N}(x)$ has a zero x_n in $(-\infty,0]$. Let $x_1<x_2<\cdots<x_{n-1}$ denote the positive zeros of $L_n^{\alpha,M,N}(x)$. Then we have*

$$0 \leq -x_n < x_1. \tag{4.4.7}$$

Proof. For $x_n=0$ (4.4.7) is trivial. So we take $x_n<0$.

In (4.2.6) we have seen that $\left\{\frac{d}{dx}L_n^{\alpha,M,N}(x)\right\}_{x=0}<0$ for $n=1,2,3,\ldots$. This implies, by using (4.4.3), that

$$\frac{(-1)^n}{n!}A_0\left[r(0)-x_nr'(0)\right]<0.$$

Since $r(0)=(-1)^{n-1}x_1x_2\cdots x_{n-1}$ and

$$\frac{r'(0)}{r(0)}=\left\{\frac{d}{dx}\ln|r(x)|\right\}_{x=0}=-\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{n-1}}\right)$$

we obtain

$$-\frac{A_0}{n!}x_1x_2\cdots x_{n-1}\left[1+x_n\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{n-1}}\right)\right]<0.$$

Now we use (4.2.3) to conclude that

$$x_n\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{n-1}}\right)>-1.$$

Since $x_n<0$ this implies

$$-\frac{1}{x_n}>\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{n-1}}\geq\frac{1}{x_1}.$$

Hence

$$-x_n < x_1.$$

This proves (4.4.7) and therefore the theorem.

4.5 More results concerning the zeros

Now we consider the inner product

$$\begin{cases} < f, g > = \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty x^\alpha e^{-x} f(x)g(x)dx + \\ \qquad\qquad + M_0 f(0)g(0) + M_1 f'(0)g'(0) + M_2 f''(0)g''(0), \\ \alpha > -1,\ M_0 \geq 0,\ M_1 \geq 0 \text{ and } M_2 \geq 0 \end{cases}$$

and the polynomials $\left\{L_n^{\alpha,M_0,M_1,M_2}(x)\right\}_{n=0}^{\infty}$ which are orthogonal with respect to this inner product. For these polynomials we prove the following theorem.

Theorem 4.4. *The polynomial $L_n^{\alpha,M_0,M_1,M_2}(x)$ has at least $n-2$ positive zeros.*

Proof. For $n \geq 1$ we have $< 1, L_n^{\alpha,M_0,M_1,M_2}(x) > = 0$. Hence

$$\frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha e^{-x} L_n^{\alpha,M_0,M_1,M_2}(x)dx + M_0 L_n^{\alpha,M_0,M_1,M_2}(0) = 0.$$

This implies that the polynomial $L_n^{\alpha,M_0,M_1,M_2}(x)$ changes sign on $(0,\infty)$ at least once.

Suppose that $x_1, x_2, \ldots, x_k$ are those zeros of $L_n^{\alpha,M_0,M_1,M_2}(x)$ which are positive and have odd multiplicity. Define

$$p(x) := k_n(x-x_1)(x-x_2)\cdots(x-x_k),$$

where k_n denotes the leading coefficient in the polynomial $L_n^{\alpha,M_0,M_1,M_2}(x)$. This implies that

$$p(x)L_n^{\alpha,M_0,M_1,M_2}(x) \geq 0 \text{ for all } x \geq 0.$$

Now we define

$$h(x) := \left(x^2 + ax + b\right) p(x)$$

in such a way that $h'(0) = 0 = h''(0)$. Hence

$$\begin{cases} 0 = bp'(0) + ap(0) \\ 0 = bp''(0) + 2ap'(0) + 2p(0). \end{cases}$$

This implies

$$\begin{cases} a = -\dfrac{2p(0)p'(0)}{2\left\{p'(0)\right\}^2 - p(0)p''(0)} \\ \\ b = \dfrac{2\left\{p(0)\right\}^2}{2\left\{p'(0)\right\}^2 - p(0)p''(0)}. \end{cases}$$

Suppose that $y_1, y_2, \ldots, y_{k-1}$ are the zeros of $p'(x)$. Then we have $x_i < y_i < x_{i+1}$ for $i = 1, 2, \ldots, k-1$. This implies

$$\begin{aligned} \frac{p'(0)}{p(0)} &= \left\{\frac{d}{dx} \ln |p(x)|\right\}_{x=0} = -\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k}\right) \\ &< -\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{k-1}}\right) \\ &\leq -\left(\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_{k-1}}\right) = \left\{\frac{d}{dx} \ln |p'(x)|\right\}_{x=0} = \frac{p''(0)}{p'(0)}. \end{aligned}$$

Hence

$$\left\{p'(0)\right\}^2 - p(0)p''(0) > 0,$$

since $p(0)$ and $p'(0)$ have opposite signs. This implies that $a > 0$ and $b > 0$. Hence

$$h(x)L_n^{\alpha,M_0,M_1,M_2}(x) \geq 0 \quad \text{for all} \quad x \geq 0.$$

This implies that

$$\begin{aligned} &< h, L_n^{\alpha,M_0,M_1,M_2} > \\ = &\frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha e^{-x} h(x) L_n^{\alpha,M_0,M_1,M_2}(x)dx + M_0 h(0) L_n^{\alpha,M_0,M_1,M_2}(0) > 0. \end{aligned}$$

Hence, degree$[h] \geq n$ which implies that $k \geq n-2$. This proves the theorem.

This theorem might suggest that the maximal number of zeros of $L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)$ which may be located outside $(0,\infty)$ equals N. The following theorem shows that this is not true.

Theorem 4.5. *Let $\{S_n(x)\}_{n=0}^\infty$ with degree$[S_n(x)] = n$ denote the polynomials which are orthogonal with respect to the inner product*

$$\begin{cases} < f, g > = \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty x^\alpha e^{-x} f(x)g(x)dx + Mf(0)g(0) + Rf^{(r)}(0)g^{(r)}(0), \\ \\ \alpha > -1, \ M \geq 0, \ R \geq 0 \ \text{ and } \ r \in \{1, 2, 3, \ldots\}. \end{cases}$$

Then the polynomial $S_n(x)$ has n real and simple zeros. If $n < r$ these zeros all are positive. If $n \geq r$ at least $n-1$ of these zeros are positive.

Note that, except for the normalization, we have, by using (2.1.10)

$$S_n(x) = L_n^{\alpha,M,\overbrace{0,0,\ldots,0}^{r-1 \text{ zeros}},R}(x).$$

Proof. As before we have for $n \geq 1: <1, S_n(x)> = 0$. Hence

$$\frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha e^{-x} S_n(x)dx + MS_n(0) = 0.$$

This implies that $S_n(x)$ changes sign on $(0,\infty)$ at least once.

Suppose that $x_1, x_2, \ldots, x_k$ are those zeros of $S_n(x)$ which are positive and have odd multiplicity. Define

$$p(x) := k_n(x-x_1)(x-x_2)\cdots(x-x_k),$$

where k_n denotes the leading coefficient in the polynomial $S_n(x)$. This implies that

$$p(x)S_n(x) \geq 0 \text{ for all } x \geq 0.$$

If $n < r$ we have $S_n^{(r)}(0) = 0$. Hence

$$<p, S_n> = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha e^{-x} p(x)S_n(x)dx + Mp(0)S_n(0) > 0.$$

This implies that degree$[p(x)] \geq n$ which proves that all zeros of $S_n(x)$ are positive in that case.

Now we consider the case that $n \geq r$. Suppose that degree$[p(x)] \leq n-2$. Then we have

$$<p(x), S_n(x)> = 0 \text{ and } <xp(x), S_n(x)> = 0.$$

Hence

$$\frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha e^{-x} p(x)S_n(x)dx + Mp(0)S_n(0) + Rp^{(r)}(0)S_n^{(r)}(0) = 0$$

and

$$\frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^{\alpha+1} e^{-x} p(x)S_n(x)dx + Rrp^{(r-1)}(0)S_n^{(r)}(0) = 0.$$

Since the integrals are positive and $Mp(0)S_n(0) \geq 0$, this implies that $R > 0$ and

$$p^{(r)}(0)S_n^{(r)}(0) < 0 \text{ and } p^{(r-1)}(0)S_n^{(r)}(0) < 0.$$

Hence

$$p^{(r-1)}(0)p^{(r)}(0)\left\{S_n^{(r)}(0)\right\}^2 > 0$$

which contradicts the fact that $p^{(r-1)}(0)p^{(r)}(0) \leq 0$. Hence degree$[p(x)] \geq n-1$ or $k \geq n-1$. This proves the theorem.

PART TWO

Chapter 5

A q-analogue of the classical Laguerre polynomials

5.1 Introduction

The second part of this work deals with some q-analogues of the generalized Laguerre polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ of the first part. The q in the word q-analogue stands for a number which acts as a base usually chosen in $(-1,1)$ or $(0,1)$.

The q-theory is based on the simple observation that

$$\lim_{q\uparrow 1}\frac{1-q^a}{1-q}=a.$$

The number $\frac{1-q^a}{1-q}$ is often called the base number of a.

Many important classical functions in analysis have one or more q-analogues. So a q-analogue of a function is not unique. For instance the exponential function has two important different q-analogues denoted by e_q and E_q. For details the reader is referred to the book [6] of G. Gasper and M. Rahman.

In this chapter we give the basic definitions and formulas which we will use in the second part of this work. Further we define one q-analogue of the classical Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$. We have chosen this particular q-analogue as an example. For other q-analogues similar results will arise.

The main idea is to show that the results of part one of this work extend to the q-case and that some results concerning the zeros of the generalized q-Laguerre polynomials essentially differ from those of part one (the limit case $q\uparrow 1$).

5.2 Some basic formulas

In this section we summarize some definitions and formulas we need from the q-theory. For details the reader is referred to [6].

We always take $0 < q < 1$ in the sequel.

The q-shifted factorial is defined by

$$\begin{cases} (a;q)_0 = 1 \\ (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),\ n = 1,2,3,\dots. \end{cases}$$

For negative subscripts the q-shifted factorial is defined by

$$(a;q)_{-n} = \frac{1}{(1-aq^{-n})(1-aq^{-n+1})\cdots(1-aq^{-1})}, \quad a \neq q, q^2, q^3, \dots, q^n,\ n = 1,2,3,\dots. \tag{5.2.1}$$

Further we have for all integers n

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

where

$$(a;q)_\infty := \prod_{k=0}^{\infty} \left(1 - aq^k\right).$$

We will use two simple formulas involving these q-shifted factorials :

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k,\ k,n = 0,1,2,\dots \tag{5.2.2}$$

and

$$(a^{-1}q^{1-n};q)_n = (-a^{-1})^n q^{-\binom{n}{2}}(a;q)_n,\ a \neq 0,\ n = 0,1,2,\dots. \tag{5.2.3}$$

We have a q-analogue of the binomial coefficient given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{5.2.4}$$

It is easy to see that

$$\lim_{q \uparrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

The basic hypergeometric series or q-hypergeometric series is defined by

$${}_r\phi_s\left(\begin{array}{c} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{array} \middle| q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} \frac{(-1)^{(1+s-r)n} q^{(1+s-r)\binom{n}{2}} z^n}{(q;q)_n},$$

where

$$(a_1, a_2, \dots, a_r; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_r;q)_n.$$

This basic hypergeometric series was first introduced by Heine in 1846. Therefore it is sometimes called Heine's series.

The q-hypergeometric series is a q-analogue of the hypergeometric series defined by (1.2.1) since

$$\lim_{q\uparrow 1} {}_r\phi_s\left(\begin{array}{c} q^{\alpha_1}, q^{\alpha_2}, \ldots, q^{\alpha_r} \\ q^{\beta_1}, q^{\beta_2}, \ldots, q^{\beta_s} \end{array} \middle| q; (q-1)^{1+s-r} z\right) = {}_rF_s\left(\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_s \end{array} \middle| z\right).$$

The q-binomial theorem

$${}_1\phi_0\left(\begin{array}{c} a \\ - \end{array} \middle| q; z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \; |z| < 1$$

is a q-analogue of Newton's binomial series

$${}_1F_0\left(\begin{array}{c} \alpha \\ - \end{array} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n = (1-z)^{-\alpha}, \; |z| < 1.$$

If $a = 0$ this leads to

$$e_q(z) := {}_1\phi_0\left(\begin{array}{c} 0 \\ - \end{array} \middle| q; z\right) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \; |z| < 1, \tag{5.2.5}$$

which can be seen as a q-analogue of the exponential function since

$$\lim_{q\uparrow 1} e_q\left((1-q)z\right) = e^z.$$

We will use another summation formula

$${}_2\phi_1\left(\begin{array}{c} q^{-n}, b \\ c \end{array} \middle| q; \frac{cq^n}{b}\right) = \frac{(\frac{c}{b};q)_n}{(c;q)_n} \tag{5.2.6}$$

which is often referred to as the q-Vandermonde summation formula since it is a q-analogue of (1.2.2).

The q-difference operator D_q is defined by

$$D_q f(x) := \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \\ \\ f'(0), & x = 0 \end{cases} \tag{5.2.7}$$

where the function f is differentiable in a neighbourhood of $x = 0$. We easily see that

$$\lim_{q\uparrow 1} D_q f(x) = f'(x).$$

For functions f analytic in a neighbourhood of $x = 0$ this implies

$$\left(D_q^n f\right)(0) := \left(D_q\left(D_q^{n-1} f\right)\right)(0) = \frac{f^{(n)}(0)}{n!} \frac{(q;q)_n}{(1-q)^n}, \; n = 1, 2, 3, \ldots. \tag{5.2.8}$$

Another easy consequence of the definition (5.2.7) is

$$D_q\left[f(\gamma x)\right] = \gamma\left(D_q f\right)(\gamma x),\ \gamma\ \text{ real}$$

or more general

$$D_q^n\left[f(\gamma x)\right] = \gamma^n\left(D_q^n f\right)(\gamma x),\ \gamma\ \text{ real and }\ n = 0,1,2,\ldots. \tag{5.2.9}$$

We easily find from (5.2.7)

$$D_q\left[f(x)g(x)\right] = f(qx)D_q g(x) + g(x)D_q f(x) \tag{5.2.10}$$

which is often referred to as the q-product rule. This q-product rule can be generalized to a q-analogue of Leibniz' rule

$$D_q^n\left[f(x)g(x)\right] = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_q^{n-k} f\right)(q^k x)\left(D_q^k g\right)(x), \tag{5.2.11}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient defined by (5.2.4).

We also have q-integrals. Here we only give the definition for the q-integral on $(0,\infty)$. More about q-integrals can be found in section 1.11 of the book [6] of Gasper and Rahman. The q-integral of a function f on $(0,\infty)$ is defined by

$$\int_0^\infty f(t)d_q t := (1-q)\sum_{k=-\infty}^{\infty} f(q^k)q^k, \tag{5.2.12}$$

provided that the sum on the right-hand side converges. This definition of the q-integral on $(0,\infty)$ is due to F.H. Jackson. See [8]. It can be shown that

$$\lim_{q\uparrow 1}\int_0^\infty f(t)d_q t = \int_0^\infty f(t)dt$$

for functions f which satisfy suitable conditions. For details the reader is referred to [1] and to references given in [6].

In [7] Jackson defined a q-analogue of the gamma function :

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{5.2.13}$$

Note that this q-gamma function $\Gamma_q(x)$ satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x),\ \Gamma_q(1) = 1.$$

Jackson also showed that

$$\lim_{q\uparrow 1}\Gamma_q(x) = \Gamma(x).$$

For details the reader is referred to section 1.10 of [6] and to [1].

In [2] R. Askey gave a proof of the following integral formula which is due to Ramanujan :

$$\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty}dx = \frac{\Gamma(-\alpha)\Gamma(\alpha+1)}{\Gamma_q(-\alpha)},\ \alpha > -1. \tag{5.2.14}$$

If $\alpha = k$ is a nonnegative integer we have to take the analytic continuation

$$\begin{aligned}\lim_{\alpha\to k}\frac{\Gamma(-\alpha)\Gamma(\alpha+1)}{\Gamma_q(-\alpha)} &= \lim_{\alpha\to k}\frac{(-\alpha+k)\Gamma(-\alpha)}{(-\alpha+k)\Gamma_q(-\alpha)}\Gamma(\alpha+1)\\ &= \frac{(-1)^k}{k!}\frac{(q^{-k};q)_k\ln q^{-1}}{(1-q)^{k+1}}\Gamma(k+1)\\ &= \frac{(q;q)_k q^{-\binom{k+1}{2}}\ln q^{-1}}{(1-q)^{k+1}}.\end{aligned}$$

For the residue of the q-gamma function the reader is referred to formula (1.10.6) in [6]. We remark that we have in view of (5.2.5)

$$\frac{1}{(-(1-q)x;q)_\infty} = e_q\left(-(1-q)x\right) \to e^{-x} \text{ as } q\uparrow 1.$$

From (5.2.14) it is clear that

$$\frac{\Gamma(-\alpha)\Gamma(\alpha+1)}{\Gamma_q(-\alpha)} > 0 \text{ for all } \alpha > -1.$$

Finally we have a basic bilateral series which is defined by

$${}_r\psi_s\left(\begin{array}{c} a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{array}\middle| q;z\right) = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}(-1)^{(s-r)n}q^{(s-r)\binom{n}{2}}z^n.$$

The special case $r = s = 1$ can be summed :

$${}_1\psi_1\left(\begin{array}{c} a\\ b\end{array}\middle| q;z\right) = \sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(q,a^{-1}b,az,a^{-1}z^{-1}q;q)_\infty}{(b,a^{-1}q,z,a^{-1}z^{-1}b;q)_\infty},\ |a^{-1}b| < |z| < 1. \tag{5.2.15}$$

This summation formula is due to Ramanujan. A proof of this summation formula can be found in [2] and [6].

5.3 The definition and properties of the q-Laguerre polynomials

In this section we state the definition and some properties of the q-Laguerre polynomials $\left\{L_n^{(\alpha)}(x;q)\right\}_{n=0}^\infty$. These q-Laguerre polynomials were studied in detail by D.S. Moak in [31]. For more details concerning these polynomials the reader is referred to [12] and [31].

Let $\alpha > -1$ and $0 < q < 1$.

The q-Laguerre polynomials $\left\{L_n^{(\alpha)}(x;q)\right\}_{n=0}^{\infty}$ are defined by

$$\begin{aligned} L_n^{(\alpha)}(x;q) &= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array}\middle| q; -(1-q)q^{n+\alpha+1}x\right) \\ &= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(1-q)^k\left(q^{n+\alpha+1}x\right)^k}{(q^{\alpha+1};q)_k(q;q)_k},\ n=0,1,2,\ldots. \quad (5.3.1) \end{aligned}$$

We easily see that

$$\lim_{q\uparrow 1} L_n^{(\alpha)}(x;q) = L_n^{(\alpha)}(x),$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomial defined by (1.3.1).

By using (5.2.3) we obtain

$$L_n^{(\alpha)}(x;q) = (-1)^n q^{n(n+\alpha)}\frac{(1-q)^n}{(q;q)_n}x^n + \text{ lower order terms},\ n=0,1,2,\ldots. \quad (5.3.2)$$

These q-Laguerre polynomials $\left\{L_n^{(\alpha)}(x;q)\right\}_{n=0}^{\infty}$ satisfy two different kinds of orthogonality relations, an absolutely continuous one and a discrete one. These orthogonality relations respectively are

$$\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\int_0^{\infty}\frac{x^{\alpha}}{(-(1-q)x;q)_{\infty}}L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)dx = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\delta_{mn}$$

and

$$\frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_{\infty}}L_m^{(\alpha)}(cq^k;q)L_n^{(\alpha)}(cq^k;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\delta_{mn},\ c>0 \quad (5.3.3)$$

where the normalization factor A equals

$$A = \sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_{\infty}}.$$

By using the fact that

$$(-c(1-q)q^k;q)_{\infty} = \frac{(-c(1-q);q)_{\infty}}{(-c(1-q);q)_k}$$

we obtain from Ramanujan's sum (5.2.15) with $a=-c(1-q)$, $b=0$ and $z=q^{\alpha+1}$:

$$A = \sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_{\infty}} = \frac{(q,-c(1-q)q^{\alpha+1},-c^{-1}(1-q)^{-1}q^{-\alpha};q)_{\infty}}{(q^{\alpha+1},-c(1-q),-c^{-1}(1-q)^{-1}q;q)_{\infty}}. \quad (5.3.4)$$

Note that (5.3.3) can also be written in terms of the q-integral defined by (5.2.12) :

$$\frac{1}{A^*}\int_0^\infty \frac{t^\alpha}{(-c(1-q)t;q)_\infty} L_m^{(\alpha)}(ct;q)L_n^{(\alpha)}(ct;q)d_qt = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\delta_{mn},\ c>0$$

where A^* equals

$$A^* := \int_0^\infty \frac{t^\alpha}{(-c(1-q)t;q)_\infty} d_qt. \tag{5.3.5}$$

As a q-analogue of (1.3.4) we have

$$L_n^{(\alpha)}(0;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n},\ n=0,1,2,\ldots. \tag{5.3.6}$$

The q-Laguerre polynomials satisfy a second order q-difference equation which can be stated in terms of the q-difference operator defined by (5.2.7) as

$$xD_q^2 L_n^{(\alpha)}(x;q) + \left[\frac{1-q^{\alpha+1}}{1-q} - q^{\alpha+2}x\right]\left(D_q L_n^{(\alpha)}\right)(qx;q) + \frac{1-q^n}{1-q}q^{\alpha+1}L_n^{(\alpha)}(qx;q) = 0, \tag{5.3.7}$$

which is a q-analogue of the Laguerre equation (1.3.5).

Further we have a three term recurrence relation

$$\begin{aligned} -xL_n^{(\alpha)}(x;q) &= \frac{1-q^{n+1}}{(1-q)q^{2n+\alpha+1}} L_{n+1}^{(\alpha)}(x;q) + \\ &- \left[\frac{1-q^{n+\alpha+1}}{(1-q)q^{2n+\alpha+1}} + \frac{1-q^n}{(1-q)q^{2n+\alpha}}\right] L_n^{(\alpha)}(x;q) + \frac{1-q^{n+\alpha}}{(1-q)q^{2n+\alpha}} L_{n-1}^{(\alpha)}(x;q) \end{aligned}$$

and a Christoffel-Darboux formula

$$\begin{aligned} &(x-y)\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^n \frac{q^k(q;q)_k L_k^{(\alpha)}(x;q)L_k^{(\alpha)}(y;q)}{(q^{\alpha+1};q)_k} \\ = &\ \frac{1-q^{n+1}}{(1-q)q^{n+\alpha+1}}\left[L_n^{(\alpha)}(x;q)L_{n+1}^{(\alpha)}(y;q) - L_{n+1}^{(\alpha)}(x;q)L_n^{(\alpha)}(y;q)\right]. \end{aligned} \tag{5.3.8}$$

If we divide by $x-y$ and let y tend to x we obtain the confluent form of the Christoffel-Darboux formula

$$\begin{aligned} &\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^n \frac{q^k(q;q)_k\left\{L_k^{(\alpha)}(x;q)\right\}^2}{(q^{\alpha+1};q)_k} \\ = &\ \frac{1-q^{n+1}}{(1-q)q^{n+\alpha+1}}\left[L_{n+1}^{(\alpha)}(x;q)\frac{d}{dx}L_n^{(\alpha)}(x;q) - L_n^{(\alpha)}(x;q)\frac{d}{dx}L_{n+1}^{(\alpha)}(x;q)\right]. \end{aligned} \tag{5.3.9}$$

The q-analogue of the differentiation formula (1.3.7) yields

$$D_q^k L_n^{(\alpha)}(x;q) = (-1)^k q^{k(\alpha+k)} L_{n-k}^{(\alpha+k)}(q^k x;q),\ k=0,1,2,\ldots,n,\ n=0,1,2,\ldots. \tag{5.3.10}$$

Finally, we prove the following q-analogue of (1.3.8) :

$$L_n^{(\alpha+1)}(x;q) = L_n^{(\alpha)}(qx;q) - D_q L_n^{(\alpha)}(x;q),\ n = 0,1,2,\ldots. \tag{5.3.11}$$

For $n = 0$ this relation is trivial. For $n \geq 1$ we have, by using (5.3.10) and the definition (5.3.1)

$$\begin{aligned}
&L_n^{(\alpha)}(qx;q) - D_q L_n^{(\alpha)}(x;q) = L_n^{(\alpha)}(qx;q) + q^{\alpha+1} L_{n-1}^{(\alpha+1)}(qx;q) \\
= &\frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} (1-q)^k \left(q^{n+\alpha+2}x\right)^k}{(q^{\alpha+1};q)_k (q;q)_k} + \\
&\qquad + q^{\alpha+1} \frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} \sum_{k=0}^{n-1} \frac{(q^{-n+1};q)_k q^{\binom{k}{2}} (1-q)^k \left(q^{n+\alpha+2}x\right)^k}{(q^{\alpha+2};q)_k (q;q)_k} \\
= &\frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} (1-q)^k \left(q^{n+\alpha+2}x\right)^k}{(q^{\alpha+1};q)_{k+1} (q;q)_k} \times \\
&\qquad \times \left[1 - q^{\alpha+k+1} + q^{\alpha+1} \frac{1-q^n}{1-q^{-n}} (1-q^{-n+k})\right] \\
= &\frac{(q^{\alpha+2};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} (1-q)^k \left(q^{n+\alpha+2}x\right)^k}{(q^{\alpha+2};q)_k (q;q)_k} = L_n^{(\alpha+1)}(x;q).
\end{aligned}$$

This proves (5.3.11).

Chapter 6

Generalizations of a q-analogue of the classical Laguerre polynomials

6.1 The definition and the orthogonality relations

We will try to determine the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product

$$\begin{cases} < f,g >_q = \dfrac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\displaystyle\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} f(x)g(x)dx + \\ \qquad\qquad\qquad + \displaystyle\sum_{\nu=0}^{N} M_\nu \left(D_q^\nu f\right)(0)\left(D_q^\nu g\right)(0), \\ \alpha > -1,\ N \in \{0,1,2,\ldots\} \text{ and } M_\nu \geq 0 \text{ for all } \nu \in \{0,1,2,\ldots,N\}. \end{cases} \tag{6.1.1}$$

We will show that these orthogonal polynomials can be defined by

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = \sum_{k=0}^{N+1} q^{-k(\alpha+k)} A_k \left(D_q^k L_n^{(\alpha)}\right)(q^{-k}x;q),\ n = 0,1,2,\ldots \tag{6.1.2}$$

for some real coefficients $\{A_k\}_{k=0}^{N+1}$. Moreover, we will prove the following two orthogonality relations

$$\begin{aligned} &\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q) L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)dx + \\ &+ \sum_{\nu=0}^{N} M_\nu \left(D_q^\nu L_m^{\alpha,M_0,M_1,\ldots,M_N}\right)(0;q)\left(D_q^\nu L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(0;q) \\ = \ &\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} A_0 \left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k\right)\delta_{mn},\ m,n = 0,1,2,\ldots \end{aligned} \tag{6.1.3}$$

and

$$\begin{aligned}
&\frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty}L_m^{\alpha,M_0,M_1,\ldots,M_N}(cq^k;q)L_n^{\alpha,M_0,M_1,\ldots,M_N}(cq^k;q)+\\
&+\sum_{\nu=0}^{N}M_\nu\left(D_q^\nu L_m^{\alpha,M_0,M_1,\ldots,M_N}\right)(0;q)\left(D_q^\nu L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(0;q)\\
=\;&\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}A_0\left(\sum_{k=0}^{N+1}q^{nk-\binom{k}{2}}A_k\right)\delta_{mn},\;m,n=0,1,2,\ldots
\end{aligned}\tag{6.1.4}$$

where A is given by (5.3.4) and $c>0$ is an arbitrary constant. This second orthogonality relation can also be written in terms of the q-integral defined by (5.2.12) as

$$\begin{aligned}
&\frac{1}{A^*}\int_0^\infty\frac{t^\alpha}{(-c(1-q)t;q)_\infty}L_m^{\alpha,M_0,M_1,\ldots,M_N}(ct;q)L_n^{\alpha,M_0,M_1,\ldots,M_N}(ct;q)d_qt+\\
&+\sum_{\nu=0}^{N}M_\nu\left(D_q^\nu L_m^{\alpha,M_0,M_1,\ldots,M_N}\right)(0;q)\left(D_q^\nu L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(0;q)\\
=\;&\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}A_0\left(\sum_{k=0}^{N+1}q^{nk-\binom{k}{2}}A_k\right)\delta_{mn},\;m,n=0,1,2,\ldots
\end{aligned}$$

where A^* is defined by (5.3.5).

First we will determine the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^\infty$ which are orthogonal with respect to the inner product (6.1.1). The Gram-Schmidt orthogonalization process assures us that such a set of polynomials exists with degree$[L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)]=n$. So we may write by using (5.3.10)

$$\begin{aligned}
L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) &= \sum_{k=0}^{n}(-1)^kA_kL_{n-k}^{(\alpha+k)}(x;q)\\
&= \sum_{k=0}^{n}q^{-k(\alpha+k)}A_k\left(D_q^kL_n^{(\alpha)}\right)(q^{-k}x;q),\;n=0,1,2,\ldots,
\end{aligned}\tag{6.1.5}$$

where $L_n^{(\alpha)}(x;q)$ denotes the q-Laguerre polynomial defined by (5.3.1) and the coefficients $\{A_k\}_{k=0}^n$ are real constants which may depend on $n,\alpha,M_0,M_1,\ldots,M_N$ and q. Moreover, each polynomial $L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)$ is unique except for a multiplicative constant. We will choose this constant such that

$$L_n^{\alpha,0,0,\ldots,0}(x;q)=L_n^{(\alpha)}(x;q).$$

By using the representation (6.1.5) and (5.3.2) we easily see that the coefficient k_n of x^n in the polynomial $L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)$ equals

$$k_n=(-1)^nq^{n(n+\alpha)}\frac{(1-q)^n}{(q;q)_n}A_0.\tag{6.1.6}$$

This implies that $A_0 \neq 0$.

Let $p(x) = x^m$. First of all we choose $L_0^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = 1$ for the moment and we will try to determine the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=1}^{\infty}$ in such a way that $< p(x), L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q = 0$ for all $m \in \{0,1,2,\ldots,n-1\}$.

We use the definition (5.3.1) of the q-Laguerre polynomials and Ramanujan's integral formula (5.2.14) to obtain for $k = 0,1,2,\ldots,n$ and $m,n = 0,1,2,\ldots$

$$\begin{aligned}
&\int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty} L_{n-k}^{(\alpha+k)}(x;q)dx \\
= \; & \frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_{n-k}} \sum_{j=0}^{n-k} \frac{(q^{-n+k};q)_j q^{\binom{j}{2}}(1-q)^j q^{(n+\alpha+1)j}}{(q^{\alpha+k+1};q)_j(q;q)_j} \int_0^\infty \frac{x^{\alpha+m+j}}{(-(1-q)x;q)_\infty}dx \\
= \; & \frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_{n-k}} \sum_{j=0}^{n-k} \frac{(q^{-n+k};q)_j q^{\binom{j}{2}}(1-q)^j q^{(n+\alpha+1)j}}{(q^{\alpha+k+1};q)_j(q;q)_j} \frac{\Gamma(-\alpha-m-j)\Gamma(\alpha+m+j+1)}{\Gamma_q(-\alpha-m-j)}.
\end{aligned}$$

Now we use the definition (5.2.13) of the q-gamma function and the identities (5.2.2) and (5.2.3) to find

$$\begin{aligned}
&\frac{\Gamma_q(-\alpha)\Gamma(-\alpha-m-j)\Gamma(\alpha+m+j+1)}{\Gamma(-\alpha)\Gamma(\alpha+1)\Gamma_q(-\alpha-m-j)} = (-1)^{m+j}(1-q)^{-m-j}\frac{(q^{-\alpha-m-j};q)_\infty}{(q^{-\alpha};q)_\infty} \\
= \; & (1-q)^{-m-j} q^{-(\alpha+1)m-\binom{m}{2}} q^{-(\alpha+m+1)j-\binom{j}{2}} (q^{\alpha+1};q)_m (q^{\alpha+m+1};q)_j.
\end{aligned}$$

Hence, by using the summation formula (5.2.6) we find

$$\begin{aligned}
&\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty} L_{n-k}^{(\alpha+k)}(x;q)dx \\
= \; & \frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_{n-k}} \frac{(q^{\alpha+1};q)_m}{(1-q)^m} q^{-(\alpha+1)m-\binom{m}{2}} {}_2\phi_1\left(\left.\begin{matrix} q^{-n+k}, q^{\alpha+m+1} \\ q^{\alpha+k+1}\end{matrix}\right| q; q^{n-m}\right) \\
= \; & \frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}} \frac{(q^{\alpha+1};q)_m}{(1-q)^m} q^{-(\alpha+1)m-\binom{m}{2}}, \; k = 0,1,2,\ldots,n, \; m,n = 0,1,2,\ldots. \qquad (6.1.7)
\end{aligned}$$

Now we have by using (6.1.5) and (6.1.7)

$$\begin{aligned}
&\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)dx \\
= \; & \frac{(q^{\alpha+1};q)_m}{(1-q)^m} q^{-(\alpha+1)m-\binom{m}{2}} \sum_{k=0}^{n} (-1)^k \frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}} A_k, \; m,n = 0,1,2,\ldots.
\end{aligned}$$

First we consider the case that $n \geq N+2$ and $N+1 \leq m \leq n-1$. Then it is clear that

$$\left(D_q^\nu p\right)(0) = 0 \;\text{ for all }\; \nu \in \{0,1,2,\ldots,N\}.$$

Since

$$(q^{k-m};q)_{n-k}=0 \quad\text{for}\quad k=0,1,2,\ldots,m \quad\text{and}\quad m<n$$

we see that $< p(x), L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q = 0$ is equivalent to

$$\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}}\sum_{k=m+1}^{n}(-1)^k\frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}}A_k=0,\ m=N+1,N+2,\ldots,n-1.$$

If we substitute $m=n-1,n-2,\ldots,N+1$ respectively we easily obtain

$$A_{N+2}=A_{N+3}=\cdots=A_n=0 \quad\text{for}\quad n\geq N+2.$$

Hence, the expression (6.1.5) reduces to (6.1.2) for $n\geq N+2$. For $n\leq N+1$ (6.1.2) is trivial. In that case the coefficients $\{A_k\}_{k=n+1}^{N+1}$ can be chosen arbitrarily. This proves that the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ can be defined by (6.1.2) for all $n\in\{0,1,2,\ldots\}$.

In order to define the coefficients $\{A_k\}_{k=0}^{N+1}$ we now have to consider for $n=1,2,3,\ldots$

$$< p(x), L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q = 0 \quad\text{for}\quad m=0,1,2,\ldots,\min(n-1,N). \tag{6.1.8}$$

Since $p(x)=x^m$ we have by using (5.2.8)

$$\left(D_q^{\nu}p\right)(0)=\frac{(q;q)_m}{(1-q)^m}\delta_{m\nu},\ \nu=0,1,2,\ldots,N.$$

Hence, (6.1.8) implies, by using (6.1.1), (6.1.2), (6.1.7), (5.3.10) and (5.3.6), that

$$\begin{aligned}&\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}}\sum_{k=m+1}^{\min(n,N+1)}(-1)^k\frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}}A_k+\\&\quad+(-1)^m\frac{(q;q)_m}{(1-q)^m}q^{m(m+\alpha)}M_m\sum_{k=0}^{\min(n,N+1)}(-1)^k\frac{(q^{\alpha+k+m+1};q)_{n-k-m}}{(q;q)_{n-k-m}}q^{mk}A_k=0,\end{aligned}$$

for $m=0,1,2,\ldots,\min(n-1,N)$. We remark that the definition (5.2.1) implies that

$$\frac{(q^{\gamma};q)_{-n}}{(q;q)_{-n}}=\frac{(1-q^{-n+1})(1-q^{-n+2})\cdots(1-q^0)}{(1-q^{\gamma-n})(1-q^{\gamma-n+1})\cdots(1-q^{\gamma-1})}=0$$

for $\gamma-n>0$ and $n=1,2,3,\ldots$. Hence

$$\frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}}=\frac{(q^{\alpha+k+m+1};q)_{n-k-m}}{(q;q)_{n-k-m}}=0$$

for $k\geq n+1$ and $m=0,1,2,\ldots,\min(n-1,N)$. Note that we have by using (5.2.4)

$$\frac{(q^{k-m};q)_{n-k}}{(q;q)_{n-k}}=\begin{bmatrix}n-m-1\\n-k\end{bmatrix}_q=\begin{bmatrix}n-m-1\\k-m-1\end{bmatrix}_q=\frac{(q^{n-k+1};q)_{k-m-1}}{(q;q)_{k-m-1}},\ m<n.$$

This allows us to write

$$\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}}\sum_{k=m+1}^{N+1}(-1)^k\frac{(q^{n-k+1};q)_{k-m-1}}{(q;q)_{k-m-1}}A_k+ \\ +(-1)^m\frac{(q;q)_m}{(1-q)^m}q^{m(m+\alpha)}M_m\sum_{k=0}^{N+1}(-1)^k\frac{(q^{\alpha+k+m+1};q)_{n-k-m}}{(q;q)_{n-k-m}}q^{mk}A_k=0,$$

for $m=0,1,2,\ldots,\min(n-1,N)$. However, we will define the coefficients $\{A_k\}_{k=0}^{N+1}$ in such a way that

$$\frac{(q^{\alpha+1};q)_m}{(q;q)_m}q^{-(\alpha+1)m-\binom{m}{2}}\sum_{k=m+1}^{N+1}(-1)^k\frac{(q^{n-k+1};q)_{k-m-1}}{(q;q)_{k-m-1}}A_k+ \\ +(-1)^m q^{m(m+\alpha)}M_m\sum_{k=0}^{N+1}(-1)^k\frac{(q^{\alpha+k+m+1};q)_{n-k-m}}{(q;q)_{n-k-m}}q^{mk}A_k=0, \qquad (6.1.9)$$

for $m=0,1,2,\ldots,N$ is valid for all $n\in\{0,1,2,\ldots\}$. For $n\geq N+1$ this is the same system of equations. For $n\leq N$ we have added the following conditions on the arbitrary coefficients $\{A_k\}_{k=n+1}^{N+1}$:

$$\frac{(q^{\alpha+1};q)_m}{(q;q)_m}q^{-(\alpha+1)m-\binom{m}{2}}\sum_{k=m+1}^{N+1}(-1)^k\frac{(q^{n-k+1};q)_{k-m-1}}{(q;q)_{k-m-1}}A_k+ \\ +(-1)^m q^{m(m+\alpha)}M_m\sum_{k=0}^{N+1}(-1)^k\frac{(q^{\alpha+k+m+1};q)_{n-k-m}}{(q;q)_{n-k-m}}q^{mk}A_k=0,$$

where $m=n,n+1,n+2,\ldots,N$. Since we have by using (5.2.3) for $k\geq n+1$

$$(q^{n-k+1};q)_{k-n-1}=(-1)^{k-n-1}q^{-\binom{k-n}{2}}(q;q)_{k-n-1}=(-1)^{k-n-1}q^{nk-\binom{k}{2}-\binom{n+1}{2}}(q;q)_{k-n-1}$$

this implies

$$\begin{cases} \dfrac{(q^{\alpha+1};q)_n}{(q;q)_n}q^{-(\alpha+1)n-\binom{n}{2}-\binom{n+1}{2}}\displaystyle\sum_{k=n+1}^{N+1}q^{nk-\binom{k}{2}}A_k=q^{n(n+\alpha)}M_nA_0 \\ \\ \dfrac{(q^{\alpha+1};q)_{n+i}}{(q;q)_{n+i}}q^{-(\alpha+1)(n+i)-\binom{n+i}{2}}\times \\ \qquad\times\displaystyle\sum_{k=n+i+1}^{N+1}(-1)^k\frac{(q^{n-k+1};q)_{k-n-i-1}}{(q;q)_{k-n-i-1}}A_k=0,\ i=1,2,3,\ldots,N-n. \end{cases}$$

This implies for $n\leq N$ that $A_{n+2}=A_{n+3}=\cdots=A_{N+1}=0$ and

$$\frac{(q^{\alpha+1};q)_n}{(q;q)_n}q^{-n(n+\alpha+1)+n(n+1)-\binom{n+1}{2}}A_{n+1}=q^{n(n+\alpha)}M_nA_0.$$

However, in the sequel we only need

$$q^{n(n+\alpha)}M_nA_0 = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}q^{-n(n+\alpha+1)}\sum_{k=n+1}^{N+1} q^{nk-\binom{k}{2}}A_k \quad \text{for} \quad n \leq N. \tag{6.1.10}$$

Now we have found the representation (6.1.2) where the coefficients $\{A_k\}_{k=0}^{N+1}$ satisfy (6.1.9). Note that we changed the choice of $L_0^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = 1$ such that (6.1.2) also holds for $n = 0$.

To complete the proof of the orthogonality relation (6.1.3) we note that it follows from (6.1.2), (5.3.2) and the orthogonality we just proved that

$$\begin{aligned} & < L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q), L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q \\ = \; & (-1)^n q^{n(n+\alpha)}\frac{(1-q)^n}{(q;q)_n}A_0 < x^n, L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q . \end{aligned}$$

Now we obtain from (6.1.1), (6.1.2) and (6.1.7) for $m = n \geq N+1$

$$\begin{aligned} < x^n, L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q &= \frac{(q^{\alpha+1};q)_n}{(1-q)^n}q^{-(\alpha+1)n-\binom{n}{2}}\sum_{k=0}^{N+1}(-1)^k\frac{(q^{k-n};q)_{n-k}}{(q;q)_{n-k}}A_k \\ &= (-1)^n\frac{(q^{\alpha+1};q)_n}{(1-q)^n}q^{-n(n+\alpha+1)}\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k. \end{aligned}$$

This proves (6.1.3) in the case that $n \geq N+1$.

For $n \leq N$ we find by using (6.1.10)

$$\begin{aligned} < x^n, L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q &= \frac{(q^{\alpha+1};q)_n}{(1-q)^n}q^{-(\alpha+1)n-\binom{n}{2}}\sum_{k=0}^{n}(-1)^k\frac{(q^{k-n};q)_{n-k}}{(q;q)_{n-k}}A_k + \\ &\qquad + (-1)^n\frac{(q;q)_n}{(1-q)^n}q^{n(n+\alpha)}M_nA_0 \\ &= (-1)^n\frac{(q^{\alpha+1};q)_n}{(1-q)^n}q^{-n(n+\alpha+1)}\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k. \end{aligned}$$

This proves (6.1.3).

Finally, to prove the second orthogonality relation (6.1.4) we only need to show that for $m = 0, 1, 2, \ldots, n$

$$\begin{aligned} & \frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty}\left(cq^k\right)^m L_{n-i}^{(\alpha+i)}(cq^k;q) \\ = \; & \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\int_0^\infty \frac{x^{\alpha+m}}{(-(1-q)x;q)_\infty}L_{n-i}^{(\alpha+i)}(x;q)dx, \; i = 0, 1, 2, \ldots, n, \end{aligned} \tag{6.1.11}$$

where A is given by (5.3.4) and $c > 0$ is an arbitrary constant.

By using the definition (5.3.1) of the q-Laguerre polynomial we find for $m = 0, 1, 2, \ldots$

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty}\left(cq^k\right)^m L_{n-i}^{(\alpha+i)}(cq^k;q)\\
=\ &\frac{(q^{\alpha+i+1};q)_{n-i}}{(q;q)_{n-i}}\sum_{j=0}^{n-i}\frac{(q^{-n+i};q)_j q^{\binom{j}{2}}(1-q)^j q^{(n+\alpha+1)j}}{(q^{\alpha+i+1};q)_j(q;q)_j}c^{m+j}\sum_{k=-\infty}^{\infty}\frac{q^{(\alpha+m+j+1)k}}{(-c(1-q)q^k;q)_\infty}.
\end{aligned}$$

Now we use (5.3.4) twice to obtain

$$\frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{(\alpha+m+j+1)k}}{(-c(1-q)q^k;q)_\infty} = \frac{(q^{\alpha+1}, -c(1-q)q^{\alpha+m+j+1}, -c^{-1}(1-q)^{-1}q^{-\alpha-m-j};q)_\infty}{(q^{\alpha+m+j+1}, -c(1-q)q^{\alpha+1}, -c^{-1}(1-q)^{-1}q^{-\alpha};q)_\infty}.$$

Hence, by using (5.2.3)

$$\begin{aligned}
\frac{c^{m+j}}{A}\sum_{k=-\infty}^{\infty}\frac{q^{(\alpha+m+j+1)k}}{(-c(1-q)q^k;q)_\infty} &= \frac{c^{m+j}(q^{\alpha+1};q)_{m+j}(-c^{-1}(1-q)^{-1}q^{-\alpha-m-j};q)_{m+j}}{(-c(1-q)q^{\alpha+1};q)_{m+j}}\\
&= \frac{(q^{\alpha+1};q)_{m+j}}{(1-q)^{m+j}q^{(\alpha+1)(m+j)}}q^{-\binom{m+j}{2}}.
\end{aligned}$$

So we have by using the q-Vandermonde summation formula (5.2.6) and (5.2.2)

$$\begin{aligned}
&\frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty}\left(cq^k\right)^m L_{n-i}^{(\alpha+i)}(cq^k;q)\\
=\ &\frac{(q^{\alpha+i+1};q)_{n-i}}{(q;q)_{n-i}}\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}}{}_2\phi_1\left(\left.\begin{matrix} q^{-n+i}, q^{\alpha+m+1}\\ q^{\alpha+i+1}\end{matrix}\right| q;q^{n-m}\right)\\
=\ &\frac{(q^{i-m};q)_{n-i}}{(q;q)_{n-i}}\frac{(q^{\alpha+1};q)_m}{(1-q)^m}q^{-(\alpha+1)m-\binom{m}{2}},\ i = 0,1,2,\ldots,n,\ m,n = 0,1,2,\ldots.
\end{aligned}$$

In view of (6.1.7) this proves (6.1.11) and therefore the orthogonality relation (6.1.4).

6.2 Another representation

In this section we will show that the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ given by (6.1.2) can also be written as

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = \sum_{k=0}^{N+1} q^{-k(\alpha+2k)}B_k x^k\left(D_q^k L_n^{(\alpha+k)}\right)(q^{-k}x;q), \tag{6.2.1}$$

where the coefficients $\{B_k\}_{k=0}^{N+1}$ are related to the coefficients $\{A_k\}_{k=0}^{N+1}$ found in the preceding section in the following way

$$A_i = q^{\binom{i+1}{2}}\sum_{k=i}^{N+1} q^{-k(\alpha+k+i)}\begin{bmatrix}k\\ i\end{bmatrix}_q \frac{(q^{n-k+1};q)_{k-i}(q^{\alpha+k};q)_i}{(1-q)^k}B_k,\ i = 0,1,2,\ldots,N+1$$

and

$$B_k=\frac{(1-q)^k}{(q^{\alpha+k};q)_k}q^{-\binom{k+1}{2}+k(\alpha+2k)}\sum_{j=k}^{N+1}(-1)^{j+k}\begin{bmatrix}j\\k\end{bmatrix}_q\frac{(q^{n-j+1};q)_{j-k}}{(q^{\alpha+2k+1};q)_{j-k}}A_j,\ k=0,1,2,\ldots,N+1,$$

where the q-binomial coefficient is defined by (5.2.4).

In order to prove this we prove the following q-analogues of (1.3.9), (2.2.1), (2.2.2) and (2.2.3)

$$\frac{1-q^n}{1-q}L_n^{(\alpha)}(qx;q)+\frac{1-q^{\alpha+1}}{(1-q)q^{\alpha+1}}D_qL_n^{(\alpha)}(x;q)=xD_qL_n^{(\alpha+1)}(x;q), \tag{6.2.2}$$

$$\begin{aligned}xD_q^{k+1}L_n^{(\alpha+1)}(x;q) &= \frac{1-q^{n-k}}{1-q}q^k\left(D_q^kL_n^{(\alpha)}\right)(qx;q)+\\ &\qquad+\frac{1-q^{\alpha+k+1}}{(1-q)q^{\alpha+k+1}}D_q^{k+1}L_n^{(\alpha)}(x;q),\end{aligned} \tag{6.2.3}$$

$$\begin{aligned}x^k\left(D_q^kL_n^{(\alpha+k)}\right)(q^{-k}x;q) &= q^{k^2}\sum_{i=0}^k\begin{bmatrix}k\\i\end{bmatrix}_q\frac{(q^{n-k+1};q)_{k-i}(q^{\alpha+k};q)_i}{(1-q)^k}\times\\ &\qquad\times q^{-\binom{i}{2}-(\alpha+k)i}\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q),\end{aligned} \tag{6.2.4}$$

and

$$\begin{aligned}&q^{-k(\alpha+k)}\left(D_q^kL_n^{(\alpha)}\right)(q^{-k}x;q)\\ &=\sum_{i=0}^k(-1)^{i+k}\begin{bmatrix}k\\i\end{bmatrix}_q\frac{(q^{n-k+1};q)_{k-i}(1-q)^i}{(q^{\alpha+i};q)_i(q^{\alpha+2i+1};q)_{k-i}}q^{-\binom{i+1}{2}}x^i\left(D_q^iL_n^{(\alpha+i)}\right)(q^{-i}x;q),\end{aligned} \tag{6.2.5}$$

respectively for $k,n=0,1,2,\ldots$.

First we prove (6.2.2). For $n=0$ this relation is trivial. So we assume that $n\geq 1$. By using (5.3.10) for $k=1$ and the definition (5.3.1) of the q-Laguerre polynomials we find

$$\begin{aligned}&\frac{1-q^n}{1-q}L_n^{(\alpha)}(qx;q)+\frac{1-q^{\alpha+1}}{(1-q)q^{\alpha+1}}D_qL_n^{(\alpha)}(x;q)\\ &=\frac{1-q^n}{1-q}L_n^{(\alpha)}(qx;q)-\frac{1-q^{\alpha+1}}{1-q}L_{n-1}^{(\alpha+1)}(qx;q)\\ &=\frac{1-q^n}{1-q}\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^n\frac{(q^{-n};q)_kq^{\binom{k}{2}}(1-q)^k\left(q^{n+\alpha+2}x\right)^k}{(q^{\alpha+1};q)_k(q;q)_k}+\\ &\qquad-\frac{1-q^{\alpha+1}}{1-q}\frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}}\sum_{k=0}^{n-1}\frac{(q^{-n+1};q)_kq^{\binom{k}{2}}(1-q)^k\left(q^{n+\alpha+2}x\right)^k}{(q^{\alpha+2};q)_k(q;q)_k}\\ &=\frac{(q^{\alpha+1};q)_n}{(q;q)_{n-1}}\sum_{k=1}^n\frac{(q^{-n+1};q)_{k-1}q^{\binom{k}{2}}(1-q)^{k-1}\left(q^{n+\alpha+2}x\right)^k}{(q^{\alpha+1};q)_{k+1}(q;q)_k}\times\end{aligned}$$

$$
\begin{aligned}
& \qquad \times \left[(1-q^{-n})(1-q^{\alpha+k+1})-(1-q^{-n+k})(1-q^{\alpha+1})\right] \\
= & -\frac{(q^{\alpha+3};q)_{n-1}}{(q;q)_{n-1}q^n}\sum_{k=0}^{n-1}\frac{(q^{-n+1};q)_k q^{\binom{k+1}{2}}(1-q)^k\left(q^{n+\alpha+2}x\right)^{k+1}}{(q^{\alpha+3};q)_k(q;q)_k} \\
= & -q^{\alpha+2}xL_{n-1}^{(\alpha+2)}(qx;q)=xD_qL_n^{(\alpha+1)}(x;q),
\end{aligned}
$$

since

$$q^{-n+\binom{k+1}{2}}\left(q^{n+\alpha+2}x\right)^{k+1}=q^{\binom{k}{2}}\left(q^{n+\alpha+3}x\right)^k q^{\alpha+2}x.$$

This completes the proof of (6.2.2).

Now we will prove relation (6.2.3) which is a generalization of relation (6.2.2). From the q-analogue of Leibniz' rule (5.2.11) we obtain

$$D_q^k\left[xD_qL_n^{(\alpha+1)}(x;q)\right]=q^kxD_q^{k+1}L_n^{(\alpha+1)}(x;q)+\frac{1-q^k}{1-q}D_q^kL_n^{(\alpha+1)}(x;q).$$

Now we use relation (6.2.2) and (5.3.11) to find

$$
\begin{aligned}
& q^kxD_q^{k+1}L_n^{(\alpha+1)}(x;q) \\
= \; & D_q^k\left[xD_qL_n^{(\alpha+1)}(x;q)\right]-\frac{1-q^k}{1-q}D_q^kL_n^{(\alpha+1)}(x;q) \\
= \; & \frac{1-q^n}{1-q}q^k\left(D_q^kL_n^{(\alpha)}\right)(qx;q)+\frac{1-q^{\alpha+1}}{(1-q)q^{\alpha+1}}D_q^{k+1}L_n^{(\alpha)}(x;q)+ \\
& \qquad -\frac{1-q^k}{1-q}q^k\left(D_q^kL_n^{(\alpha)}\right)(qx;q)+\frac{1-q^k}{1-q}D_q^{k+1}L_n^{(\alpha)}(x;q) \\
= \; & \frac{1-q^{n-k}}{1-q}q^{2k}\left(D_q^kL_n^{(\alpha)}\right)(qx;q)+\frac{1-q^{\alpha+k+1}}{(1-q)q^{\alpha+1}}D_q^{k+1}L_n^{(\alpha)}(x;q).
\end{aligned}
$$

If we divide by q^k we find (6.2.3). This completes the proof of (6.2.3).

To prove relation (6.2.4), which is another generalization of (6.2.2), we use induction on k. For $k=0$ relation (6.2.4) is trivial. For $k=1$ it reduces to (6.2.2). Now we assume that (6.2.4) is true for $k=m-1$. Hence

$$
\begin{aligned}
& x^{m-1}\left(D_q^{m-1}L_n^{(\alpha+m-1)}\right)(q^{-m+1}x;q) \\
= \; & q^{(m-1)^2}\sum_{i=0}^{m-1}\begin{bmatrix}m-1\\ i\end{bmatrix}_q\frac{(q^{n-m+2};q)_{m-i-1}(q^{\alpha+m-1};q)_i}{(1-q)^{m-1}}\times \\
& \qquad \times q^{-\binom{i}{2}-(\alpha+m-1)i}\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q). \qquad (6.2.6)
\end{aligned}
$$

By using the q-product rule (5.2.10) and (5.2.9) we obtain

$$
\begin{aligned}
& D_q\left[x^{m-1}\left(D_q^{m-1}L_n^{(\alpha+m-1)}\right)(q^{-m+1}x;q)\right] \\
= \; & x^{m-1}\left(D_q^mL_n^{(\alpha+m-1)}\right)(q^{-m+1}x;q)+\frac{1-q^{m-1}}{1-q}x^{m-2}\left(D_q^{m-1}L_n^{(\alpha+m-1)}\right)(q^{-m+1}x;q).
\end{aligned}
$$

We multiply by x and use (6.2.6) and (5.2.9) to find

$$\begin{aligned}
&x^m\left(D_q^mL_n^{(\alpha+m-1)}\right)(q^{-m+1}x;q)\\
=\;& q^{(m-1)^2}\sum_{i=0}^{m-1}\begin{bmatrix}m-1\\i\end{bmatrix}_q\frac{(q^{n-m+2};q)_{m-i-1}(q^{\alpha+m-1};q)_i}{(1-q)^{m-1}}q^{-\binom{i}{2}-(\alpha+m-1)i}\times\\
&\times\left[q^{-i}x\left(D_q^{i+1}L_n^{(\alpha)}\right)(q^{-i}x;q)-\frac{1-q^{m-1}}{1-q}\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q)\right].
\end{aligned}$$

We replace α by $\alpha+1$ and x by $q^{-1}x$ to obtain

$$\begin{aligned}
&q^{-m}x^m\left(D_q^mL_n^{(\alpha+m)}\right)(q^{-m}x;q)\\
=\;& q^{(m-1)^2}\sum_{i=0}^{m-1}\begin{bmatrix}m-1\\i\end{bmatrix}_q\frac{(q^{n-m+2};q)_{m-i-1}(q^{\alpha+m};q)_i}{(1-q)^{m-1}}q^{-\binom{i}{2}-(\alpha+m)i}\times\\
&\times\left[q^{-i-1}x\left(D_q^{i+1}L_n^{(\alpha+1)}\right)(q^{-i-1}x;q)-\frac{1-q^{m-1}}{1-q}\left(D_q^iL_n^{(\alpha+1)}\right)(q^{-i-1}x;q)\right].
\end{aligned}$$

Now we multiply by q^m and use (6.2.3), (5.3.11) and (5.2.9) to find

$$\begin{aligned}
&x^m\left(D_q^mL_n^{(\alpha+m)}\right)(q^{-m}x;q)\\
=\;& q^{m^2-m+1}\sum_{i=0}^{m-1}\begin{bmatrix}m-1\\i\end{bmatrix}_q\frac{(q^{n-m+2};q)_{m-i-1}(q^{\alpha+m};q)_i}{(1-q)^{m-1}}q^{-\binom{i}{2}-(\alpha+m)i}\times\\
&\times\left[\frac{1-q^{n-i}}{1-q}q^i\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q)+\frac{1-q^{\alpha+i+1}}{(1-q)q^{\alpha+i+1}}\left(D_q^{i+1}L_n^{(\alpha)}\right)(q^{-i-1}x;q)+\right.\\
&\left.-\frac{1-q^{m-1}}{1-q}q^i\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q)+\frac{1-q^{m-1}}{1-q}\left(D_q^{i+1}L_n^{(\alpha)}\right)(q^{-i-1}x;q)\right]\\
=\;& q^{m^2}\sum_{i=0}^{m-1}\begin{bmatrix}m-1\\i\end{bmatrix}_q\frac{(q^{n-m+2};q)_{m-i-1}(q^{\alpha+m};q)_i}{(1-q)^m}q^{-\binom{i}{2}-(\alpha+m-1)i}\times\\
&\times(1-q^{n-m-i+1})\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q)+\\
&+q^{m^2}\sum_{i=1}^{m}\begin{bmatrix}m-1\\i-1\end{bmatrix}_q\frac{(q^{n-m+2};q)_{m-i}(q^{\alpha+m};q)_i}{(1-q)^m}q^{-\binom{i}{2}-(\alpha+m)i}\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q)\\
=\;& q^{m^2}\frac{(q^{n-m+1};q)_m}{(1-q)^m}L_n^{(\alpha)}(x;q)+q^{m^2}\frac{(q^{\alpha+m};q)_m}{(1-q)^m}q^{-\binom{m}{2}-(\alpha+m)m}\left(D_q^mL_n^{(\alpha)}\right)(q^{-m}x;q)+\\
&+q^{m^2}\sum_{i=1}^{m-1}\frac{(q;q)_{m-1}}{(q;q)_i(q;q)_{m-i}}\frac{(q^{n-m+2};q)_{m-i-1}(q^{\alpha+m};q)_i}{(1-q)^m}q^{-\binom{i}{2}-(\alpha+m)i}\times\\
&\times\left[q^i(1-q^{m-i})(1-q^{n-m-i+1})+(1-q^i)(1-q^{n-i+1})\right]\left(D_q^iL_n^{(\alpha)}\right)(q^{-i}x;q)\\
=\;& q^{m^2}\frac{(q^{n-m+1};q)_m}{(1-q)^m}L_n^{(\alpha)}(x;q)+q^{m^2}\frac{(q^{\alpha+m};q)_m}{(1-q)^m}q^{-\binom{m}{2}-(\alpha+m)m}\left(D_q^mL_n^{(\alpha)}\right)(q^{-m}x;q)+
\end{aligned}$$

$$
\begin{aligned}
& + q^{m^2}\sum_{i=1}^{m-1}\begin{bmatrix} m \\ i \end{bmatrix}_q \frac{(q^{n-m+1};q)_{m-i}(q^{\alpha+m};q)_i}{(1-q)^m} q^{-\binom{i}{2}-(\alpha+m)i}\left(D_q^i L_n^{(\alpha)}\right)(q^{-i}x;q) \\
= \; & q^{m^2}\sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_q \frac{(q^{n-m+1};q)_{m-i}(q^{\alpha+m};q)_i}{(1-q)^m} q^{-\binom{i}{2}-(\alpha+m)i}\left(D_q^i L_n^{(\alpha)}\right)(q^{-i}x;q).
\end{aligned}
$$

This equals (6.2.4) for $k = m$. This completes the proof of (6.2.4).

Finally, we will prove relation (6.2.5). We define

$$
\begin{cases}
C_k(x) = (1-q)^k q^{-k^2} x^k \left(D_q^k L_n^{(\alpha+k)}\right)(q^{-k}x;q), & k = 0,1,2,\ldots \\[2ex]
D_i(x) = q^{-\binom{i}{2}} \left(D_q^i L_n^{(\alpha)}\right)(q^{-i}x;q), & i = 0,1,2,\ldots.
\end{cases}
$$

This implies that (6.2.4) can be written as

$$
C_k(x) = \sum_{i=0}^{k}\begin{bmatrix} k \\ i \end{bmatrix}_q (q^{n-k+1};q)_{k-i}(q^{\alpha+k};q)_i q^{-(\alpha+k)i} D_i(x),\ k = 0,1,2,\ldots. \tag{6.2.7}
$$

It is not difficult to see that the system defined by (6.2.7) has a unique solution for $\{D_i(x)\}_{i=0}^{\infty}$. We will show that this solution is given by

$$
D_i(x) = q^{\binom{i}{2}+(\alpha+1)i}\sum_{j=0}^{i}(-1)^{i+j}\begin{bmatrix} i \\ j \end{bmatrix}_q \frac{(q^{n-i+1};q)_{i-j}q^{\binom{j}{2}}}{(q^{\alpha+j};q)_j(q^{\alpha+2j+1};q)_{i-j}} C_j(x),\ i = 0,1,2,\ldots. \tag{6.2.8}
$$

To prove this we substitute (6.2.8) in the right-hand side of (6.2.7)

$$
S_k(x) := \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}q^{\binom{i}{2}+\binom{j}{2}-(k-1)i}\begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} i \\ j \end{bmatrix}_q \frac{(q^{n-k+1};q)_{k-i}(q^{\alpha+k};q)_i(q^{n-i+1};q)_{i-j}}{(q^{\alpha+j};q)_j(q^{\alpha+2j+1};q)_{i-j}} C_j(x)
$$

and show that this equals $C_k(x)$.

From (5.2.2), (5.2.3) and (5.2.4) we obtain

$$
(q^{n-k+1};q)_{k-i}(q^{n-i+1};q)_{i-j} = (q^{n-k+1};q)_{k-j}
$$

and

$$
(-1)^{i+j}\begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} i \\ j \end{bmatrix}_q = \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(q^{j-k};q)_{i-j}}{(q;q)_{i-j}} q^{\binom{j}{2}-\binom{i}{2}+(i-j)k}.
$$

We use this to find after changing the order of summation

$$
S_k(x) = \sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(q^{n-k+1};q)_{k-j}}{(q^{\alpha+j};q)_j} q^{2\binom{j}{2}-jk} C_j(x) \sum_{i=j}^{k} \frac{(q^{j-k};q)_{i-j}(q^{\alpha+k};q)_i}{(q^{\alpha+2j+1};q)_{i-j}(q;q)_{i-j}} q^i.
$$

Now we use (5.2.2) and the summation formula (5.2.6) to find for the second sum

$$\begin{aligned} T(j,k) &:= \sum_{i=j}^{k} \frac{(q^{j-k};q)_{i-j}(q^{\alpha+k};q)_i}{(q^{\alpha+2j+1};q)_{i-j}(q;q)_{i-j}} q^i \\ &= (q^{\alpha+k};q)_j q^j \sum_{i=0}^{k-j} \frac{(q^{j-k};q)_i(q^{\alpha+k+j};q)_i}{(q^{\alpha+2j+1};q)_i(q;q)_i} q^i \\ &= (q^{\alpha+k};q)_j q^j \, {}_2\phi_1\left(\left. \begin{array}{c} q^{j-k}, q^{\alpha+k+j} \\ q^{\alpha+2j+1} \end{array} \right| q;q \right) \\ &= (q^{\alpha+k};q)_j q^j \frac{(q^{j-k+1};q)_{k-j}}{(q^{\alpha+2j+1};q)_{k-j}}, \end{aligned}$$

which equals zero for $j = 0, 1, 2, \ldots, k-1$. So we have

$$\begin{aligned} S_k(x) &= \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(q^{n-k+1};q)_{k-j}}{(q^{\alpha+j};q)_j} q^{2\binom{j}{2}-jk} C_j(x) T(j,k) \\ &= \frac{(q^{\alpha+k};q)_k}{(q^{\alpha+k};q)_k} q^{2\binom{k}{2}-k^2+k} C_k(x) = C_k(x). \end{aligned}$$

This proves (6.2.5).

6.3 Representation as basic hypergeometric series

If we write

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{m=0}^{n} C_m q^{\binom{m}{2}} q^{(n+\alpha+1)m} \frac{(1-q)^m}{(q;q)_m} x^m$$

then it follows from (6.1.2) and (5.3.1), by using (5.2.2) and (5.2.3) that

$$\begin{aligned} C_m &= \sum_{k=0}^{N+1} \frac{(q^{-n};q)_{m+k}}{(q^{\alpha+1};q)_{m+k}} q^{nk-\binom{k}{2}} A_k \\ &= \frac{(q^{-n};q)_m}{(q^{\alpha+1};q)_{m+N+1}} \sum_{k=0}^{N+1} (q^{-n+m};q)_k (q^{\alpha+k+m+1};q)_{N+1-k} q^{nk-\binom{k}{2}} A_k. \end{aligned}$$

Note that

$$F(z) := \sum_{k=0}^{N+1} (q^{-n}z;q)_k (q^{\alpha+k+1}z;q)_{N+1-k} q^{nk-\binom{k}{2}} A_k$$

is a polynomial in z of degree at most $N+1$. The coefficient of z^{N+1} in $F(z)$ equals

$$(-1)^{N+1} q^{(N+1)(\alpha+1)+\binom{N+1}{2}} \sum_{k=0}^{N+1} q^{-(\alpha+1)k-\binom{k}{2}} A_k.$$

Note that it follows from (6.1.3) and (6.1.4) that

$$F(0)=\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\neq 0.$$

This implies that all zeros of $F(z)$ can be written as (complex) powers of q. If

$$\sum_{k=0}^{N+1} q^{-(\alpha+1)k-\binom{k}{2}}A_k\neq 0, \tag{6.3.1}$$

then the polynomial $F(z)$ has degree $N+1$. In that case we may write

$$\begin{aligned} F(q^m) &= \sum_{k=0}^{N+1}(q^{-n+m};q)_k(q^{\alpha+k+m+1};q)_{N+1-k}q^{nk-\binom{k}{2}}A_k \\ &= \left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)(1-q^{\beta_0})(1-q^{\beta_1})\cdots(1-q^{\beta_N})\times \\ &\qquad\times\frac{(q^{\beta_0+1};q)_m(q^{\beta_1+1};q)_m\cdots(q^{\beta_N+1};q)_m}{(q^{\beta_0};q)_m(q^{\beta_1};q)_m\cdots(q^{\beta_N};q)_m}\end{aligned}$$

for some complex $\beta_j,\ j=0,1,2,\ldots,N$. Hence, by using

$$(q^{\alpha+1};q)_{m+N+1}=(q^{\alpha+1};q)_{N+1}(q^{\alpha+N+2};q)_m,$$

which follows directly from (5.2.2), we have

$$\begin{aligned} L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) &= \frac{(1-q^{\beta_0})(1-q^{\beta_1})\cdots(1-q^{\beta_N})}{(q^{\alpha+1};q)_{N+1}}\left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)\times \\ &\times\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_{N+2}\phi_{N+2}\left(\begin{matrix} q^{-n},q^{\beta_0+1},q^{\beta_1+1},\ldots,q^{\beta_N+1} \\ q^{\alpha+N+2},q^{\beta_0},q^{\beta_1},\ldots,q^{\beta_N}\end{matrix}\,\middle|\, q;-(1-q)q^{n+\alpha+1}x\right).\end{aligned} \tag{6.3.2}$$

If (6.3.1) is not satisfied, then $F(z)$ is a polynomial of a degree less than $N+1$. In that case we find a representation as a ${}_k\phi_k$ basic hypergeometric series where $k<N+2$ in a similar way.

6.4 A second order q-difference equation

In this section we will show that the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ satisfy a second order q-difference equation. The method found in [9] can be applied in this case too. We prove the following theorem.

Theorem 6.1. *The polynomials* $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ *satisfy a second order q-difference equation of the form*

$$\begin{aligned} xP_2(x)D_q^2L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)-P_1(x)\left(D_qL_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(qx;q)+ \\ +\frac{1-q^n}{1-q}P_0(x)L_n^{\alpha,M_0,M_1,\ldots,M_N}(qx;q)=0\end{aligned} \tag{6.4.1}$$

where $P_0(x)$, $P_1(x)$ *and* $P_2(x)$ *are polynomials with*

$$\begin{cases} P_0(x) = q^{\alpha+1}A_0\left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+1} + \text{ lower order terms} \\ P_1(x) = q^{\alpha+2}A_0\left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+2} + \text{ lower order terms} \\ P_2(x) = A_0\left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+1} + \text{ lower order terms} \end{cases} \tag{6.4.2}$$

and

$$P_1(qx) = xD_qP_2(x) + \left[q^{\alpha+N+4}x - \frac{1-q^{\alpha+N+2}}{1-q}\right]P_2(x). \tag{6.4.3}$$

Proof. We consider the q-difference equation (5.3.7) for the q-Laguerre polynomials. By using the fact that

$$L_n^{(\alpha)}(q^{-1}x;q) = L_n^{(\alpha)}(x;q) + q^{-1}(1-q)x\left(D_qL_n^{(\alpha)}\right)(q^{-1}x;q)$$

which follows directly from (5.2.7), we write this q-difference equation (5.3.7) in the following form

$$\begin{aligned} q^{-2}x\left(D_q^2L_n^{(\alpha)}\right)(q^{-2}x;q) + \left[\frac{1-q^{\alpha+1}}{1-q} - q^{n+\alpha}x\right]\left(D_qL_n^{(\alpha)}\right)(q^{-1}x;q) + \\ + \frac{1-q^n}{1-q}q^{\alpha+1}L_n^{(\alpha)}(x;q) = 0. \end{aligned} \tag{6.4.4}$$

If we let D_q^k act on (6.4.4) and use the q-analogue of Leibniz' rule (5.2.11) we obtain

$$\begin{aligned} x\left(D_q^{k+2}L_n^{(\alpha)}\right)(q^{-k-2}x;q) + q^{k+2}\left[\frac{1-q^{\alpha+k+1}}{1-q} - q^{n+\alpha}x\right]\left(D_q^{k+1}L_n^{(\alpha)}\right)(q^{-k-1}x;q) + \\ + \frac{1-q^{n-k}}{1-q}q^{\alpha+3k+3}\left(D_q^kL_n^{(\alpha)}\right)(q^{-k}x;q) = 0,\ k = 0,1,2,\ldots. \end{aligned} \tag{6.4.5}$$

Now we consider the definition (6.1.2). We multiply by x and use (6.4.5) for $k = N-1$ to find

$$xL_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = \sum_{k=0}^{N} b_k(x)\left(D_q^kL_n^{(\alpha)}\right)(q^{-k}x;q) \tag{6.4.6}$$

where

$$\begin{cases} b_k(x) = q^{-k(\alpha+k)}A_k x, \;\; k = 0,1,2,\ldots,N-2 \\ b_{N-1}(x) = q^{-(N-1)(\alpha+N-1)}A_{N-1}x - q^{\alpha+3N-(N+1)(\alpha+N+1)}\dfrac{1-q^{n-N+1}}{1-q}A_{N+1} \\ b_N(x) = q^{-N(\alpha+N)}A_N x - q^{-(N+1)(\alpha+N)}\left[\dfrac{1-q^{\alpha+N}}{1-q} - q^{n+\alpha}x\right]A_{N+1}. \end{cases}$$

Now we multiply (6.4.6) by x and use (6.4.5) for $k = N-2$ to obtain

$$x^2 L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = \sum_{k=0}^{N-1}\tilde{b}_k(x)\left(D_q^k L_n^{(\alpha)}\right)(q^{-k}x;q)$$

where

$$\begin{cases} \tilde{b}_k(x) = x b_k(x), \;\; k = 0,1,2,\ldots,N-3 \\ \tilde{b}_{N-2}(x) = x b_{N-2}(x) - \dfrac{1-q^{n-N+2}}{1-q}q^{\alpha+3N-3}b_N(x) \\ \tilde{b}_{N-1}(x) = x b_{N-1}(x) - q^N\left[\dfrac{1-q^{\alpha+N-1}}{1-q} - q^{n+\alpha}x\right]b_N(x). \end{cases}$$

Repeating this process we finally obtain by using (6.4.5) for $k = 0$

$$x^N L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = p_0(x)L_n^{(\alpha)}(x;q) + p_1(x)\left(D_q L_n^{(\alpha)}\right)(q^{-1}x;q) \tag{6.4.7}$$

for some polynomials $p_0(x)$ and $p_1(x)$ which satisfy

$$\begin{cases} p_0(x) = A_0 x^N + \text{ lower order terms} \\ p_1(x) = q^{-(n+\alpha+1)}\left(\displaystyle\sum_{k=1}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^N + \text{ lower order terms.} \end{cases} \tag{6.4.8}$$

Now we use the q-product rule (5.2.10) to obtain from (6.4.7)

$$\begin{aligned} &\frac{1-q^N}{1-q}x^{N-1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(qx;q) + x^N D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) \\ = \;& D_q p_0(x) L_n^{(\alpha)}(qx;q) + [p_0(x) + D_q p_1(x)]\left(D_q L_n^{(\alpha)}\right)(x;q) + \\ &\qquad + q^{-1}p_1(x)\left(D_q^2 L_n^{(\alpha)}\right)(q^{-1}x;q). \end{aligned}$$

We multiply by x and replace x by $q^{-1}x$ to obtain

$$\frac{1-q^N}{(1-q)q^N}x^N L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) + q^{-N-1}x^{N+1}\left(D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-1}x;q)$$

$$\begin{aligned} = \; & q^{-1}x\left(D_q p_0\right)(q^{-1}x)L_n^{(\alpha)}(x;q) + \\ & \qquad + q^{-1}x\left[p_0(q^{-1}x) + \left(D_q p_1\right)(q^{-1}x)\right]\left(D_q L_n^{(\alpha)}\right)(q^{-1}x;q) + \\ & \qquad\qquad + q^{-2}xp_1(q^{-1}x)\left(D_q^2 L_n^{(\alpha)}\right)(q^{-2}x;q). \end{aligned}$$

Now we use (6.4.4) and (6.4.7) to find

$$x^{N+1}\left(D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-1}x;q) = r_0(x)L_n^{(\alpha)}(x;q) + r_1(x)\left(D_q L_n^{(\alpha)}\right)(q^{-1}x;q) \tag{6.4.9}$$

where

$$\begin{cases} r_0(x) = q^{N+1}\left[q^{-1}x\left(D_q p_0\right)(q^{-1}x) - \dfrac{1-q^n}{1-q}q^{\alpha+1}p_1(q^{-1}x)\right] - q\dfrac{1-q^N}{1-q}p_0(x) \\ r_1(x) = q^N x\left[p_0(q^{-1}x) + \left(D_q p_1\right)(q^{-1}x)\right] + \\ \qquad - q^{N+1}\left[\dfrac{1-q^{\alpha+1}}{1-q} - q^{n+\alpha}x\right]p_1(q^{-1}x) - q\dfrac{1-q^N}{1-q}p_1(x). \end{cases} \tag{6.4.10}$$

By using (6.4.8) and (6.4.10) we easily see that

$$\begin{cases} r_0(x) = -\dfrac{1-q^n}{1-q}q^{-n+1}\left(\displaystyle\sum_{k=1}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^N + \text{ lower order terms} \\ r_1(x) = \left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+1} + \text{ lower order terms.} \end{cases} \tag{6.4.11}$$

In the same way we obtain from (6.4.9)

$$\begin{aligned} & \frac{1-q^{N+1}}{1-q}x^N D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) + q^{-1}x^{N+1}\left(D_q^2 L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-1}x;q) \\ = \; & D_q r_0(x)L_n^{(\alpha)}(qx;q) + \left[r_0(x) + D_q r_1(x)\right]D_q L_n^{(\alpha)}(x;q) + q^{-1}r_1(x)\left(D_q^2 L_n^{(\alpha)}\right)(q^{-1}x;q). \end{aligned}$$

Multiplying by x and applying (6.4.4) again gives us by using (6.4.9)

$$x^{N+2}\left(D_q^2 L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-2}x;q) = s_0(x)L_n^{(\alpha)}(x;q) + s_1(x)\left(D_q L_n^{(\alpha)}\right)(q^{-1}x;q) \tag{6.4.12}$$

where

$$\begin{cases} s_0(x) = q^{N+2}\left[x\left(D_q r_0\right)(q^{-1}x) - \dfrac{1-q^n}{1-q}q^{\alpha+2}r_1(q^{-1}x)\right] - q^2\dfrac{1-q^{N+1}}{1-q}r_0(x) \\ s_1(x) = q^{N+2}x\left[r_0(q^{-1}x) + \left(D_q r_1\right)(q^{-1}x)\right] + \\ \qquad - q^{N+3}\left[\dfrac{1-q^{\alpha+1}}{1-q} - q^{n+\alpha}x\right]r_1(q^{-1}x) - q^2\dfrac{1-q^{N+1}}{1-q}r_1(x). \end{cases} \tag{6.4.13}$$

By using (6.4.11) we easily see that

$$\begin{cases} s_0(x) = -\dfrac{1-q^n}{1-q} q^{\alpha+3} \left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+1} + \text{ lower order terms} \\ \\ s_1(x) = q^{n+\alpha+2} \left(\displaystyle\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}} A_k \right) x^{N+2} + \text{ lower order terms.} \end{cases} \tag{6.4.14}$$

Elimination of $\left(D_q L_n^{(\alpha)}\right)(q^{-1}x;q)$ from (6.4.7), (6.4.9) and (6.4.12) gives us in view of (5.3.6)

$$\begin{cases} p_0(x)r_1(x) - p_1(x)r_0(x) = x^N P_2^*(x) \\ \\ p_0(x)s_1(x) - p_1(x)s_0(x) = x^N P_1^*(x) \\ \\ r_0(x)s_1(x) - r_1(x)s_0(x) = \dfrac{1-q^n}{1-q} x^{N+1} P_0^*(x) \end{cases} \tag{6.4.15}$$

for some polynomials $P_2^*(x)$, $P_1^*(x)$ and $P_0^*(x)$. Here we used the fact that for $n=0$ it follows from (6.4.7) that $p_0(x) = A_0 x^N$. Therefore we have from (6.4.10) and (6.4.13) : $r_0(x) = s_0(x) = 0$.

Now we conclude from (6.4.7), (6.4.9) and (6.4.12), by using (6.4.15)

$$\begin{aligned} 0 &= \begin{vmatrix} x^N L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) & p_0(x) & p_1(x) \\ \\ x^{N+1}\left(D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-1}x;q) & r_0(x) & r_1(x) \\ \\ x^{N+2}\left(D_q^2 L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-2}x;q) & s_0(x) & s_1(x) \end{vmatrix} \\ &= x^{2N+2} P_2^*(x) \left(D_q^2 L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-2}x;q) + \\ &\qquad - x^{2N+1} P_1^*(x) \left(D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-1}x;q) + \\ &\qquad\qquad + \frac{1-q^n}{1-q} x^{2N+1} P_0^*(x) L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q). \end{aligned}$$

We divide by x^{2N+1} to obtain

$$\begin{aligned} &x P_2^*(x) \left(D_q^2 L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-2}x;q) - P_1^*(x) \left(D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(q^{-1}x;q) + \\ &\qquad + \frac{1-q^n}{1-q} P_0^*(x) L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = 0. \end{aligned}$$

We replace x by $q^2 x$ and use the fact that

$$L_n^{\alpha,M_0,M_1,\ldots,M_N}(q^2x;q) = L_n^{\alpha,M_0,M_1,\ldots,M_N}(qx;q) - q(1-q)x \left(D_q L_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(qx;q)$$

which follows directly from (5.2.7), to find

$$\begin{aligned} & q^2xP_2^*(q^2x)D_q^2L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) + \\ & \quad - \left[P_1^*(q^2x) + q(1-q^n)xP_0^*(q^2x)\right]\left(D_qL_n^{\alpha,M_0,M_1,\ldots,M_N}\right)(qx;q) + \\ & \quad + \frac{1-q^n}{1-q}P_0^*(q^2x)L_n^{\alpha,M_0,M_1,\ldots,M_N}(qx;q) = 0 \end{aligned}$$

which proves (6.4.1) if we define

$$\begin{cases} q^{2N+4}P_2(x) := q^2P_2^*(q^2x) \\ q^{2N+4}P_1(x) := P_1^*(q^2x) + q(1-q^n)xP_0^*(q^2x) \\ q^{2N+4}P_0(x) := P_0^*(q^2x). \end{cases} \tag{6.4.16}$$

It easily follows from (6.4.15), (6.4.8), (6.4.11) and (6.4.14) that

$$\begin{cases} P_0^*(x) = q^{\alpha+3}A_0\left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+1} + \text{ lower order terms} \\ P_1^*(x) = q^{n+\alpha+2}A_0\left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+2} + \text{ lower order terms} \\ P_2^*(x) = A_0\left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)x^{N+1} + \text{ lower order terms.} \end{cases} \tag{6.4.17}$$

Now (6.4.2) follows from (6.4.16) and (6.4.17).

It remains to show that (6.4.3) is true. To prove this we note, by using (5.2.7) and (6.4.16), that (6.4.3) is equivalent to

$$\begin{aligned} & (1-q)\left[P_1^*(qx) + (1-q^n)xP_0^*(qx)\right] \\ = \ & q^{\alpha+N+4}P_2^*(x) - q^2P_2^*(qx) + (1-q)q^{\alpha+N+4}xP_2^*(x). \end{aligned} \tag{6.4.18}$$

Now we will prove (6.4.18).

From (6.4.10) it follows by using the definition (5.2.7) that

$$\begin{cases} (1-q)r_0(qx) = q^{N+1}p_0(x) - qp_0(qx) - (1-q^n)q^{\alpha+N+2}p_1(x) \\ (1-q)r_1(qx) = (1-q)q^{N+1}xp_0(x) + q^{\alpha+N+2}p_1(x) - qp_1(qx) + \\ \qquad\qquad + (1-q)q^{n+\alpha+N+2}xp_1(x). \end{cases} \tag{6.4.19}$$

Now we use (6.4.15) and (6.4.19) to see that

$$\begin{aligned} & x^N\left[P_1^*(qx) + (1-q^n)xP_0^*(qx)\right] \\ = \ & q^{-N}\left[p_0(qx)s_1(qx) - p_1(qx)s_0(qx)\right] + (1-q)q^{-N-1}\left[r_0(qx)s_1(qx) - r_1(qx)s_0(qx)\right] \\ = \ & \left[p_0(x) - (1-q^n)q^{\alpha+1}p_1(x)\right]s_1(qx) + \\ & \quad - \left[(1-q)xp_0(x) + q^{\alpha+1}p_1(x) + (1-q)q^{n+\alpha+1}xp_1(x)\right]s_0(qx). \end{aligned} \tag{6.4.20}$$

By using (6.4.13) and (5.2.7) we find

$$\begin{cases} (1-q)s_0(qx) = q^{N+3}r_0(x) - q^2r_0(qx) - (1-q^n)q^{\alpha+N+4}r_1(x) \\ (1-q)s_1(qx) = (1-q)q^{N+3}xr_0(x) + q^{\alpha+N+4}r_1(x) - q^2r_1(qx) + \\ \qquad\qquad + (1-q)q^{n+\alpha+N+4}xr_1(x). \end{cases} \tag{6.4.21}$$

Hence, by using (6.4.20) and (6.4.21) we obtain

$$\begin{aligned} & (1-q)x^N\left[P_1^*(qx) + (1-q^n)xP_0^*(qx)\right] \\ = \; & q^{\alpha+N+4}\left[p_0(x)r_1(x) - p_1(x)r_0(x)\right] + (1-q)q^{\alpha+N+4}x\left[p_0(x)r_1(x) - p_1(x)r_0(x)\right] + \\ & + \left[(1-q)q^2xp_0(x) + q^{\alpha+3}p_1(x) + (1-q)q^{n+\alpha+3}xp_1(x)\right]r_0(qx) + \\ & - \left[q^2p_0(x) - (1-q^n)q^{\alpha+3}p_1(x)\right]r_1(qx). \end{aligned}$$

Finally, we use (6.4.19) and (6.4.15) to find

$$\begin{aligned} & (1-q)x^N\left[P_1^*(qx) + (1-q^n)xP_0^*(qx)\right] \\ = \; & q^{\alpha+N+4}\left[p_0(x)r_1(x) - p_1(x)r_0(x)\right] + (1-q)q^{\alpha+N+4}x\left[p_0(x)r_1(x) - p_1(x)r_0(x)\right] + \\ & + \left[(1-q)q^{-N+1}r_1(qx) + q^{-N+2}p_1(qx)\right]r_0(qx) + \\ & - \left[(1-q)q^{-N+1}r_0(qx) + q^{-N+2}p_0(qx)\right]r_1(qx) \\ = \; & q^{\alpha+N+4}\left[p_0(x)r_1(x) - p_1(x)r_0(x)\right] + (1-q)q^{\alpha+N+4}x\left[p_0(x)r_1(x) - p_1(x)r_0(x)\right] + \\ & - q^{-N+2}\left[p_0(qx)r_1(qx) - p_1(qx)r_0(qx)\right] \\ = \; & x^N\left[q^{\alpha+N+4}P_2^*(x) + (1-q)q^{\alpha+N+4}xP_2^*(x) - q^2P_2^*(qx)\right]. \end{aligned}$$

This proves (6.4.18) and therefore (6.4.3).

This completes the proof of the theorem.

6.5 Recurrence relation

In this section we will prove the following theorem.

Theorem 6.2. *The polynomials* $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^\infty$ *satisfy a* $(2N+3)$*-term recurrence relation of the form*

$$\begin{aligned} & x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) \\ = \; & \sum_{k=\max(0,n-N-1)}^{n+N+1} E_k^{(n)}L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q),\; n = 0,1,2,\ldots. \end{aligned} \tag{6.5.1}$$

Proof. Since $x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)$ is a polynomial of degree $n+N+1$ we have

$$x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) = \sum_{k=0}^{n+N+1} E_k^{(n)}L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q),\; n = 0,1,2,\ldots \tag{6.5.2}$$

for some real coefficients $E_k^{(n)}$, $k=0,1,2,\ldots,n+N+1$.

Taking the inner product with $L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q)$ on both sides of (6.5.2) we find by using (6.1.1) for $n=0,1,2,\ldots$ and $m=0,1,2,\ldots,n+N+1$:

$$\begin{aligned}&< L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q), L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q \cdot E_m^{(n)}\\ &\qquad = < x^{N+1}L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q), L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q\\ &\qquad = < x^{N+1}L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q), L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q . \qquad (6.5.3)\end{aligned}$$

In view of the orthogonality property of the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ we conclude that $E_m^{(n)}=0$ for $m+N+1<n$. This proves (6.5.1).

The coefficients $\{A_k\}_{k=0}^{N+1}$ in the definition (6.1.2) depend on n. To distinguish two coefficients with the same index, but depending on a different value of n we will write $A_k(n)$ instead of A_k. Comparing the leading coefficients on both sides of (6.5.2) we obtain by using this notation and (6.1.6)

$$\begin{aligned}E_{n+N+1}^{(n)} &= \frac{k_n}{k_{n+N+1}}\\ &= (-1)^{N+1}q^{-(N+1)(2n+\alpha+N+1)}\frac{(q^{n+1};q)_{N+1}}{(1-q)^{N+1}}\frac{A_0(n)}{A_0(n+N+1)} \neq 0,\ n=0,1,2,\ldots.\end{aligned}$$

If we define

$$\Lambda_n := < L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q), L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q) >_q = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}A_0\left(\sum_{k=0}^{N+1} q^{nk-\binom{k}{2}}A_k\right)$$

then we find by using (6.5.3), (6.1.6) and the orthogonality that

$$E_{n-N-1}^{(n)} = \frac{k_{n-N-1}\Lambda_n}{k_n\Lambda_{n-N-1}} \neq 0,\ n=N+1,N+2,\ldots.$$

6.6 A Christoffel-Darboux type formula

From the recurrence relation (6.5.1) we easily obtain

$$\begin{aligned}&\left(x^{N+1}-y^{N+1}\right) L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y;q)\\ &= \sum_{m=\max(0,k-N-1)}^{k+N+1} E_m^{(k)}\left[L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y;q) + \right.\\ &\qquad \left. - L_m^{\alpha,M_0,M_1,\ldots,M_N}(y;q)L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right],\ k=0,1,2,\ldots. \qquad (6.6.1)\end{aligned}$$

We divide by Λ_k and sum over $k=0,1,2,\ldots,n$:

$$\left(x^{N+1}-y^{N+1}\right)\sum_{k=0}^{n}\frac{L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y;q)}{\Lambda_k}$$

$$= \sum_{k=0}^{n} \sum_{m=\max(0,k-N-1)}^{k+N+1} \frac{E_m^{(k)}}{\Lambda_k} \Big[L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q) L_k^{\alpha,M_0,M_1,\ldots,M_N}(y;q) + \\ - L_m^{\alpha,M_0,M_1,\ldots,M_N}(y;q) L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q) \Big]$$

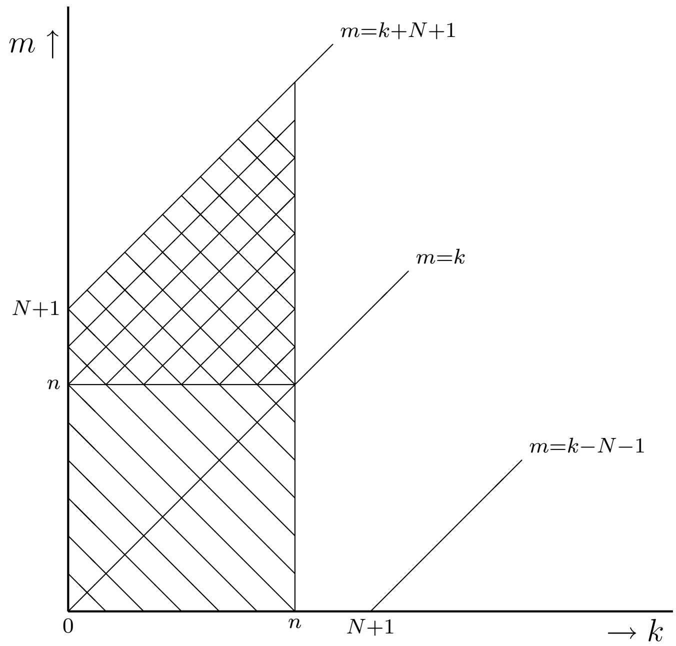
for $n = 0, 1, 2, \ldots$.

Now we use (6.5.3) to see that

$$\frac{E_m^{(k)}}{\Lambda_k} = \frac{E_k^{(m)}}{\Lambda_m},\ k-N-1 \leq m \leq k+N+1,\ k,m = 0,1,2,\ldots.$$

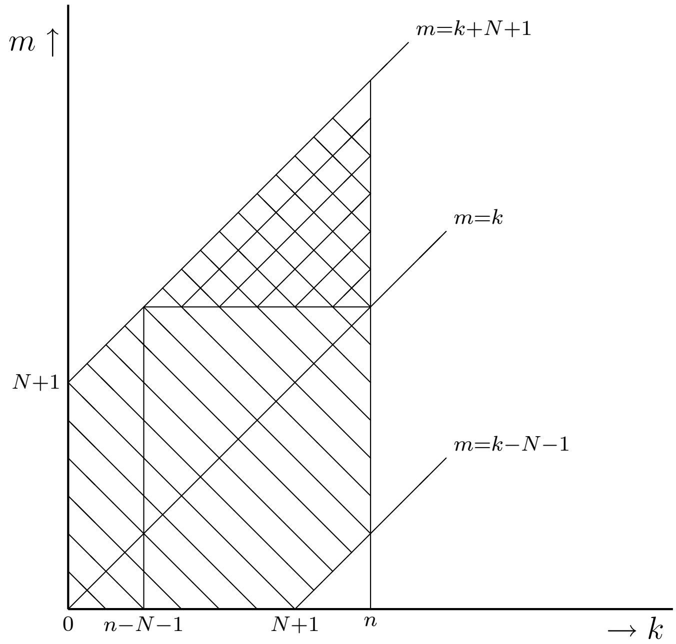
Now we have the following situations :

For $n \leq N$ we have



$$\sum_{k=0}^{n} \sum_{m=\max(0,k-N-1)}^{k+N+1} = \sum_{k=0}^{n} \sum_{m=0}^{n} + \sum_{k=0}^{n} \sum_{m=n+1}^{k+N+1} = \sum_{k=0}^{n} \sum_{m=n+1}^{k+N+1}$$

and for $n \geq N+1$ we have



$$\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{k+N+1}=\sum_{k=0}^{n}\sum_{m=\max(0,k-N-1)}^{n}+\sum_{k=n-N}^{n}\sum_{m=n+1}^{k+N+1}=\sum_{k=n-N}^{n}\sum_{m=n+1}^{k+N+1}.$$

So it follows from (6.6.1) by using this observation that

$$\begin{aligned}
&\left(x^{N+1}-y^{N+1}\right)\sum_{k=0}^{n}\frac{L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y;q)}{\Lambda_k}\\
&=\sum_{k=\max(0,n-N)}^{n}\sum_{m=n+1}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}\Big[L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q)L_k^{\alpha,M_0,M_1,\ldots,M_N}(y;q)+\\
&\qquad -L_m^{\alpha,M_0,M_1,\ldots,M_N}(y;q)L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\Big]
\end{aligned}\tag{6.6.2}$$

for $n=0,1,2,\ldots$. This can be considered as a generalization of the Christoffel-Darboux formula (5.3.8) for the q-Laguerre polynomials.

If we divide the Christoffel-Darboux type formula (6.6.2) by $x-y$ and let y tend to x then we find the confluent form

$$\begin{aligned}
&(N+1)x^N\sum_{k=0}^{n}\frac{\left\{L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}^2}{\Lambda_k}\\
&=\sum_{k=\max(0,n-N)}^{n}\sum_{m=n+1}^{k+N+1}\frac{E_m^{(k)}}{\Lambda_k}\left[L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\frac{d}{dx}L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q)+\right.\\
&\qquad\left.-L_m^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\frac{d}{dx}L_k^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right]
\end{aligned}$$

for $n=0,1,2,\ldots$. This formula can be considered as a generalization of (5.3.9).

Chapter 7

A special case

7.1 The definition, the orthogonality relations and some elementary properties

In this chapter we consider a special case of the inner product (6.1.1). We consider the inner product

$$\begin{cases} < f, g >_q = \dfrac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \displaystyle\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} f(x)g(x)dx + \\ \qquad\qquad + Mf(0)g(0) + N\left(D_q f\right)(0)\left(D_q g\right)(0), \\ \\ \alpha > -1,\ M \geq 0 \text{ and } N \geq 0. \end{cases} \tag{7.1.1}$$

In this chapter we always write M and N instead of M_0 and M_1 respectively.

Note that the definition (5.2.7) implies that $\left(D_q f\right)(0) := f'(0)$.

We have an explicit representation for the polynomials $\left\{L_n^{\alpha,M,N}(x;q)\right\}_{n=0}^\infty$ which are orthogonal with respect to this inner product (7.1.1).

For $N > 0$ the inner product (7.1.1) cannot be obtained from any weight function, since then $< 1, x^2 > \neq < x, x >$. We will investigate some properties of the polynomials $\left\{L_n^{\alpha,M,N}(x;q)\right\}_{n=0}^\infty$. In this section we give the definition, the orthogonality relations and some elementary properties and in the next section we give some results concerning the zeros of these polynomials. The system of equations (6.1.9) can quite easily be solved explicitly in this case, so that we have an explicit representation for the polynomials $\left\{L_n^{\alpha,M,N}(x;q)\right\}_{n=0}^\infty$.

The polynomials $\left\{L_n^{\alpha,M,N}(x;q)\right\}_{n=0}^\infty$ which are orthogonal with respect to this inner product (7.1.1) are defined by

$$\begin{aligned} L_n^{\alpha,M,N}(x;q) &= A_0 L_n^{(\alpha)}(x;q) + A_1 q^{-(\alpha+1)}\left(D_q L_n^{(\alpha)}\right)(q^{-1}x;q) + \\ &\quad + A_2 q^{-2(\alpha+2)}\left(D_q^2 L_n^{(\alpha)}\right)(q^{-2}x;q) \end{aligned} \tag{7.1.2}$$

where

$$\begin{cases} A_0 = 1 + M\dfrac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} + \\ \qquad + Nq^{2\alpha+3}\dfrac{(1-q^n)(1-q^{\alpha+2})-q(1-q)(1-q^{\alpha+1})}{(1-q^{\alpha+1})(1-q^{\alpha+3})}\dfrac{(q^{\alpha+3};q)_{n-2}}{(q;q)_{n-2}} + \\ \qquad + MNq^{2\alpha+3}\dfrac{(1-q)^2}{(1-q^{\alpha+1})(1-q^{\alpha+2})}\dfrac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}}\dfrac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}} \\ A_1 = M\dfrac{(q^{\alpha+1};q)_n}{(q;q)_n} + Nq^{2\alpha+2}\dfrac{(1-q^{n-1})}{(1-q^{\alpha+1})}\dfrac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} + \\ \qquad + MNq^{2\alpha+2}\dfrac{(1-q)(1-q^2)}{(1-q^{\alpha+1})^2}\dfrac{(q^{\alpha+1};q)_n}{(q;q)_n}\dfrac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}} \\ A_2 = Nq^{2\alpha+2}\dfrac{(1-q)}{(1-q^{\alpha+1})}\dfrac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} + \\ \qquad + MNq^{2\alpha+2}\dfrac{(1-q)^2}{(1-q^{\alpha+1})^2}\dfrac{(q^{\alpha+1};q)_n}{(q;q)_n}\dfrac{(q^{\alpha+3};q)_{n-1}}{(q;q)_{n-1}}. \end{cases} \tag{7.1.3}$$

For $N = 0$ this leads to

$$\begin{aligned} L_n^{\alpha,M}(x;q) := L_n^{\alpha,M,0}(x;q) &= \left[1 + M\frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}}\right] L_n^{(\alpha)}(x;q) + \\ &\qquad + M\frac{(q^{\alpha+1};q)_n}{(q;q)_n} q^{-(\alpha+1)}\left(D_q L_n^{(\alpha)}\right)(q^{-1}x;q), \end{aligned}$$

which is a q-analogue of Koornwinder's generalized Laguerre polynomial $L_n^{\alpha,M}(x)$ defined by (3.1.3).

Since $0 < q < 1$ and $\alpha > -1$ we have

$$1 - q < 1 - q^n \quad \text{for} \quad n \geq 2 \quad \text{and} \quad q(1-q^{\alpha+1}) = q - q^{\alpha+2} < 1 - q^{\alpha+2}.$$

Hence

$$(1-q^n)(1-q^{\alpha+2}) - q(1-q)(1-q^{\alpha+1}) > 0 \quad \text{for} \quad n \geq 2.$$

So it follows from (7.1.3) that

$$A_0 \geq 1, \; A_1 \geq 0 \; \text{ and } \; A_2 \geq 0. \tag{7.1.4}$$

The orthogonality relations (6.1.3) and (6.1.4) reduce to the following orthogonality relations :

$$< L_m^{\alpha,M,N}(x;q), L_n^{\alpha,M,N}(x;q) >_q = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} A_0\left(A_0 + q^n A_1 + q^{2n-1}A_2\right)\delta_{mn},$$

where the inner product $<,>_q$ is defined by (7.1.1) and

$$\begin{aligned} &\frac{1}{A}\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-c(1-q)q^k;q)_\infty} L_m^{\alpha,M,N}(cq^k;q) L_n^{\alpha,M,N}(cq^k;q) + \\ &+ ML_m^{\alpha,M,N}(0;q)L_n^{\alpha,M,N}(0;q) + N\left(D_q L_m^{\alpha,M,N}\right)(0;q)\left(D_q L_n^{\alpha,M,N}\right)(0;q) \\ = \; &\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} A_0\left(A_0 + q^n A_1 + q^{2n-1}A_2\right)\delta_{mn}, \end{aligned}$$

where A is given by (5.3.4) and $c > 0$ is an arbitrary constant. In terms of the q-integral defined by (5.2.12) the latter orthogonality relation yields

$$\begin{aligned} & \frac{1}{A^*}\int_0^\infty \frac{t^\alpha}{(-c(1-q)t;q)_\infty} L_m^{\alpha,M,N}(ct;q)L_n^{\alpha,M,N}(ct;q)d_qt + \\ & + ML_m^{\alpha,M,N}(0;q)L_n^{\alpha,M,N}(0;q) + N\left(D_qL_m^{\alpha,M,N}\right)(0;q)\left(D_qL_n^{\alpha,M,N}\right)(0;q) \\ = & \frac{(q^{\alpha+1};q)_n}{(q;q)_nq^n}A_0\left(A_0+q^nA_1+q^{2n-1}A_2\right)\delta_{mn}, \end{aligned}$$

where A^* is defined by (5.3.5).

From the definition (7.1.2) we find by using (5.3.6), (5.3.10) and (7.1.3)

$$L_n^{\alpha,M,N}(0;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\left[1 - Nq^{2\alpha+4}\frac{1-q}{1-q^{\alpha+1}}\frac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}}\right] \tag{7.1.5}$$

and

$$\left(D_qL_n^{\alpha,M,N}\right)(0;q) = -q^{\alpha+1}\frac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} - Mq^{\alpha+1}\frac{1-q}{1-q^{\alpha+1}}\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\frac{(q^{\alpha+3};q)_{n-1}}{(q;q)_{n-1}}. \tag{7.1.6}$$

The second representation (6.2.1) in this case reads

$$\begin{aligned} L_n^{\alpha,M,N}(x;q) \quad = \quad & B_0L_n^{(\alpha)}(x;q) + q^{-(\alpha+2)}B_1x\left(D_qL_n^{(\alpha+1)}\right)(q^{-1}x;q) + \\ & + q^{-2(\alpha+4)}B_2x^2\left(D_q^2L_n^{(\alpha+2)}\right)(q^{-2}x;q), \end{aligned}$$

where

$$\begin{cases} B_0 = 1 - Nq^{2\alpha+4}\dfrac{1-q}{1-q^{\alpha+1}}\dfrac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}} \\ B_1 = Mq^{\alpha+1}\dfrac{1-q}{1-q^{\alpha+1}}\dfrac{(q^{\alpha+1};q)_n}{(q;q)_n} + Nq^{3\alpha+5}\dfrac{(1-q)(1-q^{\alpha+2})}{(1-q^{\alpha+1})(1-q^{\alpha+3})}\dfrac{(q^{\alpha+3};q)_{n-2}}{(q;q)_{n-2}} \\ B_2 = Nq^{4\alpha+7}\dfrac{(1-q)^3}{(1-q^{\alpha+1})(1-q^{\alpha+2})(1-q^{\alpha+3})}\dfrac{(q^{\alpha+2};q)_{n-1}}{(q;q)_{n-1}} + \\ \qquad + MNq^{4\alpha+7}\dfrac{(1-q)^4}{(1-q^{\alpha+1})^2(1-q^{\alpha+2})(1-q^{\alpha+3})}\dfrac{(q^{\alpha+1};q)_n}{(q;q)_n}\dfrac{(q^{\alpha+3};q)_{n-1}}{(q;q)_{n-1}}. \end{cases}$$

The (formal) representation as basic hypergeometric series (6.3.2) in this case reads

$$\begin{aligned} L_n^{\alpha,M,N}(x;q) \quad = \quad & \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\left[1 - Nq^{2\alpha+4}\frac{1-q}{1-q^{\alpha+1}}\frac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}}\right] \times \\ & \times {}_3\phi_3\left(\begin{array}{c} q^{-n}, q^{\beta+1}, q^{\gamma+1} \\ q^{\alpha+3}, q^\beta, q^\gamma \end{array} \middle| q; -(1-q)q^{n+\alpha+1}x\right), \end{aligned}$$

for some complex β and γ which satisfy

$$q^\beta + q^\gamma = \frac{(q^{\alpha+1} + q^{\alpha+2})A_0 + (1 + q^{n+\alpha+2})A_1 + (q^{n-1} + q^n)A_2}{A_0 + q^n A_1 + q^{2n-1} A_2} > 0$$

and

$$q^\beta q^\gamma = \frac{q^{2\alpha+3}A_0 + q^{\alpha+2}A_1 + A_2}{A_0 + q^n A_1 + q^{2n-1}A_2} > 0.$$

The following example shows that q^β and q^γ need not to be real. If we take $\alpha = 0$, $M = 0$, $N = 1$ and $n = 1$, then it follows from (7.1.3) that $A_0 = 1$, $A_1 = 0$ and $A_2 = q^2$. So we have in that case

$$q^\beta + q^\gamma = \frac{q(1+q)^2}{1+q^3} \quad \text{and} \quad q^\beta q^\gamma = \frac{q^2(1+q)}{1+q^3}.$$

Hence

$$\left(q^\beta - q^\gamma\right)^2 = \left(q^\beta + q^\gamma\right)^2 - 4q^\beta q^\gamma = -\frac{3q^2\left(1-q^2\right)^2}{\left(1+q^3\right)^2} < 0.$$

7.2 The zeros

We will prove the following theorem concerning the zeros of the polynomial $L_n^{\alpha,M,N}(x;q)$.

Theorem 7.1. *The polynomial $L_n^{\alpha,M,N}(x;q)$ has n real and simple zeros. At least $n-1$ of these zeros are positive.*

In other words : at most one zero of $L_n^{\alpha,M,N}(x;q)$ lies in $(-\infty, 0]$.

Proof. For $n \geq 1$ we have $< 1, L_n^{\alpha,M,N}(x;q) >_q = 0$. Hence

$$\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} \int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} L_n^{\alpha,M,N}(x;q)dx + ML_n^{\alpha,M,N}(0;q) = 0.$$

This implies that the polynomial $L_n^{\alpha,M,N}(x;q)$ changes sign on $(0,\infty)$ at least once. Suppose that $x_1, x_2, \ldots, x_k$ are those zeros of $L_n^{\alpha,M,N}(x;q)$ which are positive and have odd multiplicity. Define

$$p(x) := k_n(x-x_1)(x-x_2)\cdots(x-x_k),$$

where k_n denotes the leading coefficient in the polynomial $L_n^{\alpha,M,N}(x;q)$. This implies that

$$p(x)L_n^{\alpha,M,N}(x;q) \geq 0 \quad \text{for all} \quad x \geq 0.$$

Now we define

$$h(x) := (x+d)p(x)$$

in such a way that $\left(D_q h\right)(0) = 0$. Hence, by using (5.2.7)

$$0 = \left(D_q h\right)(0) = h'(0) = dp'(0) + p(0).$$

Since
$$\frac{p'(0)}{p(0)} = \left\{\frac{d}{dx}\ln|p(x)|\right\}_{x=0} = -\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_k}\right) < 0$$
we have
$$d = -\frac{p(0)}{p'(0)} > 0.$$
Hence
$$h(x)L_n^{\alpha,M,N}(x;q) \geq 0 \text{ for all } x \geq 0.$$
This implies that
$$\begin{aligned} &< h, L_n^{\alpha,M,N} >_q \\ = \ &\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} h(x)L_n^{\alpha,M,N}(x;q)dx + Mh(0)L_n^{\alpha,M,N}(0;q) > 0.\end{aligned}$$

Hence, degree$[h] \geq n$ which implies that $k \geq n-1$. This implies that all zeros of $L_n^{\alpha,M,N}(x;q)$ must be simple.

So we have : at most one zero of $L_n^{\alpha,M,N}(x;q)$ is located outside the interval $(0,\infty)$. This immediately implies that all zeros of $L_n^{\alpha,M,N}(x;q)$ are real. This proves the theorem.

Now we examine the nonpositive zero of $L_n^{\alpha,M,N}(x;q)$ in somewhat greater detail. From (6.1.6) it follows, by using (7.1.4), that $L_n^{\alpha,M,N}(x;q) > 0$ for all $x < -B$ if $B > 0$ is sufficiently large. Hence, the polynomial $L_n^{\alpha,M,N}(x;q)$ has a zero in $(-\infty,0]$ if and only if $L_n^{\alpha,M,N}(0;q) \leq 0$. Now we use (7.1.5) to conclude that the polynomial $L_n^{\alpha,M,N}(x;q)$ has a nonpositive zero if and only if
$$1 - Nq^{2\alpha+4}\frac{1-q}{1-q^{\alpha+1}}\frac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}} \leq 0. \tag{7.2.1}$$
This implies that $N > 0$ and $n \geq 2$. We define
$$f(n) := \frac{(q^{\alpha+4};q)_{n-2}}{(q;q)_{n-2}}.$$
Then we have
$$f(n+1) = \frac{(q^{\alpha+4};q)_{n-1}}{(q;q)_{n-1}} = \frac{1-q^{n+\alpha+2}}{1-q^{n-1}}f(n) > f(n), \tag{7.2.2}$$
since $n+\alpha+2 > n+1 > n-1$. Hence $f(n)$ is an increasing function. But
$$\lim_{n\to\infty} f(n) = \frac{(q^{\alpha+4};q)_\infty}{(q;q)_\infty}.$$
Now we look at
$$F(\alpha,q,N) := 1 - Nq^{2\alpha+4}\frac{1-q}{1-q^{\alpha+1}}\frac{(q^{\alpha+4};q)_\infty}{(q;q)_\infty}. \tag{7.2.3}$$

Since

$$\lim_{q\downarrow 0} q^{2\alpha+4}\frac{1-q}{1-q^{\alpha+1}}\frac{(q^{\alpha+4};q)_\infty}{(q;q)_\infty}=0$$

for all $\alpha>-1$ this need not to be negative even for large values of N. For each N there is a value of q with $0<q<1$ such that $F(\alpha,q,N)>0$, where $F(\alpha,q,N)$ is defined by (7.2.3). This implies that we cannot guarantee the existence of a nonpositive zero for $N>0$ and n sufficiently large as in the case of the polynomials $\left\{L_n^{\alpha,M,N}(x)\right\}_{n=0}^{\infty}$. Note that

$$L_n^{\alpha,M,N}(x)=\lim_{q\uparrow 1} L_n^{\alpha,M,N}(x;q).$$

But in view of (7.2.1) and (7.2.2) it is clear that if $L_n^{\alpha,M,N}(x;q)$ has a nonpositive zero for some positive integer n, then $L_{n+1}^{\alpha,M,N}(x;q)$ has one too. Moreover, we have : the polynomial $L_n^{\alpha,M,N}(x;q)$ has a nonpositive zero for every sufficiently large n if and only if $F(\alpha,q,N)<0$, where $F(\alpha,q,N)$ is defined by (7.2.3).

Now we prove the following theorem.

Theorem 7.2. *If the polynomial $L_n^{\alpha,M,N}(x;q)$ has a nonpositive zero x_n, then we have for $M>0$:*

$$-\frac{1}{2}\sqrt{\frac{N}{M}}\leq x_n\leq 0.$$

Proof. Suppose that the polynomial $L_n^{\alpha,M,N}(x;q)$ has a nonpositive zero x_n. Then it is clear that $N>0$ and $n\geq 2$.

Let $x_1,x_2,\ldots,x_{n-1}$ be the positive zeros of $L_n^{\alpha,M,N}(x;q)$ and define

$$r(x):=(x-x_1)(x-x_2)\cdots(x-x_{n-1}).$$

Then we have in view of (6.1.6)

$$L_n^{\alpha,M,N}(x;q)=k_n r(x)(x-x_n),\ x_n\leq 0. \tag{7.2.4}$$

Since degree$[r(x)]=n-1$ we have

$$\begin{aligned} 0 &= <r(x),L_n^{\alpha,M,N}(x;q)>_q \\ &= k_n\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)}\int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty}r^2(x)(x-x_n)dx+ \\ &\qquad -Mk_nr^2(0)x_n+Nk_nr'(0)\left[r(0)-x_nr'(0)\right]. \end{aligned} \tag{7.2.5}$$

Here we used the fact that $\left(D_qr\right)(0)=r'(0)$. Since the integral in (7.2.5) is positive we conclude that

$$-Mr^2(0)x_n+Nr'(0)\left[r(0)-x_nr'(0)\right]<0.$$

Hence

$$0\leq-\left[M\left\{r(0)\right\}^2+N\left\{r'(0)\right\}^2\right]x_n\leq -Nr(0)r'(0)=N\left|r(0)r'(0)\right|,$$

since $r(0)$ and $r'(0)$ have opposite signs. Now it follows that

$$-2\sqrt{MN}\,|r(0)r'(0)|\,x_n \leq -\left[M\left\{r(0)\right\}^2 + N\left\{r'(0)\right\}^2\right]x_n \leq N\,|r(0)r'(0)|\,.$$

Hence

$$-2\sqrt{MN}x_n \leq N.$$

This implies that for $M > 0$ the zero x_n is bounded :

$$-\frac{1}{2}\sqrt{\frac{N}{M}} \leq x_n \leq 0.$$

This proves the theorem.

Finally we prove the following result.

Theorem 7.3. *Suppose that the polynomial $L_n^{\alpha,M,N}(x;q)$ has a zero x_n in $(-\infty,0]$. Let $x_1 < x_2 < \cdots < x_{n-1}$ denote the positive zeros of $L_n^{\alpha,M,N}(x;q)$. Then we have*

$$0 \leq -x_n < x_1. \tag{7.2.6}$$

Proof. For $x_n = 0$ (7.2.6) is trivial. So we assume that $x_n < 0$. From (7.1.6) it follows that $\left(D_q L_n^{\alpha,M,N}\right)(0;q) < 0$ for $n = 1, 2, 3, \ldots$. This implies, by using (7.2.4) and (5.2.7), that

$$k_n\left[r(0) - x_n r'(0)\right] < 0.$$

Since $r(0) = (-1)^{n-1}x_1x_2\cdots x_{n-1}$ and

$$\frac{r'(0)}{r(0)} = \left\{\frac{d}{dx}\ln|r(x)|\right\}_{x=0} = -\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}\right)$$

we obtain

$$(-1)^{n-1}k_n x_1 x_2 \cdots x_{n-1}\left[1 + x_n\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}\right)\right] < 0.$$

Now we use (6.1.6) and the fact that $A_0 \geq 1$ to conclude that

$$x_n\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}}\right) > -1.$$

Since $x_n < 0$ this implies

$$-\frac{1}{x_n} > \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}} \geq \frac{1}{x_1}.$$

Hence

$$-x_n < x_1.$$

This proves (7.2.6) and therefore the theorem.

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Summary

We study polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product

$$\begin{cases} < f,g > = \dfrac{1}{\Gamma(\alpha+1)}\displaystyle\int_0^\infty x^\alpha e^{-x} f(x)g(x)dx + \sum_{\nu=0}^{N} M_\nu f^{(\nu)}(0)g^{(\nu)}(0), \\ \\ \alpha > -1,\ N \in \{0,1,2,\ldots\} \ \text{ and } \ M_\nu \geq 0 \ \text{ for } \ \nu \in \{0,1,2,\ldots,N\}. \end{cases}$$

These polynomials are generalizations of the classical Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ since $L_n^{\alpha,0,0,\ldots,0}(x) = L_n^{(\alpha)}(x)$ and of Koornwinder's generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ found in [**1**] since $L_n^{\alpha,M,0,0,\ldots,0}(x) = L_n^{\alpha,M}(x)$.

Since the inner product above cannot be obtained from any weight function in general the classical theory of orthogonal polynomials cannot be applied to derive properties of these new orthogonal polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$.

In this thesis we give two definitions, an orthogonality relation and a representation as hypergeometric function for these polynomials. Moreover, we derive a second order differential equation, a recurrence relation and a Christoffel-Darboux type formula for these polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$.

In some special cases some results concerning the zeros of these orthogonal polynomials are given.

Finally, a differential equation is proved for Koornwinder's generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ which is of infinite order if $M > 0$ and $\alpha > -1$ is not an integer. For $\alpha \in \{0,1,2,\ldots\}$ this differential equation is of finite order $2\alpha+4$ provided that $M > 0$.

The second part of this thesis deals with some q-analogues $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ of the polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ described in the first part. These q-analogues are generalizations of the q-Laguerre polynomials $\left\{L_n^{(\alpha)}(x;q)\right\}_{n=0}^{\infty}$ studied by Moak in [**2**].

For these orthogonal polynomials two definitions, two orthogonality relations and a representation as basic hypergeometric function are given. Moreover, we prove a second order q-difference equation, a recurrence relation and a Christoffel-Darboux type formula for these polynomials $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$.

In a special case we derive some results concerning the zeros of these generalized q-Laguerre polynomials.

[**1**] : T.H. Koornwinder : *Orthogonal polynomials with weight function* $(1-x)^{\alpha}(1+x)^{\beta} + M\delta(x+1) + N\delta(x-1)$. Canadian Mathematical Bulletin **27**(2), 1984, 205-214.

[**2**] : D.S. Moak : *The q-analogue of the Laguerre polynomials.* Journal of Mathematical Analysis and Applications **81**, 1981, 20-47.

Samenvatting

In dit proefschrift bestuderen we polynomen $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ die orthogonaal zijn met betrekking tot het inwendig product

$$\begin{cases} < f,g > = \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty x^\alpha e^{-x} f(x)g(x)dx + \sum_{\nu=0}^{N} M_\nu f^{(\nu)}(0)g^{(\nu)}(0), \\ \\ \alpha > -1,\ N \in \{0,1,2,\ldots\} \text{ and } M_\nu \geq 0 \text{ for } \nu \in \{0,1,2,\ldots,N\}. \end{cases}$$

Deze polynomen zijn generalisaties van de klassieke Laguerre polynomen $\left\{L_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$, want $L_n^{\alpha,0,0,\ldots,0}(x) = L_n^{(\alpha)}(x)$ en van de gegeneraliseerde Laguerre polynomen $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ die gevonden werden door Koornwinder in **[1]** omdat $L_n^{\alpha,M,0,0,\ldots,0}(x) = L_n^{\alpha,M}(x)$.

Omdat het bovenstaand inwendig product in het algemeen niet kan worden verkregen uit een gewichtsfunctie kan de klassieke theorie van de orthogonale polynomen niet worden toegepast om eigenschappen van deze nieuwe orthogonale polynomen af te leiden.

In dit proefschrift geven we twee definities, een orthogonaliteitsrelatie en een representatie als hypergeometrische functie. Bovendien leiden we een tweede orde differentiaalvergelijking af voor deze polynomen $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$, evenals een recurrente betrekking en een soort Christoffel-Darboux formule.

In sommige speciale gevallen worden enkele resultaten betreffende de nulpunten van deze orthogonale polynomen afgeleid.

Tenslotte wordt nog een differentiaalvergelijking voor de polynomen $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ van Koornwinder bewezen. Deze differentiaalvergelijking is van oneindige orde als $M > 0$ en $\alpha > -1$ geen geheel getal is. Voor $\alpha \in \{0,1,2,\ldots\}$ is deze differentiaalvergelijking van de (eindige) orde $2\alpha + 4$ als $M > 0$.

Deel twee van dit proefschrift gaat over q-uitbreidingen $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$ van de polynomen $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x)\right\}_{n=0}^{\infty}$ uit het eerste deel. Deze q-analoga zijn generalisaties van de q-Laguerre polynomen $\left\{L_n^{(\alpha)}(x;q)\right\}_{n=0}^{\infty}$ die werden bestudeerd door Moak in **[2]**.

Voor deze orthogonale polynomen worden twee definities, twee orthogonaliteitsrelaties en een representatie als q-hypergeometrische functie bewezen. Bovendien worden een tweede orde q-differentie vergelijking, een recurrente betrekking en een soort Christoffel-Darboux formule afgeleid voor deze polynomen $\left\{L_n^{\alpha,M_0,M_1,\ldots,M_N}(x;q)\right\}_{n=0}^{\infty}$.

In een speciaal geval leiden we nog enkele resultaten af omtrent de nulpunten van deze gegeneraliseerde q-Laguerre polynomen.

[**1**] : T.H. Koornwinder : *Orthogonal polynomials with weight function* $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$. Canadian Mathematical Bulletin **27**(2), 1984, 205-214.

[**2**] : D.S. Moak : *The q-analogue of the Laguerre polynomials.* Journal of Mathematical Analysis and Applications **81**, 1981, 20-47.

Curriculum Vitae

De schrijver van dit proefschrift werd geboren op 25 oktober 1963 in Leiden. Op 4 juni 1982 ontving hij het diploma Voorbereidend Wetenschappelijk Onderwijs aan het Menso Alting College te Hoogeveen.

In hetzelfde jaar startte hij zijn studie Wiskunde aan de Rijksuniversiteit van Groningen. Op 31 augustus 1983 behaalde hij het propedeutisch examen Wiskunde en op 27 november 1986 behaalde hij het doctoraal examen Wiskunde alsmede de onderwijsbevoegdheid Wiskunde.

Vanaf 1 december 1986 was hij werkzaam als Assistent in Opleiding bij de Faculteit der Technische Wiskunde en Informatica van de Technische Universiteit van Delft. Hij deed daar onder andere onderzoek naar *generalisaties van de klassieke Laguerre polynomen* hetgeen uiteindelijk heeft geleid tot dit proefschrift.

Daarnaast was hij van 1 november 1987 tot 1 augustus 1990 werkzaam als docent Wiskunde aan de Polytechnische Faculteit van de Hogeschool Rotterdam en omstreken.