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Matrix Representation for Combinatorics

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Denumerably infinite matrices are introduced for the representation of combinatorial quantities. The purpose is to simplify and unify the presentation and to provide a method for the discovery of combinatorial identities and for proving them. The key concept is the derivative of a matrix.

THE SHIFT OPERATORS

Combinatorial quantities will be represented by denumerably infinite matrices

$$M = (M_{ij}) \qquad \text{with} \quad i, j = 0, 1, 2, 3, \ldots .$$

Use will also be made of row and column vectors whose indices are natural numbers.

Care must be taken in using infinite matrices, since the associative and distributive laws may fail. However, there is a simple sufficient condition for the normal matrix laws to apply: The computation of each component of a product involves only a finite number of non-zero terms. Only such products will occur in this paper.

A matrix M is *diagonal* (dg) if the only non-zero components are on the main diagonal, i.e., when $i = j$. It is *lower triangle* (lt) if the components are zero for $j > i$, and it is *almost-triangular* (alt) if components are zero for $j > i + 1$. All these matrices are row-finite (i.e., have only a finite number of non-zero entries in each row) and hence may be multiplied freely.

It should be noted that each dg matrix is lt, and a lt matrix is alt. The product of dg matrices is dg, and they commute. The product of lt matrices is lt. A lt matrix times an alt is alt.

Let us now introduce certain key matrices. The identity matrix I (dg with 1's on diagonal) plays the same role as in the finite case. Zero matrices and vectors will be denoted by 0—the meaning being clear from the context.

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The key operators are two shift matrices.

$$S_{ij} = 1 \qquad \text{if} \quad j = i + 1, \qquad 0 \text{ otherwise.} \tag{1}$$

This matrix has ones just above the main diagonal. We let

$$Z = S^T \tag{2}$$

and Z has ones just below the main diagonal.

The role of these shifts is shown by the following relations. For any matrix M

$$\begin{aligned}
(SM)_{ij} &= M_{i+1,j} && \text{removes row 0}\\
(ZM)_{ij} &= M_{i-1,j} && \text{adds a null row 0}\\
(MS)_{ij} &= M_{i,j-1} && \text{adds a null column 0}\\
(MZ)_{ij} &= M_{i,j+1} && \text{removes column 0.}
\end{aligned}$$

Various combinations are important. One occurs so frequently that we introduce the notation

$$\hat{M} = SMZ. \tag{3}$$

This operation removes both row and column 0. In effect, it shifts M up-and-left.

We also need the vectors:

$$\text{Row vector } e, \qquad e_j = 1 \qquad \text{if} \quad j = 0, \qquad 0 \text{ otherwise} \tag{4}$$

$$\text{Column vector } f, \qquad f_i = 1 \qquad \text{if} \quad i = 0, \qquad 0 \text{ otherwise} \tag{5}$$

$$f = e^T. \tag{6}$$

It should be noted that $ef = 1$ is a number, but fe is a matrix, with a 1 in the $(0, 0)$ component. Also, eM is the 0th row of M, while Mf is the 0th column.

Simple computations show that

$$SZ = I \tag{7}$$

$$ZS = I - fe \tag{8}$$

$$eZ = 0 \qquad \text{and} \qquad Sf = 0. \tag{9}$$

We shall use 1 to stand for a column vector of all ones. Then $M1$ gives the row-sums of M. For example, $e1 = 1$.

$$S1 = 1 \qquad \text{and} \qquad Z1 = 1 - f. \tag{10}$$

THE DERIVATIVE

For any matrix M we define a *derivative*:

$$M' = SM - MS. \tag{11}$$

Similarly, for a vector h we define

$$h' = Sh - h. \tag{12}$$

In component form we have

$$M'_{ij} = M_{i+1,j} - M_{i,j-1}$$
$$h'_i = h_{i+1} - h_i$$

where it is understood that a matrix component with a negative subscript is 0.

THEOREM 1. *The derivative has the properties*

(a) $(cM)' = cM'$

(b) $(M+N)' = M' + N'$

(c) $(MN)' = M'N + MN'$.

Proof. The first two results follow immediately from the definition.

$$\begin{aligned}(MN)' &= SMN - MNS \\ &= SMN - MSN + MSN - MNS \\ &= M'N + MN'. \quad \blacksquare\end{aligned}$$

The corresponding theorem for vectors is proved similarly.

THEOREM 2.

(a) $(ch)' = ch'$

(b) $(h+k)' = h' + k'$

(c) $(Mh)' = M'h + Mh'$.

We need an observation about multiplication by a lt matrix. In

$$(MN)_{ij} = \sum_k M_{ik} N_{kj}$$

if M is lt, the sum only goes to $k = i$, and if N is lt then it starts with $k = j$.

The following theorem will be the key tool for proving identities.

THEOREM 3. *For given r, M, $A^{(n)}$, and $B^{(n)}$, the conditions*

$$\text{(a)} \quad eX = r$$

$$\text{(b)} \quad X' = \sum_n A^{(n)} X B^{(n)} + M$$

determine a unique matrix X, if the matrices $A^{(n)}$ are lt.

Proof. The 0th row of X is determined by (a). Let us assume that the rows up to row i have been determined (uniquely).

$$SX = X' + XS. \tag{13}$$

This relation determines row $i+1$. If the A matrices are lt, then the sum in (b) uses only rows of X up to the ith, and XS also involves only row i. Hence, by the inductive assumption, row $i+1$ is uniquely determined. Hence by induction. ∎

We shall refer to (a) and (b) as *defining relations.* It should be noted that (13) provides an effective method for computing a finite segment of X. Some additional information may be obtained from the relations:

THEOREM 4. *If in the defining relations*

(a) *r, A, B, M are non-negative, then X is non-negative*;

(b) *$r = ce$ and B and M are alt, the X is lt.*

Proof. Each is proved by induction on the rows of X, as in the previous theorem. Part (a) is trivial. For (b) we must verify that there are only zeros above the main diagonal.

The condition $r = ce$ assures that only entry 0 is (possibly) non-zero in row 0. Assume the proposition true up to row i. From (13) we must show that in row $i+1$ the components are 0 for $j > i+1$. We shall show this term by term. It is true for M, since it is alt. By assumption, X is lt up to row i. Hence XS is alt. That leaves the terms AXB. As pointed out above, in

$$\sum_{k,l} A_{ik} X_{kl} B_{lj}$$

k is at most i, and hence l is at most i. Since B is alt, the term will be 0 if $j > i+1$. Hence by induction. ∎

The defining relations for vectors are given by

THEOREM 5. *The conditions*

$$\text{(a)}\quad ex = c$$

$$\text{(b)}\quad x' = \sum A^{(n)}x + h$$

determine a unique vector x, *if the* $A^{(n)}$ *are lt.*

Let us write down the defining relations for the vectors and matrices we have so far. (A summary and numerical values for a piece of each matrix will be found in the Appendix.)

$$\begin{aligned} eI &= e & I' &= 0 \\ eS &= (0, 1, 0, \ldots) & S' &= 0 \\ eZ &= 0 & Z' &= fe \\ ef &= 1 & f' &= -f \\ e1 &= 1 & 1' &= 0. \end{aligned} \tag{14}$$

Each of these follows easily from the definition of the derivative and the fact that eM is row 0 of M.

From here on we shall introduce new vectors and matrices either by specifying their defining relations or by forming them out of known matrices by elementary matrix operations. For example,

$$eD = 0 \qquad D' = S. \tag{15}$$

It will be helpful to the reader to compute a segment of D from the definition (15), by using (13). The result is that D is dg, with $D_{ii} = i$.

In working with dg matrices, the double shift is particularly useful. It changes a lt matrix into a lt, and a dg into another dg. If we multiply (13) by Z on the right, we obtain

$$\hat{X} = X + X'Z. \tag{16}$$

Thus

$$\hat{I} = I \qquad \hat{S} = S \qquad \hat{Z} = Z. \tag{17}$$

(From now on we shall use $SZ = I$ and $ZS = I - fe$ without mention.) Thus (15) changes to

$$\hat{D} = D + I \qquad D_{00} = 0. \tag{18}$$

Thus we see that starting with 0, 1 is added to the diagonal entry on each new row.

THEOREM 6. *M is dg if and only if $eM = ce$ and $M'Z$ is dg.*

Proof. A simple induction using (16). ∎

$$eF = e \qquad F' = FDS. \tag{19}$$

Hence

$$\hat{F} = F + FD = F\hat{D}. \tag{20}$$

Again a dg, where the entry $(i+1, i+1)$ is obtained from (i, i) by multiplying by $i+1$. This is the well-known recursion for the factorials. Hence F has $i!$ on the main diagonal. It will be useful to note that

$$Df = 0 \qquad Ff = f. \tag{21}$$

Next

$$E = I + DZ. \tag{22}$$

This will be a useful auxiliary matrix. Since $eI = e$, $eD = 0$, $eE = e$. For the derivative we have

$$E' = I' + D'Z + DZ' = 0 + SZ + Dfe = I.$$

Thus the defining relations are

$$eE = e \qquad E' = I. \tag{23}$$

An interesting vector is t, with components

$$t_i = 2^i.$$

It is easily seen that

$$et = 1 \qquad t' = t. \tag{24}$$

Thus t is the unique vector starting with 1 and being equal to its derivative. The matrix with similar properties will play a key role in this paper. (See later.)

To give an application for the Uniqueness Theorem, we introduce the matrix T as the lt matrix all of whose entries on and below the main diagonal are 1. We encourage the reader to derive the defining relations

$$eT = e \qquad T' = 1e. \tag{25}$$

This matrix is used to give partial column sums:

$$(TM)_{ij} = \sum_{k=0}^{i} M_{kj} \tag{26}$$

We shall study the product $T(I-Z)$.

$$eT(I-Z)=e(I-Z)=e$$
$$(T(I-Z))'=T'(I-Z)+T(I-Z)'=1e(I-Z)+T(-fe)$$
$$=1e-1e=0.$$

We conclude that

$$T(I-Z)=I. \tag{27}$$

Why? The defining relations for I are $eX=e$ and $X'=0$. We have shown that our matrix also satisfies these relations. Hence, by uniqueness, it is equal to I. This will be the key use of derivatives and defining relations!

A final observation on the derivative: $M'f=SMf$, which furnishes column 0 of M, except for the top entry. The latter is obtained from eM.

Inverses

The concept of an inverse for an infinite matrix is not, in general, well defined. (See, for example, (7) and (8).) But it has a simple meaning for lt matrices. Let us refer to the submatrix consisting of the first rows and columns of a lt matrix as a "subtriangle." The equation $XY=I$ states that a subtriangle of X is an inverse, in the finite sense, of the corresponding subtriangle of Y. The rest follows from well-known properties of finite triangular matrices.

A finite lt matrix has an inverse if and only if the diagonal entries are non-zero. The inverse is lt, and corresponding subtriangles are inverses of each other. There is a simple algorithm for computing the next row of the inverse. We may think of the same process for infinite lt matrices, but the algorithm requires an infinite number of steps.

The following results are a direct consequence of the properties of finite lt matrices.

THEOREM 7. *A lt matrix X has an inverse if and only if all components on the main diagonal are non-zero.*

(a) $XX^{-1}=X^{-1}X=I$.

(b) *If M and N have inverses,* $(MN)^{-1}=N^{-1}M^{-1}$.

Thus Eq. (27) may be restated as

$$T=(I-Z)^{-1}. \tag{28}$$

Since we may think of T as $I + Z + Z^2 + \cdots$, Eq. (28) gives the usual sum for a geometric series!

THEOREM 8. *$Y = X^{-1}$ if and only if*

$$e(XY) = e \qquad \textit{and} \qquad (XY)' = 0.$$

Proof. These are the conditions that assure that XY satisfies the defining relation of I. ∎

Inverses of dg matrices are dg, and very simple to find. The corresponding diagonal entries are reciprocals. Since $eD = 0$, D does not have an inverse. But

$$\hat{D}_{ii}^{-1} = 1/(i+1) \qquad F_{ii}^{-1} = 1/i! \tag{29}$$

In general there is no simple way from the defining relations of a matrix to find relations for its inverse. It should be pointed out that defining relations are differential equations (with initial condition eX), and the same matrix may satisfy many such equations. We shall show how to solve this problem if the defining relations are of one of two special forms:

(I) $eX = e \qquad X' = MX + NXS, \qquad M$ and N lt.

(II) $eX = e \qquad X' = XV + XWS, \qquad V$ and W lt.

THEOREM 9. *If X is defined as in* (I) *and $(N + I)^{-1}$ exists, or as in* (II) *and $(W + I)^{-1}$ exists, then $Y = X^{-1}$ exists.*

Furthermore,

(a) *if X satisfies* (I), *Y satisfies* (II) *with $V = -(N + I)^{-1}M$ and $W = (N + I)^{-1} - I$;*

(b) *if X satisfies* (II), *then Y satisfies* (I) *with $M = -(W + I)^{-1}V$ and $N = (W + I)^{-1} - I$.*

Proof. We shall prove the first case; the other is similar. We must verify that if X is given by (I) and Y by (a), then (XY) satisfies a set of defining relations for I.

$$e(XY) = eY = e. \tag{30}$$

$$\begin{aligned}(XY)' &= X'Y + XY' = (MX + NXS)\,Y + X(YV + YWS)\\ &= M(XY) + NX(Y' + YS) + (XY)(V + WS)\\ &= M(XY) + NX(YV + YWS + YS) + (XY)(V + WS)\\ &= M(XY) + N(XY)(V + WS + S) + (XY)(V + WS).\end{aligned} \tag{31}$$

If on the right we replace (XY) by I, we obtain

$$M + N(V + WS + S) + V + WS = M + NS + (N + I)(V + WS)$$
$$= M + NS - M + (I - (N + I))\, S \qquad \text{from (a)}$$
$$= 0 = I'.$$

Hence (30) and (31) are defining relations for I. ∎

While our development has been entirely for lt matrices, an identical treatment can be given for "upper triangular" matrices, i.e., the transposes of lt matrices. These are column finite and may be used as operators on the right. The usual connection holds:

$$(M^T)^{-1} = (M^{-1})^T. \tag{32}$$

The Double Shift

We need a number of additional facts about $\hat{M} = SMZ$.

THEOREM 10.

(a) $\widehat{(cM)} = c\hat{M}$.

(b) $\widehat{(M + N)} = \hat{M} + \hat{N}$.

(c) *If* $eM = ce$, *then* $Z\hat{M} = MZ$.

(d) *If* $Mf = cf$, *then* $\hat{M}S = SM$ *and* $(\hat{M})' = \widehat{(M')}$.

(e) *If* N *is lt, then* $\widehat{(MN)} = \hat{M}\hat{N}$.

(f) *If* M, N *are lt,* $N = M^{-1}$, *then* $\hat{N} = (\hat{M})^{-1}$.

Proof. Parts (a) and (b) follow directly from the definition.

(c) $ZSMZ = MZ - feMZ$, and $eMZ = ceZ = 0$.

(d) Similar.

(e) If N is lt, part (c) applies.

$$\hat{M}\hat{N} = SMZ\hat{N} = SMNZ = \widehat{(MN)}.$$

(f) $\hat{M}\hat{N} = \widehat{(MN)} = \hat{I} = I$. ∎

It is important for the use of this theorem to remember that all lt matrices satisfy (c). And dg matrices satisfy both (c) and (d). This allows us to find many relations, of which we list only a few.

$$Z\hat{D} = DZ \qquad \hat{D}S = SD \qquad DZS = D \tag{33}$$

$$DZ\hat{D}^{-1} = Z \qquad \hat{D}^{-1}SD = S \tag{34}$$

$$Z\hat{F} = FZ \qquad \hat{F}S = SF \tag{35}$$

$$F^{-1}Z\hat{F} = Z \qquad \hat{F}SF^{-1} = S \tag{36}$$

$$\hat{D}FS = \hat{F}S = SF \qquad FS = \hat{D}^{-1}SF \tag{37}$$

$$DZF = Z\hat{D}F = Z\hat{F} = FZ \tag{38}$$

$$E = I + DZ = I + Z\hat{D} \tag{39}$$

$$EF = F + DZF = F(I + Z). \tag{40}$$

$$\text{Since } \hat{D} = D + I,\ (D + I)\,\hat{D}^{-1} = I,\ D\hat{D}^{-1} = I - \hat{D}^{-1}. \tag{41}$$

It may be helpful for the reader to work a few of these relations in component form.

We now have our basic tools and are ready to tackle combinatorial identities.

Stirling Numbers

For each real number x we define a pair of column vectors:

$$p_i(x) = x^i \qquad p_0(x) = 1 \tag{42}$$

$$r_i(x) = x^{(i)} = x(x-1)\cdots(x-i+1) \qquad r_0(x) = 1. \tag{43}$$

Let us calculte the derivatives.

$$p_{i+1} - p_i = x^{i+1} - x^i = (x-1)\,x^i = (x-1)\,p_i$$

$$r_{i+1} - r_i = x(x-1)\cdots(x-i+1)[(x-i)-1] = (x-1-i)\,r_i.$$

Thus the defining relations are

$$ep(x) = e \qquad p(x)' = [(x-1)\,I]\,p(x) \tag{44}$$

$$er(x) = e \qquad r(x)' = [(x-1)\,I - D]\,r(x). \tag{45}$$

Stirling numbers of the first kind express $r(x)$ in terms of $p(x)$, and the second kind do the reverse:

$$\sum_{k=0}^{i} s_{ik}x^k = x^{(i)} \qquad \sum_{k=0}^{i} t_{ik}x^{(k)} = x^i.$$

We let $K=(s_{ij})$, $L=(t_{ij})$. They are, by definition, lt matrices. The definitions in matrix form are

$$Kp(x)=r(x) \qquad Lr(x)=p(x) \qquad eK=eL=e. \tag{46}$$

If we restrict x to natural numbers, we may introduce matrices that correspond to $p(x)$ and $r(x)$.

$$\Pi_{ij}=j^i \qquad R_{ij}=i^{(j)}. \tag{47}$$

Each of these needs a comment. We have used the Greek letter Π as a warning that the matrix is neither row nor column finite. It may be operated on by a row finite matrix from the left or a column finite matrix from the right. But we must not, for example, multiply two such matrices. For R the order of the indices may appear unnatural. But with our choice $R_{ij}=0$ if $j>i$ since one of the factors will be 0. Thus R is lt. Furthermore, $R_{ii}=i!$; hence R has an inverse.

$$(K\Pi)_{ij}=j^{(i)}=R_{ji} \qquad \text{or} \qquad K\Pi=R^T. \tag{48}$$

The other relation in (46) is

$$LR^T=\Pi. \tag{49}$$

Combining the two relations,

$$KLR^T=R^T$$

and since R and thus R^T is invertible,

$$KL=I. \tag{50}$$

Thus K and L are triangular matrices that are inverses of each other. Let us find their defining relations. We shall use (44)–(46), omitting some x's.

$$\begin{aligned}
&p'=L'r+Lr'\\
&[(x-1)I]\,p=[(x-1)I]\,Lr=L'r+L[(x-1)I-D]\,r\\
&L'r(x)=LDr(x).
\end{aligned}$$

And since we may replace x by any natural number,

$$L'R^T=LDR^T$$

and hence

$$eL=e \qquad L'=LD. \tag{51}$$

From Theorem 4 we can verify that L is lt, and conclude that L is non-negative. Since (51) is of form (II), we can apply Theorem 9.

$$V = D,\ W = 0; \qquad \text{hence} \quad (W + I)^{-1} = I.$$

K has relations of form (I), with $M = -D$ and $N = 0$.

$$eK = e \qquad K' = -DK. \tag{52}$$

We again verify that K is lt, but it does have negative entries. We shall see later that these occur on alternate diagonals.

Since $Df = 0$, $L'f = 0$. Thus $Lf = f$. And $Kf = KLf = f$. The only non-zero entry in either row 0 or column 0 is the 1 in component $(0, 0)$—for both K and L. We mention this because Stirling numbers are more commonly defined only for positive subscripts, and the defining sums start with $k = 1$. Because of the special form of row and column 0, for $i > 0$ we have not changed the usual definition, and the equations for $i = 0$ are the normal convention that $x^0 = x^{(0)} = 1$. Thus while $\hat{K}$ and $\hat{L}$ might more properly be called the Stirling matrices, we would miss some important relations—as will be seen later.

From (16),

$$\hat{L} = L + LDZ = LE. \tag{53}$$

And, using that L and K are inverses,

$$E = K\hat{L} \tag{54}$$

$$E^{-1} = \hat{K}L. \tag{55}$$

We conclude the section by finding the derivative of R.

$$\begin{aligned} \text{If } j > 0, \qquad R_{i+1,j} - R_{i,j-1} &= (i + 1)^{(j)} - i^{(j-1)} \\ &= (i + 1 - 1)\, i^{(j-1)} = i \cdot i^{(j-1)}. \\ \text{If } j = 0, \qquad R_{i+1,0} &= (i + 1)^{(0)} = 1. \end{aligned}$$

Hence

$$eR = e \qquad R' = DRS + 1e. \tag{56}$$

Again we verify that R is lt and non-negative.

BINOMIAL COEFFICIENTS

Let

$$B_{ij} = \binom{i}{j}.$$

The well-known recursion equations are

$$\binom{0}{j} = 1 \quad \text{if} \quad j = 0, \qquad 0 \quad \text{if} \quad j > 0 \text{ or } eB = e$$

$$\binom{i+1}{j} = \binom{i}{j-1} + \binom{i}{j} \qquad \text{or } SB = BS + B.$$

Hence the defining relations are

$$eB = e \qquad B' = B. \tag{57}$$

From Theorem 4 we conclude that Pascal's Triangle is a triangle, and that it is non-negative. It is the unique lt matrix starting with 1 and equal to its own derivative. It is the analogue of the exponential function in calculus—an analogy that will be strengthened in the next section.

There are many variants of the recursion equation. For example,

$$(S - I)\,B = BS \qquad \text{or} \qquad \binom{i+1}{j} - \binom{i}{j} = \binom{i}{j-1}. \tag{58}$$

From (16)

$$\hat{B} = B + BZ \tag{59}$$

and from Theorem 10(c)

$$BZ = Z\hat{B} = ZB + ZBZ. \tag{60}$$

We shall prove a number of identities by uniqueness (Theorems 3 and 5) without specifically quoting a theorem. In each case, we shall either verify that a new quantity satisfies the defining relations of a known entity, or that two new quantities satisfy the same defining relations. Then we conclude equality.

$$e(Bf) = ef = 1$$

$$(Bf)' = Bf + B(-f) = 0.$$

Hence

$$Bf = 1. \tag{61}$$

The 0th column of B consists of 1's.

$$e(B1) = e1 = 1$$
$$(B1)' = B1 + B \cdot 0 = (B1).$$

Hence

$$B1 = t. \tag{62}$$

The sum of row i of B is 2^i.

$$e(TB) = eB = e$$
$$(TB)' = 1eB + TB = (TB) + 1e$$
$$e[B(I+Z)] = e(I+Z) = e$$
$$[B(I+Z)]' = B(I+Z) + Bfe = B(I+Z) + 1e.$$

Hence

$$TB = B(I+Z) \tag{63}$$

and from (59),

$$TB = \hat{B}. \tag{64}$$

In traditional notation:

$$\sum_{k=j}^{i} \binom{k}{j} = \binom{i+1}{j+1}. \tag{65}$$

Since all the lt matrices in our identities satisfy $eM = e$, and the vectors $eh = 1$, and since this is true for products of such entities, we shall from now on omit part (a) of Theorems 3 and 5, and just show that (b) is satisfied.

From (44)

$$[Bp(x)]' = Bp(x) + B(x-1)p(x) = xBp(x)$$
$$[p(x+1)]' = xp(x+1).$$

Hence

$$Bp(x) = p(x+1). \tag{66}$$

This states that

$$\sum_{k=0}^{i} \binom{i}{k} x^k = (x+1)^i.$$

Since x is any real number, we may replace it by y/z, and then multiply the equation by z^i.

$$z^i \sum_{k=0}^{i} \binom{i}{k} (y/z)^k = z^i(y/z+1)^i$$

$$\sum_{k=0}^{i} \binom{i}{k} y^k z^{i-k} = (y+z)^i.$$

This is the Binomial Theorem. Please note that its essence is the identity (66).

Let

$$C_{ij} = B_{i,i-j}.$$

Then $eC = e$ and

$$C'_{ij} = C_{i+1,j} - C_{i,j-1} = B_{i+1,i+1-j} - B_{i,i+1-j} = B_{i,i-j}$$

the last equality following from (58). Thus $C' = C$ and hence

$$C = B \qquad \text{or} \qquad \binom{i}{i-j} = \binom{i}{j}. \tag{67}$$

In particular this shows, with (61), that the main diagonal of B has all ones.

$$(EB)' = IB + EB = (EB) + B$$

$$(BE)' = BE + BI = (BE) + B.$$

Thus

$$EB = BE \tag{68}$$

and subtracting B from both sides,

$$DZB = BDZ \qquad \text{or} \qquad i \cdot \binom{i-1}{j} = \binom{i}{j+1} \cdot (j+1). \tag{69}$$

We will need the following for a later proof:

$$\begin{aligned} DB &= DZSB && \text{by (33)} \\ &= DZB(I+S) && \text{by the recursion formula} \\ &= BDZ(I+S) && \text{by (69)} \\ &= BD(I+Z) && \text{by (33)} \end{aligned}$$

$$DB = BD(I+Z) \tag{70}$$

$$\begin{aligned} D(S-I)\,B &= DBS && \text{by (58)} \\ &= BD(I+Z)\,S && \text{by (70)} \\ &= BD(I+S) && \text{by (33)} \end{aligned}$$

$$D(S-I)\,B = BD(I+S). \tag{71}$$

We shall now make some connections to the previous section. The matrix version of (66) is

$$B\Pi = \Pi Z. \tag{72}$$

Concerning R we found that its defining relations are $eX = e$ and $X' = DXS + 1e$.

$$\begin{aligned} (BF)' &= BF + BFDS \\ DBFS &= BD(I+Z)\,FS && \text{by (70)} \\ &= BDFS + BFZS && \text{by (38)} \\ &= BFDS + BF - BFfe \\ &= (BF)' - Bfe \\ &= (BF)' - 1e && \text{by (61)} \\ (BF)' &= D(BF)\,S + 1e \end{aligned}$$

$$R = BF \tag{73}$$

$$B = RF^{-1} \qquad \text{or} \qquad \binom{i}{j} = i^{(j)}/j!. \tag{74}$$

This is the usual formula for computing B.

Now let us bring in Stirling numbers. From (49),

$$LFB^T = \Pi \tag{75}$$

and transposing,

$$BFL^T = \Pi^T \tag{76}$$

$$(BL)' = BL + BLD = (BL)\,\hat{D}$$

$$(\hat{L})' = (\widehat{LD}) = \hat{L}\hat{D}$$

$$BL = \hat{L}. \tag{77}$$

And multiplying by $K = L^{-1}$ on the right,

$$B = \hat{L}K. \tag{78}$$

We have expressed the binomial coefficients in terms of Stirling numbers.

Duality

For every matrix M we define a *dual*

$$\bar{M}_{ij} = M_{ij}(-1)^{i-j} = M_{ij}(-1)^{i+j}. \tag{79}$$

The effect is to change the sign on alternate diagonals, with the main diagonal left unchanged. Similarly we define for row and column vectors

$$\bar{r}_j = r_j(-1)^j \qquad \bar{h}_i = h_i(-1)^i. \tag{80}$$

THEOREM 11. *Duality has the following properties*:

(a) $\overline{(cM)} = c\bar{M}$

(b) $\overline{(M+N)} = \bar{M} + \bar{N}$

(c) $\bar{\bar{M}} = M$

(d) $\overline{M^T} = \bar{M}^T$

(e) *If M is dg, then $\bar{M} = M$*

(f) $\bar{S} = -S \qquad \bar{Z} = -Z \qquad \bar{e} = e \qquad \bar{f} = f$

(g) $\overline{(MN)} = \bar{M} \cdot \bar{N}$

(h) $\bar{M}' = -\overline{M'}$

(i) $\hat{\bar{M}} = \bar{\hat{M}}$.

Proof. Parts (a)–(d) are obvious. Parts (e) and (f) follow from the alternate diagonal interpretation of the dual.

(g) $\sum_k M_{ik}(-1)^{i-k} N_{kj}(-1)^{k-j} = \sum_k M_{ik} N_{kj}(-1)^{i-j}.$

The proofs involving vectors are similar.

$$\begin{aligned}(\text{h})\quad \overline{M'} &= \overline{SM} - \overline{MS} && \text{by (a), (b), (g)}\\ &= -S\bar{M} + \bar{M}S && \text{by (f)}\\ &= -\bar{M}'.\end{aligned}$$

Part (i) follows from (g) and (f). ∎

As an example of the use of this theorem,

$$\bar{E} = \bar{I} + \overline{DZ} = I + D\bar{Z} = I - DZ. \tag{81}$$

For any of our results we obtain a dual. For example, the dual of (40) is

$$\bar{E}F = F(I - Z) = FT^{-1}. \tag{82}$$

Hence

$$\bar{E} = FT^{-1}F^{-1} \qquad \text{and} \qquad \bar{E}^{-1} = FTF^{-1}. \tag{83}$$

COROLLARY. *If $eM = e$, then $e\bar{M} = e$.*

Proof. $e\bar{M} = \overline{eM} = \overline{(eM)} = \bar{e} = e.$ ∎

Let us explore the properties of $\bar{B}$, the dual binomial coefficients. From (h) and the corollary,

$$e\bar{B} = e \qquad \bar{B}' = -\bar{B}. \tag{84}$$

If B is the analogue of e^x, then $\bar{B}$ is of e^{-x}. That analogy is strengthened by

$$(B\bar{B})' = B\bar{B} + B(-\bar{B}) = 0$$

$$\bar{B} = B^{-1} \tag{85}$$

$$e(\bar{B}1) = 1 \qquad (\bar{B}1)' = -\bar{B}1$$

$$\bar{B}1 = f. \tag{86}$$

The row sum of B, with alternating signs, is 0 (except for row 0). The dual of (69) is

$$\overline{DZB} = \overline{BDZ}$$

and by cancelling a minus sign,

$$DZ\bar{B} = \bar{B}DZ \tag{87}$$

$$\overline{p_i(x)} = (-1)^i x^i = (-x)^i$$

$$\overline{p(x)} = p(-x). \tag{88}$$

And using (66),

$$\bar{B}p(x) = \overline{B\overline{p(x)}} = \overline{Bp(-x)} = \overline{p(-x+1)} = p(x-1). \tag{89}$$

If we apply (85) to (64),

$$T = \hat{B}\bar{B} \tag{90}$$

or

$$\sum_{k=j}^{i} \binom{i+1}{k+1}\binom{k}{j}(-1)^{k-j} = 1 \qquad \text{if} \quad j \leqslant i \text{ and 0 otherwise.}$$

Similarly, from (72), (73), and (77)

$$\Pi = \bar{B}\Pi Z \tag{91}$$

$$R^{-1} = F^{-1}\bar{B} \tag{92}$$

$$\bar{B}\hat{L} = L. \tag{93}$$

And using that

$$\begin{aligned} \hat{K} &= \hat{L}^{-1} \\ \bar{B} &= L\hat{K}. \end{aligned} \tag{94}$$

By comparison to (78) we see that $\hat{L}K$ and $L\hat{K}$ are duals. Finally, from (76)

$$FL^T = \bar{B}\Pi^T \tag{95}$$

or

$$\sum_{k=0}^{i} \binom{i}{k}(-1)^{i-k} k^j = i!\, L_{ji}.$$

This seems to be a very interesting identity. Please note that the sum is 0 when $j < i$.

Dual Stirling Numbers

By Theorem 11 and its corollary, the duals of K and L have the defining relations

$$e\bar{K} = e \qquad \bar{K}' = DK \tag{96}$$

$$e\bar{L} = e \qquad \bar{L}' = -LD \tag{97}$$

$$\bar{K} = \bar{L}^{-1}. \tag{98}$$

From Theorem 4 we see that we have a pair of lt matrices, and that it is $\bar{K}$ that is non-negative. This proves that K has its negative entries in alternate diagonals. Conversely, $\bar{L}$ has alternate non-positive diagonals.

From (78) and the dual of (94)

$$B = \hat{L}K = \overline{L\hat{\bar{K}}} \tag{99}$$

$$\bar{B} = L\hat{K} = \overline{\hat{\bar{L}}K} \tag{100}$$

and from (53)

$$\hat{\bar{L}} = \overline{LE} \qquad \bar{L} = \overline{\hat{L}E}^{-1} = \hat{\bar{L}}FTF^{-1}. \tag{101}$$

In place of these duals, it will be convenient to work with

$$L^* = \bar{L}F \qquad K^* = F^{-1}\bar{K}.$$

Clearly, they are lt matrices, inverses, and K^* is non-negative. From (101) we have the alternate form

$$L^* = \hat{\bar{L}}FT. \tag{102}$$

Let us find the defining relations.

$$eL^* = e \qquad L^{*\prime} = -\bar{L}DF + \bar{L}FDS = -L^*D + L^*DS. \tag{103}$$

This is of form (II), with $V = -D$ and $W = D$. Hence Theorem 9 yields

$$\begin{gathered}(W+I)^{-1} = \hat{D}^{-1}, \qquad M = D\hat{D}^{-1} = I - \hat{D}^{-1}, \qquad N = \hat{D}^{-1} - I \\ eK^* = e \qquad K^{*\prime} = (I - \hat{D}^{-1})\,K^*(I - S).\end{gathered} \tag{104}$$

To see an example of how the starred quantities turn up naturally, let us take the dual of (95) and transpose it:

$$L^* = \bar{\Pi}B^T. \tag{105}$$

From (100),

$$\begin{aligned} \overline{BL} &= \hat{\bar{L}} \\ \bar{B}L^* &= \hat{\bar{L}}F \\ \bar{B}L^*T &= \hat{\bar{L}}FT = L^* \\ L^*T &= BL^*. \end{aligned} \tag{106}$$

And, using the inverse,

$$TK^* = K^*B. \tag{107}$$

From (28)

$$\begin{aligned} L^* &= BL^*(I-Z) \\ (B-I)\,L^* &= BL^*Z \end{aligned} \tag{108}$$

$$S(B-I)\,L^* = SBL^*Z = \hat{B}\widehat{L^*}. \tag{109}$$

From (108) we also have

$$\begin{aligned} \bar{B}L^* &= L^*(I-Z) \\ (I-\bar{B})\,L^* &= L^*Z \\ S(I-\bar{B})\,L^* &= \widehat{L^*}. \end{aligned} \tag{110}$$

Let us introduce the pair of dual matrices

$$A = S(B-I) \qquad \bar{A} = S(I-\bar{B}).$$

The defining relations are

$$eA = e \qquad A' = SB = A + S \tag{111}$$

$$e\bar{A} = e \qquad \bar{A}' = S\bar{B} = S - \bar{A}. \tag{112}$$

We see that they are lt and that A is non-negative. (This is a good example of the use of Theorem 4 to save work.) Using these matrices we rewrite (109) and (110)

$$AL^* = \hat{B}\widehat{L^*} \tag{113}$$

$$\bar{A}L^* = \widehat{L^*}. \tag{114}$$

Let

$$P = L^*\widehat{K^*}.$$

As the product of lt matrices P is lt. Our relations become

$$AP = \hat{B} \tag{115}$$

$$\bar{A}P = I. \tag{116}$$

The significance of the matrix P will be the subject of the next section.

There are a very large number of different relations among these matrices. We shall derive only a few, either because of their intrinsic interest, or because we will need them later. Let us start with the dual of P.

$$\begin{aligned} A\bar{P} &= I \\ A(P-\bar{P}) &= \hat{B} - I = S(B-I)\,Z = AZ \qquad (117) \\ P-\bar{P} &= Z \\ \bar{P} &= P - Z. \qquad (118) \end{aligned}$$

Many more follow from the fact that A and $\bar{A}$ are inverses of $\bar{P}$ and P, respectively. For example, from (115)

$$A = \hat{B}\bar{A}$$

$$\bar{A} = P^{-1} = \widehat{L^*K^*}$$

and its dual

$$\begin{aligned} A &= \hat{L}\hat{F}\cdot F^{-1}K = \hat{L}\hat{D}K \\ \bar{P} &= A^{-1} = L\hat{D}^{-1}\hat{K} \\ A\bar{B} &= S(B-I)\,\bar{B} = S(I-\bar{B}) = \bar{A} \\ \bar{A}B &= A \end{aligned}$$

and since P is the inverse of $\bar{A}$,

$$PA = B$$

$$P = B\bar{P} = BL\hat{D}^{-1}\hat{K}$$

and from (77),

$$P = \hat{L}\hat{D}^{-1}\hat{K} \qquad (119)$$

which expresses P in terms of the original Stirling numbers.

We need one more application of the Uniqueness Theorem.

$$\begin{aligned} (L^*B)' &= -L^*DB + L^*DSB + L^*B \\ &= L^*D(S-I)\,B + L^*B \\ &= L^*BD(I+S) + L^*B \qquad \text{by (71)} \\ &= (L^*B)(I+D+DS) \\ (BLF)' &= BLF + BLDF + BLFDS \\ &= (BLF)(I+D+DS) \\ L^*B &= BLF. \qquad (120) \end{aligned}$$

We also establish an alternate role for K^*.
From (46)

$$\begin{aligned} Kp(-x) &= r(-x) \\ [Kp(-x)]_i &= (-x)^{(i)} = (-x)(-x-1)\cdots(-x-(i-1)) \\ &= (-1)^i (x+i-1)^{(i)}. \end{aligned}$$

Taking duals,

$$[\bar{K}p(x)]_i = (x+i-1)^{(i)}.$$

Let $x = j+1$,

$$[\bar{K}p(j+1)]_i = (i+j)^{(i)}$$

$$(1/i!)[\bar{K}p(j+1)]_i = \binom{i+j}{i} \tag{121}$$

$$(K^*\Pi Z)_{ij} = \binom{i+j}{i}.$$

The right side does not change if i and j are interchanged; hence the matrix on the left is symmetric.

$$(K^*\Pi Z)^T = K^*\Pi Z.$$

Its present form does not explain the symmetry of the matrix, but the following alternate form does:

$$\begin{aligned} K^*\Pi Z &= K^*B\Pi && \text{by (72)} \\ &= K^*BLFB^T && \text{by (75)} \\ &= K^*L^*BB^T && \text{by (120)} \\ &= BB^T. \end{aligned}$$

With (121) we have proved the identity

$$\sum_k \binom{i}{k}\binom{j}{k} = \binom{i+j}{i}. \tag{122}$$

Sums of Powers

We wish to prove the well-known result that the sum of the ith powers of the first n positive integers can be written as a polynomial in n (of degree $i+1$). And the coefficients are given by the matrix P.

$$\sum_{l=1}^{n} l^i = \sum_{k=0}^{i} P_{ik} n^{k+1}. \tag{123}$$

Since both sides are 0 when $n=0$, we may let $n=j+1$. Then the matrix form is

$$\Pi Z T^T = P\hat{\Pi}. \tag{124}$$

We can derive this from the results of the previous section.

$$K^*\Pi Z = K^*B\Pi = TK^*\Pi \qquad \text{by (107)}$$

$$(K^*\Pi Z)\, Z = T(K^*\Pi Z)$$

transposing and using the symmetry of the matrix

$$S(K^*\Pi Z) = (K^*\Pi Z)\, T^T$$

$$\widehat{K^*\Pi} = K^*\Pi Z T^T$$

and multiplying by L^* on the left we obtain (124).

A more "traditional" proof would proceed as follows.[1]

$$\begin{aligned}
\Pi Z T^T &= B\Pi T^T && \text{by (72)}\\
&= BLR^T T^T && \text{by (49)}\\
&= \hat{L}(TR)^T && \text{by (77)}\\
&= \hat{L}(\hat{R}\hat{D}^{-1})^T && \text{by (64), (74)}\\
&= \hat{L}\hat{D}^{-1}\hat{R}^T && \\
&= \hat{L}\hat{D}^{-1}\hat{K}\hat{\Pi} && \text{by (48)}\\
&= P\hat{\Pi} && \text{by (119).}
\end{aligned}$$

Definition. M is *even* if $M = \bar{M}$. M is *odd* if $M = -\bar{M}$.

An even (odd) matrix has alternate diagonals equal to 0. An odd matrix

[1] The author is indebted to R. Z. Norman for this suggestion.

has the main diagonal equal to 0, while an even matrix would have the diagonals above and below the main diagonal equal to 0. Thus, e.g., I is even, while S and Z are odd. The motivation for the terminology comes from the following:

THEOREM 12. *Let $M_i(x)=\sum_k M_{ik}x^k$. Then*

(a) *if M is even, then M_i is an even polynomial for i even (and odd otherwise);*

(b) *if M is odd, then M_i is an odd polynomial for i even (and even otherwise).*

Proof. $\bar{M}_i(x)=\sum_k M_{ik}(-1)^{i+k}x^k=(-1)^i M_i(-x)$.

(a) If $M=\bar{M}$, then $M_i(x)=(-1)^i M_i(-x)$. Thus for i even $M_i(x)=M_i(-x)$ while for i odd $M_i(x)=-M_i(x)$.

The proof of (b) is similar. ∎

As a direct consequence of Theorem 11(h) we have

THEOREM 13. *The derivative of an even matrix is odd, of an odd matrix even.*

Finally the very easy result:

THEOREM 14. *For any matrix M, $M^+=\frac{1}{2}(M+\bar{M})$ is even, $M^-=\frac{1}{2}(M-\bar{M})$ is odd, and $M=M^++M^-$.* ∎

These theorems are analogous to similar theorems in calculus. We wish to apply them to the matrix P and to the polynomials

$$P_i(n)=\sum_{k=0}^{i} P_{ik}n^{k+1}. \tag{125}$$

The matrix P is neither even nor odd; however, it is nearly even. From (118)

$$P^+=P-\tfrac{1}{2}Z. \tag{126}$$

Thus the diagonal below the main one consists entirely of $\frac{1}{2}$'s, and it is the only non-zero "odd" diagonal. $\bar{P}$ differs from P by having $-\frac{1}{2}$'s on this diagonal.

For the polynomials $P_i(x)$ we now have, remembering the "extra" n-factor in each term of (125),

$$P_i(n)-\tfrac{1}{2}n^i \text{ is odd for even } i, \text{ even for odd } i. \tag{127}$$

Bernoulli Numbers

Bernoulli numbers (b_i) are defined by

$$\sum_{k=0}^{n} \binom{n+1}{k} b_k = 1 \qquad \text{if} \quad n=0, \qquad 0 \text{ otherwise.}$$

Note that the sum goes to n and not $n+1$. Hence the matrix version is

$$S(B-I)\,b = f \qquad \text{or} \qquad Ab = f. \tag{128}$$

Using (117) we can solve for b.

$$b = \bar{P}f. \tag{129}$$

Thus the Bernoulli numbers are given by the 0th column of $\bar{P}$. From the above discussion of the parity of P we may conclude that $b_1 = -\frac{1}{2}$, and $b_i = 0$ for all other odd i. The even entries of b are the even entries in the 0th column of P.

We hope that the visual simplicity of the matrix formulas and the ease with which some traditional results are derived justify the approach of this paper.

APPENDIX

Triangular Matrices[a]

M	M'	Identification
A	$A+S$	$S(B-I)$
$\bar{A}$	$S-\bar{A}$	P^{-1}
B	B	Binomial coefficients
$\bar{B}$	$-\bar{B}$	B^{-1}
$\hat{D}$	S	$D+I$
E	I	$I+DZ$
F	FDS	Factorials (dg)
I	0	Identity
$I-Z$	$-fe$	T^{-1}
K	$-DK$	Stirling numbers, first kind
K^*	$(I-\hat{D}^{-1})K^*(I-S)$	$F^{-1}\bar{K}$
L	LD	Stirling numbers, second kind
L^*	$L^*D(S-I)$	$\bar{L}F$
P		$L^*\widehat{K^*}$, sums of powers
R	$DRS+1e$	$i^{(j)}$
T	$1e$	All ones

[a] Each is invertible. Each has $eM = e$.

Numerical Values

$$I = \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 &&& \\ & 0 & 1 && \\ && 0 & 1 & \\ &&& 0 & 1 \\ &&&& 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 0 &&&& \\ 1 & 0 &&& \\ & 1 & 0 && \\ && 1 & 0 & \\ &&& 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 &&&& \\ & 1 &&& \\ && 2 && \\ &&& 3 & \\ &&&& 4 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 &&&& \\ & 2 &&& \\ && 6 && \\ &&& 24 & \\ &&&& 120 \end{pmatrix} \qquad E \doteq \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ & 2 & 1 && \\ && 3 & 1 & \\ &&& 4 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 1 & 1 && \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 2 & 2 && \\ 1 & 3 & 6 & 6 & \\ 1 & 4 & 12 & 24 & 24 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 2 & 1 && \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \qquad \bar{B} = \begin{pmatrix} 1 &&&& \\ -1 & 1 &&& \\ 1 & -2 & 1 && \\ -1 & 3 & -3 & 1 & \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} 1 &&&& \\ 0 & 1 &&& \\ 0 & -1 & 1 && \\ 0 & 2 & -3 & 1 & \\ 0 & -6 & 11 & -6 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 &&&& \\ 0 & 1 &&& \\ 0 & 1 & 1 && \\ 0 & 1 & 3 & 1 & \\ 0 & 1 & 7 & 6 & 1 \end{pmatrix}$$

$$\hat{K} = \begin{pmatrix} 1 &&&& \\ -1 & 1 &&& \\ 2 & -3 & 1 && \\ -6 & 11 & -6 & 1 & \\ 24 & -50 & 35 & -10 & 1 \end{pmatrix} \qquad \hat{L} = \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 3 & 1 && \\ 1 & 7 & 6 & 1 & \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix}$$

Table continued

Numerical Values (*continued*)

$$K^* = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & \frac{1}{2} & \frac{1}{2} & & \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & \\ 0 & \frac{1}{4} & \frac{11}{24} & \frac{1}{4} & \frac{1}{24} \end{pmatrix} \qquad L^* = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & -1 & 2 & & \\ 0 & 1 & -6 & 6 & \\ 0 & -1 & 14 & -36 & 24 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & & \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \end{pmatrix} \qquad A = \begin{pmatrix} 1 & & & & \\ 1 & 2 & & & \\ 1 & 3 & 3 & & \\ 1 & 4 & 6 & 4 & \\ 1 & 5 & 10 & 10 & 5 \end{pmatrix}$$

$$\Pi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 9 & 16 \\ 0 & 1 & 8 & 27 & 64 \\ 0 & 1 & 16 & 81 & 256 \end{pmatrix} \qquad e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad t = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{6} \\ 0 \\ -\frac{1}{30} \end{pmatrix}$$

Notes. For a thorough treatment of combinatorial quantities and recursion relations see [2].

Approaches similar to this paper may be found in the recent literature. There is a strong "family resemblance" to the approach of Chapter 2 of [3] and to the recursive matrices of [1]. While these authors obtain powerful results, elegantly, it does not appear that either the concept of the matrix derivative or the method of proving identities is a special case of these results.

References

1. M. Barnabei, A. Brini, and G. Nicoletti, Recursive matrices and umbral calculus, *J. Algebra* **75** (1982), 546–573.
2. L. Comtet, "Advanced Combinatorics," Reidel, Dordrecht, 1974.
3. G-C. Rota, "Finite Operator Calculus," Academic Press, New York, 1975.