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The Andrews–Stanley partition function and Al-Salam–Chihara polynomials

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Abstract

For any partition λ let $\omega(\lambda)$ denote the four parameter weight

$$\omega(\lambda)=a^{\sum_{i\ge 1}\lceil\lambda_{2i-1}/2\rceil}b^{\sum_{i\ge 1}\lfloor\lambda_{2i-1}/2\rfloor}c^{\sum_{i\ge 1}\lceil\lambda_{2i}/2\rceil}d^{\sum_{i\ge 1}\lfloor\lambda_{2i}/2\rfloor},$$

and let $\ell(\lambda)$ be the length of λ. We show that the generating function $\sum\omega(\lambda)z^{\ell(\lambda)}$, where the sum runs over all ordinary (resp. strict) partitions with parts each $\le N$, can be expressed by the Al-Salam–Chihara polynomials. As a corollary we derive Andrews' result by specializing some parameters and Boulet's results by letting $N\to+\infty$. In the last section we prove a Pfaffian formula for the weighted sum $\sum\omega(\lambda)z^{\ell(\lambda)}P_\lambda(x)$ where $P_\lambda(x)$ is Schur's P-function and the sum runs over all strict partitions.

Keywords: Andrews–Stanley partition function; Basic hypergeometric series; Al-Salam–Chihara polynomials; Minor summation formula of Pfaffians; Schur's Q-functions

1. Introduction

For any integer partition λ, denote by λ' its conjugate and $\ell(\lambda)$ the number of its parts. Let $\mathcal{O}(\lambda)$ denote the number of odd parts of λ and $|\lambda|$ the sum of its parts. Stanley [16] has shown that if $t(n)$ denotes the number of partitions λ of n for which $\mathcal{O}(\lambda)\equiv\mathcal{O}(\lambda')\pmod 4$, then

$$t(n)=\frac{1}{2}\left(p(n)+f(n)\right),$$

where $p(n)$ is the total number of partitions of n, and $f(n)$ is defined by

$$\sum_{n=0}^{\infty}f(n)q^n=\prod_{i\ge 1}\frac{(1+q^{2i-1})}{(1-q^{4i})(1+q^{4i-2})}.$$

Motivated by Stanley's problem, Andrews [1] assigned the weight $z^{\mathcal{O}(\lambda)}y^{\mathcal{O}(\lambda')}q^{|\lambda|}$ to each partition λ and computed the corresponding generating function of all partitions with parts each less than or equal to N (see Corollary 4.4).

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The following more general weight first appeared in Stanley's problem [17]. Let a, b, c and d be commuting indeterminates. For each partition λ, define the *Andrews–Stanley partition functions* $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor}, \tag{1.1}$$

where $\lceil x\rceil$ (resp. $\lfloor x\rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. Actually it is more convenient to define the above weight through the Ferrers diagram of λ: one fills the ith row of the Ferrers diagram alternatively by a and b (resp. c and d) if i is *odd* (resp. *even*), the weight $w(\lambda)$ is then equal to the product of all the entries in the diagram. For example, if $\lambda = (5, 4, 4, 1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ.

a	b	a	b	a
c	d	c	d	
a	b	a	b	
c				

In [2] Boulet has obtained results for the generating functions of all ordinary partitions and all strict partitions with respect to the weight (1.1) (see Corollaries 3.6 and 4.5). On the other hand, Sills [15] has given a combinatorial proof of Andrews' result, which has been further generalized by Yee [19] by restricting the sum over partitions with parts each $\leq N$ and length $\leq M$.

In this paper we shall generalize Boulet's results by summing the weight function $\omega(\lambda)z^{\ell(\lambda)}$ over all the ordinary (resp. strict) partitions with parts each $\leq N$. It turns out that the corresponding generating functions are related to the basic hypergeometric series, namely the Al-Salam–Chihara polynomials and the associated Al-Salam–Chihara polynomials (see Corollaries 3.4 and 4.3).

This paper can be regarded as a succession of [6], in which the first author gave a Pfaffian formula for the weighted sum $\sum \omega(\lambda)s_\lambda(x)$ of the Schur functions $s_\lambda(x)$, where the sum runs over all ordinary partitions λ, and settled an open problem [17] by Stanley. Though it is not possible to specialize the Schur functions to $z^{\ell(\lambda)}$, we show in this paper that this approach still works, i.e., we can evaluate the weighted sum $\sum \omega(\lambda)z^{\ell(\lambda)}$ by using Pfaffians and minor summation formulas as tools (see [8,9]), but, as an afterthought, we also provide alternative combinatorial proofs.

In the last section we show that the weighted sum $\sum \omega(\mu)z^{\ell(\mu)}P_\mu(x)$ of Schur's P-functions $P_\mu(x)$ (when $z = 2$, this equals the weighted sum $\sum \omega(\mu)Q_\mu(x)$ of Schur's Q-functions $Q_\mu(x)$) can be expressed by a Pfaffian where μ runs over all strict partitions (with parts each $\leq N$).

2. Preliminaries

A q-shifted factorial is defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n = 1, 2, \ldots.$$

We also define $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$. Since products of q-shifted factorials occur very often, to simplify them we shall use the compact notations

$$(a_1, \ldots, a_m; q)_n = (a_1;q)_n \cdots (a_m;q)_n,$$
$$(a_1, \ldots, a_m; q)_\infty = (a_1;q)_\infty \cdots (a_m;q)_\infty.$$

We define a ${}_{r+1}\phi_r$ *basic hypergeometric series* by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n.$$

The *Al-Salam–Chihara polynomial* $Q_n(x) = Q_n(x;\alpha,\beta|q)$ is, by definition (cf. [11, p. 80]),

$$Q_n(x;\alpha,\beta|q) = \frac{(\alpha\beta;q)_n}{\alpha^n}\ {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha u, \alpha u^{-1} \\ \alpha\beta, 0 \end{matrix}; q, q\right),$$

$$= (\alpha u; q)_n u^{-n} \ {}_2\phi_1 \left({q^{-n}, \beta u^{-1} \atop \alpha^{-1} q^{-n+1} u^{-1}} ; \ q, \alpha^{-1} q u \right),$$
$$= (\beta u^{-1}; q)_n u^n \ {}_2\phi_1 \left({q^{-n}, \alpha u \atop \beta^{-1} q^{-n+1} u} ; \ q, \beta^{-1} q u^{-1} \right),$$

where $x = \frac{u+u^{-1}}{2}$. This is a specialization of the Askey–Wilson polynomials (see [3]), and satisfies the three-term recurrence relation

$$2x Q_n(x) = Q_{n+1}(x) + (\alpha + \beta) q^n Q_n(x) + (1 - q^n)(1 - \alpha\beta q^{n-1}) Q_{n-1}(x), \tag{2.1}$$

with $Q_{-1}(x) = 0$, $Q_0(x) = 1$.

We also consider a more general recurrence relation:

$$2x \widetilde{Q}_n(x) = \widetilde{Q}_{n+1}(x) + (\alpha + \beta) t q^n \widetilde{Q}_n(x) + (1 - t q^n)(1 - t\alpha\beta q^{n-1}) \widetilde{Q}_{n-1}(x), \tag{2.2}$$

which we call the *associated Al-Salam–Chihara recurrence relation*. Put

$$\widetilde{Q}_n^{(1)}(x) = u^{-n} \, (t\alpha u; q)_n \ {}_2\phi_1 \left({t^{-1} q^{-n}, \beta u^{-1} \atop t^{-1} \alpha^{-1} q^{-n+1} u^{-1}} ; \ q, \alpha^{-1} q u \right), \tag{2.3}$$

$$\widetilde{Q}_n^{(2)}(x) = u^n \, \frac{(tq; q)_n (t\alpha\beta; q)_n}{(t\beta u q; q)_n} \ {}_2\phi_1 \left({t q^{n+1}, \alpha^{-1} q u \atop t\beta q^{n+1} u} ; \ q, \alpha u \right), \tag{2.4}$$

where $x = \frac{u+u^{-1}}{2}$. In [10], Ismail and Rahman have presented two linearly independent solutions of the associated Askey–Wilson recurrence equation (see also [4,5]). By specializing the parameters, we conclude that $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ are two linearly independent solutions of the associated Al-Salam–Chihara equation (2.2) (see [10, p. 203]). Here, we use this fact and omit the proof. The series (2.3) and (2.4) are convergent if we assume $|u| < 1$ and $|q| < |\alpha| < 1$ (see [10, p. 204]).

Let

$$W_n = \widetilde{Q}_n^{(1)}(x) \widetilde{Q}_{n-1}^{(2)}(x) - \widetilde{Q}_{n-1}^{(1)}(x) \widetilde{Q}_n^{(2)}(x) \tag{2.5}$$

denote the Casorati determinant of the equation (2.2). Since $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ both satisfy recurrence equation (2.2), it is easy to see that W_n satisfies the recurrence equation

$$W_{n+1} = (1 - t q^n)(1 - t\alpha\beta q^{n-1}) W_n.$$

Using this equation recursively, we obtain

$$W_{n+1} = (tq, t\alpha\beta; q)_n W_1,$$

which implies

$$W_1 = \frac{\lim_{n\to\infty} W_{n+1}}{(tq, t\alpha\beta; q)_\infty}.$$

Using (2.3) and (2.4), we obtain

$$\lim_{n\to\infty} W_{n+1} = \frac{u^{-1} (t\alpha u, tq, t\alpha\beta, \beta u; q)_\infty}{(t\beta u q, \alpha u; q)_\infty}$$

(for the details, see [10]). Thus we conclude that

$$W_1 = \frac{u^{-1} (t\alpha u, \beta u; q)_\infty}{(\alpha u, t\beta u q; q)_\infty}. \tag{2.6}$$

In the following sections we need to find a polynomial solution of the recurrence equation (2.2) which satisfies a given initial condition, say $\widetilde{Q}_0(x) = \widetilde{Q}_0$ and $\widetilde{Q}_1(x) = \widetilde{Q}_1$. Since $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ are linearly independent solutions of (2.2), this $\widetilde{Q}_n(x)$ can be written as a linear combination of these functions, say

$$\widetilde{Q}_n(x) = C_1 \, \widetilde{Q}_n^{(1)}(x) + C_2 \, \widetilde{Q}_n^{(2)}(x).$$

If we substitute the initial condition $\widetilde{Q}_0(x) = \widetilde{Q}_0$ and $\widetilde{Q}_1(x) = \widetilde{Q}_1$ into this equation and solve the linear equation, then we obtain

$$C_1 = \frac{1}{W_1}\left\{\widetilde{Q}_1\widetilde{Q}_0^{(2)}(x) - \widetilde{Q}_0\widetilde{Q}_1^{(2)}(x)\right\},$$
$$C_2 = \frac{1}{W_1}\left\{\widetilde{Q}_0\widetilde{Q}_1^{(1)}(x) - \widetilde{Q}_1\widetilde{Q}_0^{(1)}(x)\right\}.$$

By (2.6), we obtain

$$\widetilde{Q}_n(x) = \frac{u(\alpha u, t\beta uq; q)_\infty}{(t\alpha u, \beta u; q)_\infty}\left[\left\{\widetilde{Q}_1\widetilde{Q}_0^{(2)}(x) - \widetilde{Q}_0\widetilde{Q}_1^{(2)}(x)\right\}\widetilde{Q}_n^{(1)}(x) + \left\{\widetilde{Q}_0\widetilde{Q}_1^{(1)}(x) - \widetilde{Q}_1\widetilde{Q}_0^{(1)}(x)\right\}\widetilde{Q}_n^{(2)}(x)\right] \tag{2.7}$$

with

$$\widetilde{Q}_0^{(1)}(x) = {}_2\phi_1\left({t^{-1}, \beta u^{-1} \atop t^{-1}\alpha^{-1}u^{-1}q}; q, \alpha^{-1}uq\right),$$
$$\widetilde{Q}_1^{(1)}(x) = u^{-1}(1-\alpha tu)\ {}_2\phi_1\left({t^{-1}q^{-1}, \beta u^{-1} \atop t^{-1}\alpha^{-1}u^{-1}}; q, \alpha^{-1}uq\right),$$
$$\widetilde{Q}_0^{(2)}(x) = {}_2\phi_1\left({tq, \alpha^{-1}uq \atop t\beta uq}; q, \alpha u\right),$$
$$\widetilde{Q}_1^{(2)}(x) = \frac{u(1-tq)(1-t\alpha\beta)}{(1-t\beta uq)}\ {}_2\phi_1\left({tq^2, \alpha^{-1}uq \atop t\beta uq^2}; q, \alpha u\right).$$

Since

$$\lim_{n\to\infty} u^n\,\widetilde{Q}_n^{(1)}(x) = \frac{(t\alpha u, \beta u; q)_\infty}{(u^2; q)_\infty},$$
$$\lim_{n\to\infty} u^n\,\widetilde{Q}_n^{(2)}(x) = 0,$$

if we take the limit $\lim_{n\to\infty} u^n\,\widetilde{Q}_n(x)$, then we have

$$\lim_{n\to\infty} u^n\,\widetilde{Q}_n(x) = \frac{u(t\beta uq, \alpha u; q)_\infty}{(u^2; q)_\infty}\left\{\widetilde{Q}_1\widetilde{Q}_0^{(2)}(x) - \widetilde{Q}_0\widetilde{Q}_1^{(2)}(x)\right\}. \tag{2.8}$$

In the latter half of this section, we briefly recall our tools, i.e. partitions and Pfaffians. We follow the notation in [14] concerning partitions and the symmetric functions. For more information about the general theory of determinants and Pfaffians, the reader can refer to [12,13,9] since, in this paper, we sometimes omit the details and give sketches of proofs.

Let n be a non-negative integer and assume that we are given a $2n$ by $2n$ skew-symmetric matrix $A = (a_{ij})_{1\le i,j\le 2n}$, (i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring. The *Pfaffian* of A is, by definition,

$$\mathrm{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n})\, a_{\sigma_1\sigma_2}\ldots a_{\sigma_{2n-1}\sigma_{2n}},$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \ldots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-element blocks, and where $\epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n})$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2n} \end{pmatrix}.$$

We call a partition $\sigma = \{\{\sigma_1, \sigma_2\}_<, \ldots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-element blocks *a perfect matching* or *1-factor* of $[2n]$, and let $\mathcal{F}_n$ denote the set of all perfect matchings of $[2n]$. We represent a perfect matching σ graphically by embedding the points $i \in [2n]$ along the x-axis in the coordinate plane and representing each block $\{\sigma_{2i-1}, \sigma_{2i}\}_<$ by the curve connecting σ_{2i-1} to σ_{2i} in the upper half plane. The graphical representation of $\sigma = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$

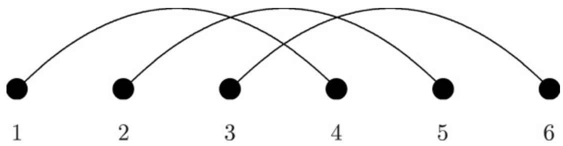


Fig. 1. A perfect matching.

is given in Fig. 1. If we write $\mathrm{wt}(\sigma) = \epsilon(\sigma)\prod_{i=1}^{n} a_{\sigma_{2i-1}\sigma_{2i}}$ for each perfect matching σ, then we can restate our definition as

$$\mathrm{Pf}(A) = \sum_{\sigma\in\mathcal{F}_n} \mathrm{wt}(\sigma). \tag{2.9}$$

A skew-symmetric matrix $A = (a_{ij})_{1\le i,j\le n}$ is uniquely determined by its upper triangular entries $(a_{ij})_{1\le i<j\le n}$. So we sometimes define a skew-symmetric matrix by describing its upper triangular entries. One of the most important formulas for Pfaffians is the expansion formula by minors. While the Laplacian determinant expansion formula by minors should be well known to everybody, the reader might not be so familiar with the Pfaffian expansion formula by minors so that we cite the formula here. For $1 \le i < j \le 2n$, let $(A; \{i, j\}, \{i, j\})$ denote the $(2n-2)\times(2n-2)$ skew-symmetric matrix obtained by removing both the ith and jth rows and both the ith and jth columns of A. Let us define $\gamma(i, j)$ by

$$\gamma(i, j) = (-1)^{j-i-1}\,\mathrm{Pf}(A; \{i, j\}, \{i, j\}). \tag{2.10}$$

Then the following identities are called the Laplacian Pfaffian expansions by minors:

$$\delta_{i,j}\,\mathrm{Pf}\,(A) = \sum_{k=1}^{2n} a_{kj}\gamma(k, i), \tag{2.11}$$

$$\delta_{i,j}\,\mathrm{Pf}\,(A) = \sum_{k=1}^{2n} a_{ik}\gamma(j, k). \tag{2.12}$$

(See [8,9].) We call the formula (2.11) the Pfaffian expansion along the jth column, and the formula (2.12) the Pfaffian expansion along the ith row. Especially, if we put $i = 1$ in (2.12), then we obtain the expansion formula along the first row:

$$\mathrm{Pf}\,(A) = \sum_{k=2}^{2n} (-1)^k a_{1,k}\,\mathrm{Pf}(A; \{1, k\}, \{1, k\}). \tag{2.13}$$

Let $O_{m,n}$ denote the $m\times n$ zero matrix and let E_n denote the identity matrix $(\delta_{i,j})_{1\le i,j\le n}$ of size n. Here $\delta_{i,j}$ denotes the Kronecker delta. We use the abbreviation O_n for $O_{n,n}$.

For any finite set S and any non-negative integer r, let $\binom{S}{r}$ denote the set of all r-element subsets of S. For example, $\binom{[n]}{r}$ stands for the set of all multi-indices $\{i_1, \dots, i_r\}$ such that $1 \le i_1 < \dots < i_r \le n$. Let m, n and r be integers such that $r \le m, n$ and let T be an m by n matrix. For any index sets $I = \{i_1, \dots, i_r\} \in \binom{[m]}{r}$ and $J = \{j_1, \dots, j_r\} \in \binom{[n]}{r}$, let $\Delta^I_J(A)$ denote the submatrix obtained by selecting the rows indexed by I and the columns indexed by J. If $r = m$ and $I = [m]$, we simply write $\Delta_J(A)$ for $\Delta^{[m]}_J(A)$. Similarly, if $r = n$ and $J = [n]$, we write $\Delta^I(A)$ for $\Delta^I_{[n]}(A)$. It is essential that the weight $\omega(\lambda)$ can be expressed by a Pfaffian, which is a fact proved in [6]:

Theorem 2.1. *Let n be a non-negative integer. Let $\lambda = (\lambda_1, \dots, \lambda_{2n})$ be a partition such that $\ell(\lambda) \le 2n$, and put $l = (l_1, \dots, l_{2n}) = \lambda + \delta_{2n}$, where $\delta_m = (m-1, m-2, \dots, 1, 0)$ for a non-negative integer m. Define a skew-symmetric matrix $A = (\alpha_{ij})_{i,j\ge 0}$ by*

$$\alpha_{ij} = a^{\lceil (j-1)/2\rceil} b^{\lfloor (j-1)/2\rfloor} c^{\lceil i/2\rceil} d^{\lfloor i/2\rfloor}$$

for $i < j$. Then we have

$$\mathrm{Pf}\left[\Delta_{I(\lambda)}^{I(\lambda)}(A)\right]_{1\le i,j\le 2n} = (abcd)^{\binom{n}{2}}\omega(\lambda),$$

where $I(\lambda) = \{l_{2n}, \dots, l_1\}$.

A variation of this theorem for strict partitions is as follows.

Theorem 2.2. *Let n be a non-negative integer. Let $\mu = (\mu_1, \dots, \mu_n)$ be a strict partition such that $\mu_1 > \cdots > \mu_n \ge 0$. Let $K(\mu) = \{\mu_n, \dots, \mu_1\}$. Define a skew-symmetric matrix $B = (\beta_{ij})_{i,j\ge -1}$ by*

$$\beta_{ij} = \begin{cases} 1, & \text{if } i = -1 \text{ and } j = 0,\\ a^{\lceil j/2\rceil} b^{\lfloor j/2\rfloor} z, & \text{if } i = -1 \text{ and } j \ge 1,\\ a^{\lceil j/2\rceil} b^{\lfloor j/2\rfloor} z, & \text{if } i = 0,\\ a^{\lceil j/2\rceil} b^{\lfloor j/2\rfloor} c^{\lceil i/2\rceil} d^{\lfloor i/2\rfloor} z^2, & \text{if } i > 0, \end{cases} \tag{2.14}$$

for $-1 \le i < j$.

(i) *If n is even, then we have*

$$\mathrm{Pf}\left[\Delta_{K(\mu)}^{K(\mu)}(B)\right] = \omega(\mu) z^{\ell(\mu)}. \tag{2.15}$$

(ii) *If n is odd, then we have*

$$\mathrm{Pf}\left[\Delta_{\{-1\}\uplus K(\mu)}^{\{-1\}\uplus K(\mu)}(B)\right] = \omega(\mu) z^{\ell(\mu)}. \quad \square \tag{2.16}$$

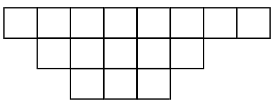
These theorems are easy consequences of the following lemma which has been proved in [8, Section 4, Lemma 7].

Lemma 2.3. *Let x_i and y_j be indeterminates, and let n be a non-negative integer. Then*

$$\mathrm{Pf}[x_i y_j]_{1\le i<j\le 2n} = \prod_{i=1}^{n} x_{2i-1} \prod_{i=1}^{n} y_{2i}. \quad \square$$

3. Strict partitions

A partition μ is *strict* if all its parts are distinct. One represents the associated shifted diagram of μ as a diagram in which the ith row from the top has been shifted to the right by i places so that the first column becomes a diagonal. A strict partition can be written uniquely in the form $\mu = (\mu_1, \dots, \mu_{2n})$ where n is an non-negative integer and $\mu_1 > \mu_2 > \cdots > \mu_{2n} \ge 0$. The *length* $\ell(\mu)$ is, by definition, the number of non-zero parts of μ. We define the weight function $\omega(\mu)$ similarly as in (1.1). For example, if $\mu = (8, 5, 3)$, then $\ell(\mu) = 3$, $\omega(\mu) = a^6 b^5 c^3 d^2$ and its shifted diagram is as follows.



Let

$$\Psi_N = \Psi_N(a, b, c, d; z) = \sum \omega(\mu) z^{\ell(\mu)}, \tag{3.1}$$

where the sum is over all strict partitions μ such that each part of μ is less than or equal to N. For example, we have

$$\begin{aligned} \Psi_0 &= 1,\\ \Psi_1 &= 1 + az,\\ \Psi_2 &= 1 + a(1+b)z + abcz^2,\\ \Psi_3 &= 1 + a(1+b+ab)z + abc(1+a+ad)z^2 + a^3bcdz^3. \end{aligned}$$

In fact, the only strict partition such that $\ell(\mu) = 0$ is $\emptyset$, the strict partitions μ such that $\ell(\mu) = 1$ and $\mu_1 \le 3$ are the following three:

$$\begin{array}{|c|}\hline a\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline a & b\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline a & b & a\\\hline\end{array},$$

the strict partitions μ such that $\ell(\mu) = 2$ and $\mu_1 \le 3$ are the following three:

$$\begin{array}{|c|c|}\hline a & b\\\hline \multicolumn{1}{c|}{} & c\\\cline{2-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline a & b & a\\\hline \multicolumn{1}{c|}{} & c & \multicolumn{1}{|c}{}\\\cline{2-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline a & b & a\\\hline \multicolumn{1}{c|}{} & c & d\\\cline{2-3}\end{array},$$

and the strict partition μ such that $\ell(\mu) = 3$ and $\mu_1 \le 3$ is the following one:

$$\begin{array}{|c|c|c|}\hline a & b & a\\\hline \multicolumn{1}{c|}{} & c & d\\\cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & a\\\cline{3-3}\end{array}.$$

The sum of the weights of these strict partitions is equal to Ψ_3. In this section we always assume $|a|, |b|, |c|, |d| < 1$. One of the main results of this section is that the even terms and the odd terms of Ψ_N respectively satisfy the associated Al-Salam–Chihara recurrence relation:

Theorem 3.1. *Set $q = abcd$. Let $\Psi_N = \Psi_N(a, b, c, d; z)$ be as in (3.1) and put $X_N = \Psi_{2N}$ and $Y_N = \Psi_{2N+1}$. Then X_N and Y_N satisfy*

$$X_{N+1} = \left\{1 + ab + a(1 + bc)z^2q^N\right\} X_N - ab(1 - z^2q^N)(1 - acz^2q^{N-1})X_{N-1}, \tag{3.2}$$

$$Y_{N+1} = \left\{1 + ab + abc(1 + ad)z^2q^N\right\} Y_N - ab(1 - z^2q^N)(1 - acz^2q^N)Y_{N-1}, \tag{3.3}$$

where $X_0 = 1$, $Y_0 = 1 + az$, $X_1 = 1 + a(1 + b)z + abcz^2$ and

$$Y_1 = 1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3.$$

Especially, if we put $X'_N = (ab)^{-\frac{N}{2}} X_N$ and $Y'_N = (ab)^{-\frac{N}{2}} Y_N$, then X'_N and Y'_N satisfy

$$\left\{(ab)^{\frac12} + (ab)^{-\frac12}\right\} X'_N = X'_{N+1} - a^{\frac12}b^{-\frac12}(1 + bc)z^2q^N X'_N + (1 - z^2q^N)(1 - acz^2q^{N-1})X'_{N-1}, \tag{3.4}$$

$$\left\{(ab)^{\frac12} + (ab)^{-\frac12}\right\} Y'_N = Y'_{N+1} - a^{\frac12}b^{\frac12}c(1 + ad)z^2q^N Y'_N + (1 - z^2q^N)(1 - a^2bc^2dz^2q^{N-1})Y'_{N-1}, \tag{3.5}$$

where $X'_0 = 1$, $Y'_0 = 1 + az$, $X'_1 = (ab)^{-\frac12} + a^{\frac12}b^{-\frac12}(1 + b)z + (ab)^{\frac12}cz^2$ and

$$Y'_1 = (ab)^{-\frac12} + a^{\frac12}b^{-\frac12}(1 + b + ab)z + a^{\frac12}b^{\frac12}c(1 + a + ad)z^2 + a^{\frac52}b^{\frac12}cdz^3.$$

Thus (3.4) agrees with the associated Al-Salam–Chihara recurrence relation (2.2) where $u = a^{\frac12}b^{\frac12}$, $\alpha = -a^{\frac12}b^{\frac12}c$, $\beta = -a^{\frac12}b^{-\frac12}$ and $t = z^2$, and (3.5) also agrees with (2.2) where $u = a^{\frac12}b^{\frac12}$, $\alpha = -a^{\frac12}b^{\frac12}c$, $\beta = -a^{\frac32}b^{\frac12}cd$ and $t = z^2$. One concludes that, when $|a|, |b|, |c|, |d| < 1$, the solutions of (3.2) and (3.3) are expressed by the linear combinations of (2.3) and (2.4) as follows.

Theorem 3.2. *Assume $|a|, |b|, |c|, |d| < 1$ and set $q = abcd$. Let $\Psi_N = \Psi_N(a, b, c, d; z)$ be as in (3.1).*

(i) *Put $X_N = \Psi_{2N}$. Then we have*

$$\begin{aligned} X_N = \frac{(-az^2q, -abc; q)_\infty}{(-a, -abcz^2; q)_\infty} \Bigg\{ & (s^X_0 X_1 - s^X_1 X_0)(-abcz^2; q)_N \; {}_2\phi_1\left(\begin{matrix} q^{-N}z^{-2}, -b^{-1} \\ -(abc)^{-1}q^{-N+1}z^{-2} \end{matrix}; q, -c^{-1}q\right) \\ & + (r^X_1 X_0 - r^X_0 X_1)(ab)^N \frac{(qz^2, acz^2; q)_N}{(-aqz^2; q)_N} \; {}_2\phi_1\left(\begin{matrix} q^{N+1}z^2, -c^{-1}q \\ -aq^{N+1}z^2 \end{matrix}; q, -abc\right)\Bigg\}, \end{aligned} \tag{3.6}$$

where

$$r_0^X = {}_2\phi_1\left(\begin{matrix} z^{-2}, -b^{-1} \\ -(abc)^{-1}z^{-2}q \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^X = {}_2\phi_1\left(\begin{matrix} z^2q, -c^{-1}q \\ -az^2q \end{matrix}; q, -abc\right),$$

$$r_1^X = (1+abcz^2)\ {}_2\phi_1\left(\begin{matrix} z^{-2}q^{-1}, -b^{-1} \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_1^X = \frac{ab(1-z^2q)(1-acz^2)}{1+az^2q}\ {}_2\phi_1\left(\begin{matrix} z^2q^2, -c^{-1}q \\ -az^2q^2 \end{matrix}; q, -abc\right).$$

(ii) *Put* $Y_N = \Psi_{2N+1}$. *Then we have*

$$Y_N = \frac{(-aq^2z^2, -abc; q)_\infty}{(-aq, -abcz^2; q)_\infty}\left\{(s_0^Y Y_1 - s_1^Y Y_0)(-abcz^2; q)_N\ {}_2\phi_1\left(\begin{matrix} q^{-N}z^{-2}, -acd \\ -(abc)^{-1}q^{-N+1}z^{-2} \end{matrix}; q, -c^{-1}q\right)\right.$$
$$\left. + (r_1^Y Y_0 - r_0^Y Y_1)(ab)^N \frac{(qz^2, acqz^2; q)_N}{(-aq^2z^2; q)_N}\ {}_2\phi_1\left(\begin{matrix} q^{N+1}z^2, -c^{-1}q \\ -aq^{N+2}z^2 \end{matrix}; q, -abc\right)\right\}, \tag{3.7}$$

where

$$r_0^Y = {}_2\phi_1\left(\begin{matrix} z^{-2}, -acd \\ (-abc)^{-1}qz^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$r_1^Y = (1+abcz^2)\ {}_2\phi_1\left(\begin{matrix} q^{-1}z^{-2}, -acd \\ -(abc)^{-1}z^{-2} \end{matrix}; q, -c^{-1}q\right),$$

$$s_0^Y = {}_2\phi_1\left(\begin{matrix} z^2q, -c^{-1}q \\ -aq^2z^2 \end{matrix}; q, -abc\right),$$

$$s_1^Y = \frac{ab(1-z^2q)(1-acqz^2)}{1+aq^2z^2}\ {}_2\phi_1\left(\begin{matrix} z^2q^2, -c^{-1}q \\ -aq^3z^2 \end{matrix}; q, -abc\right).$$

If we take the limit $N \to \infty$ in (3.6) and (3.7), then by using (2.8), we obtain the following generalization of Boulet's result (see Corollary 3.6).

Corollary 3.3. *Assume* $|a|, |b|, |c|, |d| < 1$ *and set* $q = abcd$. *Let* s_i^X, s_i^Y, X_i, Y_i $(i = 0, 1)$ *be as in the above theorem. Then we have*

$$\sum_\mu \omega(\mu) z^{\ell(\mu)} = \frac{(-abc, -az^2q; q)_\infty}{(ab; q)_\infty}(s_0^X X_1 - s_1^X X_0)$$
$$= \frac{(-abc, -az^2q^2; q)_\infty}{(ab; q)_\infty}(s_0^Y Y_1 - s_1^Y Y_0), \tag{3.8}$$

where the sum runs over all strict partitions and the first terms are as follows:

$$1 + \frac{a(1+b)}{1-ab}z + \frac{abc(1+a+ad+abd)}{(1-ab)(1-q)}z^2 + \frac{a^2q(1+b)(1+bc+abc+bq)}{(1-ab)(1-q)(1-abq)}z^3 + O(z^4).$$

On the other hand, by plugging $z = 1$ into (3.6) and (3.7), we conclude that the solutions of the recurrence relations (3.4) and (3.5) with the above initial condition are exactly the Al-Salam–Chihara polynomials, which gives two finite versions of Boulet's result.

Corollary 3.4. *Put* $u = \sqrt{ab}$, $x = \frac{u+u^{-1}}{2}$ *and* $q = abcd$. *Let* $\Psi_N(a, b, c, d; z)$ *be as in* (3.1).

(i) *The polynomial* $\Psi_{2N}(a,b,c,d;1)$ *is given by*

$$\begin{aligned}\Psi_{2N}(a,b,c,d;1) &= (ab)^{\frac{N}{2}} Q_N(x; -a^{\frac{1}{2}}b^{\frac{1}{2}}c, -a^{\frac{1}{2}}b^{-\frac{1}{2}}|q),\\ &= (-a;q)_N\ {}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N+1}\end{matrix}; q, -bq\right).\end{aligned} \tag{3.9}$$

(ii) *The polynomial* $\Psi_{2N+1}(a,b,c,d;1)$ *is given by*

$$\begin{aligned}\Psi_{2N+1}(a,b,c,d;1) &= (1+a)(ab)^{\frac{N}{2}} Q_N(x; -a^{\frac{1}{2}}b^{\frac{1}{2}}c, -a^{\frac{3}{2}}b^{\frac{1}{2}}cd|q)\\ &= (-a;q)_{N+1}\ {}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N}\end{matrix}; q, -b\right).\end{aligned} \tag{3.10}$$

Substituting $a = zyq$, $b = z^{-1}yq$, $c = zy^{-1}q$ and $d = z^{-1}y^{-1}q$ into Corollary 3.4 (see [2]), then we immediately obtain the strict version of Andrews' result (see Corollary 4.4).

Corollary 3.5.

$$\sum_{\substack{\mu \text{ strict partitions}\\ \mu_1\le 2N}} z^{\mathcal{O}(\mu)} y^{\mathcal{O}(\mu')} q^{|\mu|} = \sum_{j=0}^{N} \begin{bmatrix} N \\ j\end{bmatrix}_{q^4} (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}, \tag{3.11}$$

and

$$\sum_{\substack{\mu \text{ strict partitions}\\ \mu_1\le 2N+1}} z^{\mathcal{O}(\mu)} y^{\mathcal{O}(\mu')} q^{|\mu|} = \sum_{j=0}^{N} \begin{bmatrix} N \\ j\end{bmatrix}_{q^4} (-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}, \tag{3.12}$$

where

$$\begin{bmatrix} N \\ j\end{bmatrix}_q = \begin{cases} \dfrac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-j+1})}{(1-q^j)(1-q^{j-1})\cdots(1-q)}, & \text{for } 0\le j\le N,\\ 0, & \text{if } j<0 \text{ and } j>N.\end{cases}$$

Letting $N \to \infty$ in Corollary 3.4 or setting $z = 1$ in (3.8), we obtain the following result of Boulet (cf. [2, Corollary 2]).

Corollary 3.6 (*Boulet*). *Let* $q = abcd$, *then*

$$\sum_{\mu} \omega(\mu) = \frac{(-a;q)_\infty(-abc;q)_\infty}{(ab;q)_\infty}, \tag{3.13}$$

where the sum runs over all strict partitions.

To prove Theorem 3.1, we need several steps. Our strategy is as follows: write the weight $\omega(\mu)z^{\ell(\mu)}$ as a Pfaffian (Theorem 2.2) and apply the minor summation formula (Lemma 3.7) to make the sum of the weights into a single Pfaffian (Theorem 3.8). Then we make use of the Pfaffian to derive a recurrence relation (Proposition 3.9). We also give another proof of the recurrence relation by a combinatorial argument (Remark 3.10).

Let J_n denote the square matrix of size n whose (i,j)th entry is $\delta_{i,n+1-j}$. We simply write J for J_n when there is no fear of confusion on the size n. We need the following result on a sum of Pfaffians [18, Theorem of Section 4].

Lemma 3.7. *Let n be a positive integer. Let $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$ be skew-symmetric matrices of size n. Then*

$$\sum_{t=0}^{\lfloor n/2\rfloor} z^t \sum_{I\in\binom{[n]}{2t}} \gamma^{|I|}\,\mathrm{Pf}\left(\Delta_I^I(A)\right)\mathrm{Pf}\left(\Delta_I^I(B)\right) = \mathrm{Pf}\begin{bmatrix} J_n\,{}^tA\,J_n & J_n \\ -J_n & C\end{bmatrix}, \tag{3.14}$$

where $|I| = \sum_{i\in I} i$ *and* $C = (C_{ij})_{1\le i,j\le n}$ *is given by* $C_{ij} = \gamma^{i+j}b_{ij}z$.

This lemma is a special case of Lemma 5.4, so a proof will be given later.

Let S_n denote the $n \times n$ skew-symmetric matrix whose (i, j)th entry is 1 for $0 \le i < j \le n$. As a corollary of Lemma 3.7, we obtain the following expression of the sum of the weight $\omega(\mu)$ by a single Pfaffian.

Theorem 3.8. *Let N be a non-negative integer.*

$$\Psi_N(a, b, c, d; z) = \operatorname{Pf}\begin{bmatrix} S_{N+1} & J_{N+1} \\ -J_{N+1} & B \end{bmatrix}, \tag{3.15}$$

where $B = (\beta_{ij})_{0 \le i < j \le N}$ is the $(N+1) \times (N+1)$ skew-symmetric matrix whose (i, j)th entry β_{ij} is defined as in (2.14).

Proof. Here we assume that the row/column indices start at 0. Note that any strict partition μ is written uniquely as $\mu = (\mu_1, \ldots, \mu_{2t})$ with $\mu_1 > \cdots > \mu_{2t} \ge 0$. Here $2t = \ell(\mu)$ if $\ell(\mu)$ is even, and $2t = \ell(\mu) + 1$ and $\mu_{2t} = 0$ if $\ell(\mu)$ is odd. Thus, using Theorem 2.2 (2.15), we obtain

$$\begin{aligned}\Psi_N(a, b, c, d; z) &= \sum_{\substack{\mu \text{ strict} \\ \mu_1 \le N}} \omega(\mu) z^{\ell(\mu)} = \sum_{t=0}^{\lfloor (N+1)/2 \rfloor} \sum_{\substack{\mu=(\mu_1,\ldots,\mu_{2t}) \\ N \ge \mu_1 > \cdots > \mu_{2t} \ge 0}} \omega(\mu) z^{\ell(\mu)} \\ &= \sum_{t=0}^{\lfloor (N+1)/2 \rfloor} \sum_{\substack{\mu=(\mu_1,\ldots,\mu_{2t}) \\ N \ge \mu_1 > \cdots > \mu_{2t} \ge 0}} \operatorname{Pf}\left(\Delta_{K(\mu)}^{K(\mu)}(B)\right) = \sum_{t=0}^{\lfloor (N+1)/2 \rfloor} \sum_{I \in \binom{[0,N]}{2t}} \operatorname{Pf}\left(\Delta_I^I(B)\right).\end{aligned}$$

If we put $n = N + 1$, $z = \gamma = 1$ and $A = S_{N+1}$ into (3.14), then we obtain

$$\sum_{t=0}^{\lfloor (N+1)/2 \rfloor} \sum_{I \in \binom{[0,N]}{2t}} \operatorname{Pf}\left(\Delta_I^I(B)\right) = \operatorname{Pf}\begin{bmatrix} J_{N+1}\, {}^t S_{N+1}\, J_{N+1} & J_{N+1} \\ -J_{N+1} & C \end{bmatrix},$$

since $\operatorname{Pf}\left(\Delta_I^I(S_{N+1})\right) = 1$ holds for any subset $I \subseteq [0, N]$ of even cardinality. (For detailed arguments on sub-Pfaffians, see [9]). In this case, $C = (C_{ij})$ in Lemma 3.7 is equal to $B = (b_{ij})$ in (2.14) because of $z = \gamma = 1$. It is also easy to check that $J_{N+1}\, {}^t S_{N+1}\, J_{N+1} = S_{N+1}$. Thus we easily obtain the desired formula (3.15) from these identities. This completes the proof. $\square$

For example, if $N = 3$, then the skew-symmetric matrix in the right-hand side of (3.15) is

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & az & abz & a^2bz \\ 0 & 0 & -1 & 0 & -az & 0 & abcz^2 & a^2bcz^2 \\ 0 & -1 & 0 & 0 & -abz & -abcz^2 & 0 & a^2bcdz^2 \\ -1 & 0 & 0 & 0 & -a^2bz & -a^2bcz^2 & -a^2bcdz^2 & 0 \end{array}\right], \tag{3.16}$$

whose Pfaffian equals $\Psi_3 = 1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3$.

By performing elementary transformations on rows and columns of the matrix, we obtain the following recurrence relation:

Proposition 3.9. *Let $\Psi_N = \Psi_N(a, b, c, d; z)$ be as above. Then we have*

$$\Psi_{2N} = (1 + b)\,\Psi_{2N-1} + (a^N b^N c^N d^{N-1} z^2 - b)\,\Psi_{2N-2}, \tag{3.17}$$

$$\Psi_{2N+1} = (1 + a)\,\Psi_{2N} + (a^{N+1} b^N c^N d^N z^2 - a)\,\Psi_{2N-1}, \tag{3.18}$$

for any positive integer N.

Proof. Let A denote the $2(N+1)\times 2(N+1)$ skew-symmetric matrix $\begin{bmatrix} S_{N+1} & J_{N+1} \\ -J_{N+1} & B \end{bmatrix}$ in the right-hand side of (3.15). Here we assume that row/column indices start at 0. So, for example, the row indices for the upper $(N+1)$ rows are i, $i=0,\dots,N$, and the row indices for the lower $(N+1)$ rows are $i+N+1$, $i=0,\dots,N$. If $N=3$, then A is as in Eq. (3.16), and the row/column indices are $0,\dots,7$ in which $0,\dots,3$ are called upper and $4,\dots,7$ are called lower. Now, subtract a times $(j+N)$th column from $(j+N+1)$th column if j is odd, or subtract b times $(j+N)$th column from $(j+N+1)$th column if j is even, for $j=N,N-1,\dots,1$. To make our matrix skew-symmetric, subtract a times $(i+N)$th row from $(i+N+1)$th row if i is odd, or subtract b times $(i+N)$th row from $(i+N+1)$th row if i is even, for $i=N,N-1,\dots,1$. To make things clear, we take $N=3$ case as an example. If $N=3$, then we first subtract a times 6th column from 7th column of the matrix (3.16), then we subtract b times 5th column from 6th column of the resulting matrix, and lastly we subtract a times 4th column from 5th column of the resulting matrix. Thus we obtain the skew-matrix

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & -a \\ -1 & -1 & 0 & 1 & 0 & 1 & -b & 0 \\ -1 & -1 & -1 & 0 & 1 & -a & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & az & 0 & 0 \\ 0 & 0 & -1 & 0 & -az & a^2z & abcz^2 & 0 \\ 0 & -1 & 0 & 0 & -abz & a^2bz-abcz^2 & ab^2cz^2 & a^2bcdz^2 \\ -1 & 0 & 0 & 0 & -a^2bz & a^3bz-a^2bcz^2 & a^2b^2cz^2-a^2bcdz^2 & a^3bcdz^2 \end{array}\right]. \tag{3.19}$$

Next we perform the same operations on rows to make the matrix skew-symmetric, i.e., subtracting a times 6th row from 7th row of the matrix (3.19), then subtracting b times 5th row from 6th row of the resulting matrix, and so on. Then we obtain

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & -a \\ -1 & -1 & 0 & 1 & 0 & 1 & -b & 0 \\ -1 & -1 & -1 & 0 & 1 & -a & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & az & 0 & 0 \\ 0 & 0 & -1 & a & -az & 0 & abcz^2 & 0 \\ 0 & -1 & b & 0 & 0 & -abcz^2 & 0 & a^2bcdz^2 \\ -1 & a & 0 & 0 & 0 & 0 & -a^2bcdz^2 & 0 \end{array}\right]. \tag{3.20}$$

In the next step, we subtract $(j+1)$th column from jth column for $j=0,1,\dots,N-1$, then we also subtract $(i+1)$th row from ith row for $i=0,1,\dots,N-1$. If $N=3$, then this step is as follows. First, we subtract 1st column from 0th column of the matrix (3.20), then we subtract 2nd column from 1st column of the resulting matrix, and finally we subtract 3rd column from 2nd column of the resulting matrix. We perform the same operations on rows. Then the resulting matrix looks as follows:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1+a \\ -1 & 0 & 1 & 0 & 0 & -1 & 1+b & -a \\ 0 & -1 & 0 & 1 & -1 & 1+a & -b & 0 \\ 0 & 0 & -1 & 0 & 1 & -a & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & 0 & az & 0 & 0 \\ 0 & 1 & -1-a & a & -az & 0 & abcz^2 & 0 \\ 1 & -1-b & b & 0 & 0 & -abcz^2 & 0 & a^2bcdz^2 \\ -1-a & a & 0 & 0 & 0 & 0 & -a^2bcdz^2 & 0 \end{array}\right]. \tag{3.21}$$

Let A' denote the resulting matrix after these transformations. Then, in general, the resulting skew-symmetric matrix A' is written as

$$A' = \begin{bmatrix} P & Q \\ -{}^tQ & R \end{bmatrix} \tag{3.22}$$

with the $(N+1)\times(N+1)$ matrices $P=(\delta_{i+1,j})_{0\le i<j\le N}$, $Q=(q_{ij})_{0\le i<j\le N}$ and $R=(r_{ij})_{0\le i<j\le N}$ whose entries are given by

$$q_{ij}=\begin{cases}-1 & \text{if } i+j=N-1,\\ 1 & \text{if } i=N \text{ and } j=0,\\ 1+a^{\chi(j\text{ is odd})}b^{\chi(j\text{ is even})} & \text{if } i+j=N \text{ and } j\ge 1,\\ -a^{\chi(j\text{ is odd})}b^{\chi(j\text{is even})} & \text{if } i+j=N+1,\\ 0 & \text{otherwise,}\end{cases}$$

$$r_{ij}=\begin{cases}az\delta_{1,j} & \text{if } i=0,\\ a^{\lceil (i+1)/2\rceil}b^{\lfloor (i+1)/2\rfloor}c^{\lceil i/2\rceil}d^{\lfloor i/2\rfloor}z^2\delta_{i+1,j} & \text{if } i>0.\end{cases}$$

Here $\chi(A)$ stands for 1 if the statement A is true and 0 otherwise. If we apply the expansion formula (2.13) to $\mathrm{Pf}(A')$, then we easily obtain the desired formula, i.e. (3.17) if N is even, and (3.18) if N is odd. We illustrate this expansion by the above example. If we expand the Pfaffian of the skew-symmetric matrix (3.21) along the first row, then we obtain

$$\Psi_3=\mathrm{Pf}\left[\begin{array}{cc|cccc}0&1&-1&1+a&-b&0\\-1&0&1&-a&0&0\\\hline 1&-1&0&az&0&0\\-1-a&a&-az&0&abcz^2&0\\b&0&0&-abcz^2&0&a^2bcdz^2\\0&0&0&0&-a^2bcdz^2&0\end{array}\right]$$
$$+\,\mathrm{Pf}\left[\begin{array}{ccc|ccc}0&1&0&0&-1&-a\\-1&0&1&-1&1+a&0\\0&-1&0&1&-a&0\\\hline 0&1&-1&0&az&0\\1&-1-a&a&-az&0&0\\a&0&0&0&0&0\end{array}\right]$$
$$+(1+a)\,\mathrm{Pf}\left[\begin{array}{ccc|ccc}0&1&0&0&-1&1+b\\-1&0&1&-1&1+a&-b\\0&-1&0&1&-a&0\\\hline 0&1&-1&0&az&0\\1&-1-a&a&-az&0&abcz^2\\-1-b&b&0&0&-abcz^2&0\end{array}\right].$$

By expanding the first Pfaffian along the last column, we obtain that this Pfaffian equals $a^2bcdz^2\,\Psi_1$. Similarly, by expanding the second Pfaffian along the last column, we also obtain that this Pfaffian equals $-a\,\Psi_1$. The third Pfaffian is evidently equal to Ψ_2. Thus we obtain $\Psi_3=(a^2bcdz^2-a)\,\Psi_1+(1+a)\,\Psi_2$. The general argument is similar based on the above expression of (3.22). The details are left to the reader. This completes the proof. □

Remark 3.10. Proposition 3.9 can be also proved by a combinatorial argument as follows.

Combinatorial proof of Proposition 3.9. By definition, the generating function for strict partitions $\mu=(\mu_1,\mu_2,\ldots)$ such that $\mu_1=2N$ and $\mu_2\le 2N-2$ is equal to

$$b(\Psi_{2N-1}-\Psi_{2N-2}).$$

This, for strict partitions such that $\mu_1=2N$ and $\mu_2=2N-1$, is equal to

$$a^Nb^Nc^Nd^{N-1}z^2\,\Psi_{2N-2}.$$

Finally the generating function of strict partitions such that $\mu_1\le 2N-1$ is equal to Ψ_{2N-1}. Summing up we get (3.17). The same argument can be used to prove (3.18). □

Note that one can immediately derive Theorem 3.1 from Proposition 3.9 by substitution. Thus, if one uses (2.7), then he immediately derives Theorem 3.2 by a simple computation.

Proof of Theorem 3.2. Let $u = \sqrt{ab}$, $t = z^2$ and $q = abcd$. By (3.4), X'_N satisfies the associated Al-Salam–Chihara recurrence relation (2.2) with $\alpha = -a^{\frac{1}{2}} b^{\frac{1}{2}} c$ and $\beta = -a^{\frac{1}{2}} b^{-\frac{1}{2}}$. Note that $|u| < 1$ and $|q| < |\alpha| < 1$ hold. Thus, by (2.7), we conclude that X_N is given by (3.6). A similar argument shows that Y'_N satisfies (2.2) with $\alpha = -a^{\frac{3}{2}} b^{\frac{1}{2}} c$ and $\beta = -a^{\frac{1}{2}} b^{\frac{1}{2}} cd$, which implies Y_N is given by (3.7). □

Proof of Corollary 3.4. First, substituting z by 1 in (3.6), we have

$$\begin{aligned}
r_0^X &= 1,\\
s_0^X &= \sum_{n=0}^{\infty} \frac{(1+aq^{n+1})(-c^{-1}q;q)_n}{(-aq;q)_{n+1}}(-abc)^n,\\
r_1^X &= 1+abc+a(1+b),\\
s_1^X &= ab(1-ac)\sum_{n=0}^{\infty} \frac{(1-q^{n+1})(-c^{-1}q;q)_n}{(-aq;q)_{n+1}}(-abc)^n.
\end{aligned}$$

Since $X_0 = 1$ and $X_1 = 1 + a(1 + b) + abc$ for $z = 1$, we derive $r_1^X X_0 - r_0^X X_1 = 0$ and

$$\begin{aligned}
s_0^X X_1 - s_1^X X_0 &= (1+a)\sum_{n=0}^{\infty} \frac{(-c^{-1}q;q)_n}{(-aq;q)_{n+1}}(-abc)^n\{a+abc+a(1+b)q^{n+1}\}\\
&= (1+a)\left\{\sum_{n=0}^{\infty} \frac{(-c^{-1}q;q)_n}{(-aq;q)_n}(-abc)^n - \sum_{n=0}^{\infty} \frac{(-c^{-1}q;q)_{n+1}}{(-aq;q)_{n+1}}(-abc)^{n+1}\right\}\\
&= 1+a.
\end{aligned}$$

Therefore, when $z = 1$, Eq. (3.6) reduces to

$$X_N = (-abc;q)_N\ {}_2\phi_1\left(\begin{matrix} q^{-N}, -b^{-1} \\ -(abc)^{-1}q^{-N+1}\end{matrix}; q, -c^{-1}q\right).$$

This establishes Eq. (3.9). A similar computation shows that we can derive (3.10) from (3.7) by specializing z to 1. The details are left to the reader. □

Proof of Corollary 3.5. We first claim that

$$\Psi_{2N}(a,b,c,d;1) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-a;q)_k(-c;q)_{N-k}(ab)^{N-k}. \tag{3.23}$$

Then (3.11) is an easy consequence of (3.23) by substituting $a \leftarrow zyq$, $b \leftarrow z^{-1}yq$, $c \leftarrow zy^{-1}q$ and $d \leftarrow z^{-1}y^{-1}q$. In fact, using $(q^{-N};q)_k = \frac{(q;q)_N}{(q;q)_{N-k}}(-1)^k q^{\binom{k}{2}-Nk}$, we have

$${}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N+1}\end{matrix}; q, -bq\right) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q \frac{(-c;q)_{N-k}}{(-a^{-1}q^{-N+1};q)_{N-k}} q^{\binom{N-k}{2}-N(N-k)}(bq)^{N-k}.$$

Substitute $(-a^{-1}q^{-N+1};q)_{N-k} = \frac{(-a;q)_N}{(-a;q)_k} a^{-N+k} q^{-\binom{N}{2}+\binom{k}{2}}$ into this identity to show that the right-hand side equals

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q \frac{(-a;q)_k(-c;q)_{N-k}}{(-a;q)_N}(ab)^{N-k}.$$

Finally, use (3.9) to obtain (3.23). The proof of (3.12) reduces to

$$\Psi_{2N+1}(a,b,c,d;1) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-a;q)_{k+1}(-c;q)_{N-k}(ab)^{N-k}, \tag{3.24}$$

which is derived from (3.10) similarly. □

Proof of Corollary 3.6. By replacing k by $N-k$ and letting N to $+\infty$ in (3.23), we get

$$\lim_{N\to\infty}\Psi_{2N}(a,b,c,d;1)=(-a;q)_\infty\sum_{k=0}^{\infty}\frac{(-c;q)_k}{(q;q)_k}(ab)^k=\frac{(-a;q)_\infty(-abc;q)_\infty}{(ab;q)_\infty},$$

where the last equality follows from the q-binomial formula (see [3]). Similarly we can derive the limit from (3.24).

Note that we can also derive (3.13) from (3.8) by the same argument as in the proof of Corollary 3.4. □

4. Ordinary partitions

First we present a generalization of Andrews' result in [1]. Let us consider

$$\Phi_N=\Phi_N(a,b,c,d;z)=\sum_{\substack{\lambda\\ \lambda_1\le N}}\omega(\lambda)z^{\ell(\lambda)},\tag{4.1}$$

where the sum runs over all partitions λ such that each part of λ is less than or equal to N. For example, the first few terms can be computed directly as follows:

$$\Phi_0=1,$$
$$\Phi_1=\frac{1+az}{1-acz^2},$$
$$\Phi_2=\frac{1+a(1+b)z+abcz^2}{(1-acz^2)(1-qz^2)},$$
$$\Phi_3=\frac{1+a(1+b+ab)z+abc(1+a+ad)z^2+a^3bcdz^3}{(1-z^2ac)(1-z^2q)(1-z^2acq)},$$

where $q=abcd$ as before. If one compares these with the first few terms of Ψ_N, one can easily guess that the following theorem holds:

Theorem 4.1. *For non-negative integer N, let $\Phi_N=\Phi_N(a,b,c,d;z)$ be as in (4.1) and $q=abcd$. Then we have*

$$\Phi_N(a,b,c,d;z)=\frac{\Psi_N(a,b,c,d;z)}{(z^2q;q)_{\lfloor N/2\rfloor}(z^2ac;q)_{\lceil N/2\rceil}},\tag{4.2}$$

where $\Psi_N=\Psi_N(a,b,c,d;z)$ is the generating function defined in (3.1). Note that Ψ_N is explicitly given in terms of basic hypergeometric functions in Theorem 3.2.

In fact, the main purpose of this section is to prove this theorem. Here we give two proofs, i.e. an algebraic proof (see Propositions 4.6 and 4.7) and a bijective proof (see Remark 4.8). Before we proceed to the proofs of this theorem we state the corollaries immediately obtained from this theorem and the results in Section 3. First of all, as an immediate corollary of Theorem 4.1 and Corollary 3.3, we obtain the following generalization of Boulet's result (Corollary 4.5).

Corollary 4.2. *Assume $|a|,|b|,|c|,|d|<1$ and set $q=abcd$. Let s_i^X, s_i^Y, X_i, Y_i $(i=0,1)$ be as in Theorem 3.2. Then we have*

$$\sum_{\lambda}\omega(\lambda)z^{|\mu|}=\frac{(-abc,-az^2q;q)_\infty}{(ab,acz^2,z^2q;q)_\infty}(s_0^XX_1-s_1^XX_0)$$
$$=\frac{(-abc,-a^2bcdz^2q;q)_\infty}{(ab,acz^2,z^2q;q)_\infty}(s_0^YY_1-s_1^YY_0),\tag{4.3}$$

where the sum runs over all partitions λ.

Theorem 4.1 and Corollary 3.4 also give the following corollary:

Corollary 4.3. *Put $x=\frac{(ab)^{\frac12}+(ab)^{-\frac12}}{2}$ and $q=abcd$. Let $\Phi_N=\Phi_N(a,b,c,d;z)$ be as in (4.1).*

(i) *The generating function* $\Phi_{2N}(a,b,c,d;1)$ *is given by*

$$\Phi_{2N}(a,b,c,d;1)=\frac{(ab)^{\frac{N}{2}}Q_N(x;-a^{\frac12}b^{\frac12}c,-a^{\frac12}b^{-\frac12}|q)}{(q;q)_N(ac;q)_N}$$
$$=\frac{(-a;q)_N}{(q;q)_N(ac;q)_N}\,{}_2\phi_1\left(\begin{matrix}q^{-N},-c\\-a^{-1}q^{-N+1}\end{matrix};q,-bq\right). \tag{4.4}$$

(ii) *The generating function* $\Phi_{2N}(a,b,c,d;1)$ *is given by*

$$\Phi_{2N+1}(a,b,c,d;1)=\frac{(1+a)(ab)^{\frac{N}{2}}Q_N(x;-a^{\frac12}b^{\frac12}c,-a^{\frac32}b^{\frac12}cd|q)}{(q;q)_N(ac;q)_{N+1}}$$
$$=\frac{(-a;q)_{N+1}}{(q;q)_N(ac;q)_{N+1}}\,{}_2\phi_1\left(\begin{matrix}q^{-N},-c\\-a^{-1}q^{-N}\end{matrix};q,-b\right). \tag{4.5}$$

Let $S_N(n,r,s)$ denote the number of partitions π of n where each part of π is $\le N$, $\mathcal{O}(\pi)=r$, $\mathcal{O}(\pi')=s$. As before we immediately deduce the following result of Andrews (cf. [1, Theorem 1]) from Corollary 4.3.

Corollary 4.4 (*Andrews*).

$$\sum_{n,r,s\ge 0}S_{2N}(n,r,s)q^nz^ry^s=\frac{\sum_{j=0}^{N}\begin{bmatrix}N\\j\end{bmatrix}_{q^4}(-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^4;q^4)_N}, \tag{4.6}$$

and

$$\sum_{n,r,s\ge 0}S_{2N+1}(n,r,s)q^nz^ry^s=\frac{\sum_{j=0}^{N}\begin{bmatrix}N\\j\end{bmatrix}_{q^4}(-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^4;q^4)_{N+1}}. \tag{4.7}$$

Similarly, as in the strict case, we obtain immediately Boulet's corresponding result for ordinary partitions (cf. [2, Theorem 1]).

Corollary 4.5 (*Boulet*). *Let* $q=abcd$, *then*

$$\sum_{\lambda}\omega(\lambda)=\frac{(-a;q)_\infty(-abc;q)_\infty}{(q;q)_\infty(ab;q)_\infty(ac;q)_\infty}, \tag{4.8}$$

where the sum runs over all partitions.

In order to prove Theorem 4.1 we first derive a recurrence formula for $\Phi_N(a,b,c,d;z)$.

Proposition 4.6. *Let* $\Phi_N=\Phi_N(a,b,c,d;z)$ *be as before and* $q=abcd$. *Then the following recurrences hold for any positive integer* N.

$$(1-z^2q^N)\Phi_{2N}=(1+b)\Phi_{2N-1}-b\Phi_{2N-2}, \tag{4.9}$$
$$(1-z^2acq^N)\Phi_{2N+1}=(1+a)\Phi_{2N}-a\Phi_{2N-1}. \tag{4.10}$$

Proof. It suffices to prove that

$$\Phi_{2N}=\Phi_{2N-1}+b(\Phi_{2N-1}-\Phi_{2N-2})+z^2q^N\Phi_{2N}, \tag{4.11}$$
$$\Phi_{2N+1}=\Phi_{2N}+a(\Phi_{2N}-\Phi_{2N-1})+z^2acq^N\Phi_{2N+1}. \tag{4.12}$$

Let $\mathcal{L}_N$ denote the set of partitions λ such that $\lambda_1\le N$. The generating function of $\mathcal{L}_N$ with weight $\omega(\lambda)z^{\ell(\lambda)}$ is $\Phi_N=\Phi_N(a,b,c,d;z)$. We divide $\mathcal{L}_N$ into three disjoint subsets:

$$\mathcal{L}_N=\mathcal{L}_{N-1}\uplus\mathcal{M}_N\uplus\mathcal{N}_N,$$

where $\mathcal{M}_N$ denote the set of partitions λ such that $\lambda_1 = N$ and $\lambda_2 < N$, and $\mathcal{N}_N$ denote the set of partitions λ such that $\lambda_1 = \lambda_2 = N$. When $N = 2r$ is even, it is easy to see that the generating function of $\mathcal{M}_{2r}$ equals $b(\Phi_{2r-1} - \Phi_{2r-2})$, and the generating function of $\mathcal{N}_{2r}$ equals $z^2 q^r \Phi_{2r}$. This proves (4.11). When $N = 2r + 1$ is odd, the same division proves (4.12). □

By simple computation, one can derive the following identities from (4.9) and (4.10).

Proposition 4.7. *If we put*

$$\Phi_N(a, b, c, d; z) = \frac{F_N(a, b, c, d; z)}{(z^2 q; q)_{\lfloor N/2 \rfloor}(z^2 ac; q)_{\lceil N/2 \rceil}}, \tag{4.13}$$

then,

$$F_{2N} = (1 + b)F_{2N-1} - b(1 - z^2 acq^{N-1})F_{2N-2}, \tag{4.14}$$

$$F_{2N+1} = (1 + a)F_{2N} - a(1 - z^2 q^N)F_{2N-1} \tag{4.15}$$

hold for any positive integer N.

Proof. Substitute (4.13) into (4.9) and (4.10), and compute directly to obtain (4.14) and (4.15). □

Proof of Theorem 4.1. From (4.14) and (4.15), one easily sees that $F_{2N}(a, b, c, d; z)$ and $F_{2N+1}(a, b, c, d; z)$ satisfy exactly the same recurrence in Theorem 3.1. Further, from the above example, we see

$$\begin{aligned}
&F_0 = 1,\\
&F_1 = 1 + az,\\
&F_2 = 1 + a(1 + b)z + abcz^2,\\
&F_3 = 1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3 bcdz^3,\\
&F_4 = 1 + a(1 + b)(1 + ab)z + abc(1 + a + ab + ad + abd + abcd)z^2\\
&\qquad + a^3 bcd(1 + b)(1 + bc)z^3 + a^3 b^3 c^3 dz^4.
\end{aligned}$$

Thus the first few terms of $F_N(a, b, c, d; z)$ agree with those of $\Psi_N(a, b, c, d; z)$. We immediately conclude that $F_N(a, b, c, d; z) = \Psi_N(a, b, c, d; z)$ for all N. □

Remark 4.8. Here we also give another proof of Theorem 4.1 by a bijection, which has already been used by Boulet [2] in the infinite case.

Bijective proof of Theorem 4.1. Let $\mathcal{P}_N$ (resp. $\mathcal{D}_N$) denote the set of partitions (resp. strict partitions) whose parts are less than or equal to N and let $\mathcal{E}_N$ denote the set of partitions whose parts appear an even number of times and are less than or equal to N. We shall establish a bijection $g : \mathcal{P}_N \longrightarrow \mathcal{D}_N \times \mathcal{E}_N$ with $g(\lambda) = (\mu, \nu)$ defined as follows. Suppose λ has k parts equal to i. If k is even then ν has k parts equal to i, and if k is odd then ν has $k - 1$ parts equal to i. The parts of λ which were not removed to form ν, at most one of each cardinality, give μ. It is clear that under this bijection, $\omega(\lambda) = \omega(\mu)\omega(\nu)$. It is easy to see that the generating function of $\mathcal{E}_N$ is equal to

$$\prod_{j=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{1 - z^2 q^j} \times \prod_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{1 - z^2 acq^j},$$

where $q = abcd$. As $\lfloor \frac{N-1}{2} \rfloor = \lceil \frac{N}{2} \rceil - 1$, we obtain (4.13). □

At the end of this section we state another enumeration of ordinary partitions, which is not directly related to Andrews' result, but obtained as an application of the minor summation formula of Pfaffians. Let

$$\Phi_{N,M} = \Phi_{N,M}(a, b, c, d) = \sum_{\substack{\lambda \\ \lambda_1 \le N,\ \ell(\lambda) \le M}} \omega(\lambda),$$

where the sum runs over all partitions λ such that λ has at most M parts and each part of λ is less than or equal to N.

Again we use Lemma 3.7 and Theorem 2.1 to obtain the following theorem.

Theorem 4.9. *Let N be a positive integer and set $q = abcd$. Then we have*

$$\sum_{t=0}^{\lfloor N/2\rfloor} \Phi_{N-2t,2t}(a,b,c,d)\, z^t q^{\binom{t}{2}} = \operatorname{Pf}\begin{bmatrix} S_N & J_N \\ -J_N & C \end{bmatrix}, \tag{4.16}$$

where $S = (1)_{0\le i<j\le N-1}$ and $C = (a^{\lceil (j-1)/2\rceil} b^{\lfloor (j-1)/2\rfloor} c^{\lceil i/2\rceil} d^{\lfloor i/2\rfloor} z)_{0\le i<j\le N-1}$.

Proof. As in the proof of Theorem 3.8, we take $n = N$, $\gamma = 1$ and $A = S_N$ in (3.14), then we obtain

$$\sum_{t=0}^{\lfloor N/2\rfloor} z^t \sum_{I\in\binom{[0,N-1]}{2t}} \operatorname{Pf}\left(\Delta_I^I(B)\right) = \operatorname{Pf}\begin{bmatrix} J_N\, {}^t S_N J_N & J_N \\ -J_N & C \end{bmatrix},$$

where $C = (b_{ij} z)_{0\le i,j\le N-1}$. If we take $b_{ij} = a^{\lceil (j-1)/2\rceil} b^{\lfloor (j-1)/2\rfloor} c^{\lceil i/2\rceil} d^{\lfloor i/2\rfloor}$, then Theorem 2.1 implies

$$\operatorname{Pf}\left(\Delta_I^I(B)\right) = \omega(\lambda) q^{\binom{t}{2}},$$

where $I(\lambda) = I$. Thus, using $J_N\, {}^t S_N J_N = S_N$ and the above formulas, we obtain

$$\sum_{t=0}^{\lfloor N/2\rfloor} z^t q^{\binom{t}{2}} \sum_{I\in\binom{[0,N-1]}{2t}} \omega(\lambda) = \operatorname{Pf}\begin{bmatrix} S_N & J_N \\ -J_N & C \end{bmatrix}.$$

Now (4.16) follows since, when I runs over all $2t$-subsets of $[0, N-1]$, λ runs over all partitions with at most $2t$ parts and each part is less than or equal to $N - 2t$. $\square$

For example, if $N = 4$, then the right-hand side of (4.16) becomes

$$\operatorname{Pf}\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & z & az & abz \\ 0 & 0 & -1 & 0 & -z & 0 & acz & abcz \\ 0 & -1 & 0 & 0 & -az & -acz & 0 & abcdz \\ -1 & 0 & 0 & 0 & -abz & -abcz & -abcdz & 0 \end{array}\right].$$

Let $\tilde{\Phi}_N = \tilde{\Phi}_N(a,b,c,d;z) = \operatorname{Pf}\begin{bmatrix} S & J \\ -J & C \end{bmatrix}$ denote the right-hand side of (4.16). For example, we have $\tilde{\Phi}_1 = 1$, $\tilde{\Phi}_2 = 1 + z$, $\tilde{\Phi}_3 = 1 + (1 + a + ac)z$ and $\tilde{\Phi}_4 = 1 + (1 + a + ab + ac + abc + abcd)z + abcdz^2$. Note that the partitions λ such that $\ell(\lambda) \le 2$ and $\lambda_1 \le 2$ are the following six:

$$\emptyset \qquad \boxed{a} \qquad \boxed{a}\boxed{b} \qquad \begin{array}{|c|}\hline a \\ \hline c \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}.$$

The sum of their weights is equal to $[z]\tilde{\Phi}_4 = 1 + a + ab + ac + abc + abcd$.

The same argument as in the proof of Proposition 3.9 can be used to prove the following proposition.

Proposition 4.10. *Let $\tilde{\Phi}_N = \tilde{\Phi}_N(a,b,c,d;z)$ be as above. Then we have*

$$\tilde{\Phi}_{2N} = (1+b)\tilde{\Phi}_{2N-1} + (a^{N-1}b^{N-1}c^{N-1}d^{N-1}z - b)\tilde{\Phi}_{2N-2}, \tag{4.17}$$

$$\tilde{\Phi}_{2N+1} = (1+a)\tilde{\Phi}_{2N} + (a^{N}b^{N-1}c^{N}d^{N-1}z - a)\tilde{\Phi}_{2N-1}, \tag{4.18}$$

for any positive integer N.

Proof. Perform the same elementary transformations of rows and columns on $\begin{bmatrix} S & J \\ -J & C \end{bmatrix}$ as we did in the proof of Proposition 3.9, and expand it along the last row/column. The details are left to the reader. $\square$

Remark 4.11. The recurrence equations (4.17) and (4.18) also can be proved combinatorially.

Proof of Proposition 4.10. Consider the generating function of partitions:

$$\sum_{\substack{\lambda \\ \ell(\lambda)\le 2t \\ \lambda_1\le 2j+1-2t}} w(\lambda) = \sum_{\substack{\lambda \\ \ell(\lambda)\le 2t \\ \lambda_1\le 2j-2t}} w(\lambda) + \sum_{\substack{\lambda \\ \ell(\lambda)\le 2t \\ \lambda_1= 2j+1-2t}} w(\lambda). \tag{4.19}$$

Split the partitions λ in the second sum of the right side into two subsets: $\lambda_2 < \lambda_1$, and $\lambda_2 = \lambda_1$. Now

$$\sum_{\substack{\lambda:\lambda_1>\lambda_2 \\ \ell(\lambda)\le 2t \\ \lambda_1= 2j+1-2t}} w(\lambda) = a\left(\sum_{\substack{\lambda \\ \ell(\lambda)\le 2t \\ \lambda_1\le 2j-2t}} w(\lambda) - \sum_{\substack{\lambda \\ \ell(\lambda)\le 2t \\ \lambda_1\le 2j-1-2t}} w(\lambda)\right), \tag{4.20}$$

and

$$\sum_{\substack{\lambda:\lambda_1=\lambda_2 \\ \ell(\lambda)\le 2t \\ \lambda_1= 2j+1-2t}} w(\lambda) = acq^{j-t} \sum_{\substack{\lambda \\ \ell(\lambda)\le 2t-2 \\ \lambda_1\le 2j+1-2t}} w(\lambda). \tag{4.21}$$

Plugging (4.20) and (4.21) into (4.19) and then multiplying by $z^t q^{\binom{t}{2}}$ and summing over t we get (4.18). Similarly we can prove (4.17). $\square$

Proposition 4.12. *Set $U_N = \tilde{\Phi}_{2N}$ and $V_N = \tilde{\Phi}_{2N+1}$, then, for $N \ge 1$,*

$$U_{N+1} = \left\{1 + ab + ac(1+bd)q^{N-1}z\right\} U_N - a(b - zq^{N-1})(1 - czq^{N-1})U_{N-1}, \tag{4.22}$$

$$V_{N+1} = \left\{1 + ab + (1+ac)zq^{N}\right\} V_N - a(b - zq^{N})(1 - czq^{N-1})V_{N-1}, \tag{4.23}$$

where $U_0 = 1$, $V_0 = 1$, $U_1 = 1 + z$, $V_1 = 1 + (1 + a + ac)z$.

Thus U_N and V_N are also expressed by the solutions of the associated Al-Salam–Chihara polynomials.

5. A weighted sum of Schur's P-functions

We use the notation $X = X_n = (x_1, \dots, x_n)$ for the finite set of variables $x_1, \dots, x_n$. The aim of this section is to give some Pfaffian and determinantal formulas for the weighted sum $\sum \omega(\mu) z^{\ell(\mu)} P_\mu(x)$ where $P_\mu(x)$ is Schur's P-function.

Let A_n denote the skew-symmetric matrix

$$\left(\frac{x_i - x_j}{x_i + x_j}\right)_{1\le i,j\le n}$$

and for each strict partition $\mu = (\mu_1, \dots, \mu_l)$ of length $l \le n$, let Γ_μ denote the $n \times l$ matrix $\left(x_j^{\mu_i}\right)$. Let

$$A_\mu(x_1,\dots,x_n) = \begin{pmatrix} A_n & \Gamma_\mu J_l \\ -J_l\, {}^t\Gamma_\mu & O_l \end{pmatrix}$$

which is a skew-symmetric matrix of $(n + l)$ rows and columns. Define $\mathrm{Pf}_\mu(x_1, \dots, x_n)$ to be $\mathrm{Pf}\, A_\mu(x_1, \dots, x_n)$ if $n+l$ is even, and to be $\mathrm{Pf}\, A_\mu(x_1, \dots, x_n, 0)$ if $n+l$ is odd. By [14, Ex.13, p. 267], Schur's P-function $P_\mu(x_1, \dots, x_n)$ is defined to be

$$\frac{\mathrm{Pf}_\mu(x_1,\dots,x_n)}{\mathrm{Pf}_\emptyset(x_1,\dots,x_n)},$$

where it is well known that $\mathrm{Pf}_{\emptyset}(x_1,\dots,x_n)=\prod_{1\le i<j\le n}\frac{x_i-x_j}{x_i+x_j}$. Meanwhile, by [14, (8.7), p. 253], Schur's Q-function $Q_\mu(x_1,\dots,x_n)$ is defined to be $2^{\ell(\lambda)}P_\mu(x_1,\dots,x_n)$. In this section, we consider a weighted sum of Schur's P-functions and Q-functions, i.e.,

$$\xi_N(a,b,c,d;X_n)=\sum_{\substack{\mu\\ \mu_1\le N}}\omega(\mu)P_\mu(x_1,\dots,x_n),$$

$$\eta_N(a,b,c,d;X_n)=\sum_{\substack{\mu\\ \mu_1\le N}}\omega(\mu)Q_\mu(x_1,\dots,x_n),$$

where the sums run over all strict partitions μ such that each part of μ is less than or equal to N. More generally, we can unify these problems to find the following sum:

$$\zeta_N(a,b,c,d;z;X_n)=\sum_{\substack{\mu\\ \mu_1\le N}}\omega(\mu)z^{\ell(\mu)}P_\mu(x_1,\dots,x_n),\tag{5.1}$$

where the sum runs over all strict partitions μ such that each part of μ is less than or equal to N. One of the main results of this section is that $\zeta_N(a,b,c,d;z;X_n)$ can be expressed by a Pfaffian (see Corollary 5.6). Further, let us put

$$\zeta(a,b,c,d;z;X_n)=\lim_{N\to\infty}\zeta_N(a,b,c,d;z;X_n)=\sum_{\mu}\omega(\mu)z^{\ell(\mu)}P_\mu(X_n),\tag{5.2}$$

where the sum runs over all strict partitions μ. We also write

$$\xi(a,b,c,d;X_n)=\zeta(a,b,c,d;1;X_n)=\sum_{\mu}\omega(\mu)P_\mu(X_n),$$

where the sum runs over all strict partitions μ. Then we have the following theorem:

Theorem 5.1. *Let n be a positive integer. Then*

$$\zeta(a,b,c,d;z;X_n)=\begin{cases}\mathrm{Pf}\left(\gamma_{ij}\right)_{1\le i<j\le n}/\,\mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is even},\\ \mathrm{Pf}\left(\gamma_{ij}\right)_{0\le i<j\le n}/\,\mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is odd},\end{cases}\tag{5.3}$$

where

$$\gamma_{ij}=\frac{x_i-x_j}{x_i+x_j}+u_{ij}z+v_{ij}z^2\tag{5.4}$$

with

$$u_{ij}=\frac{a\det\begin{pmatrix}x_i+bx_i^2 & 1-abx_i^2\\ x_j+bx_j^2 & 1-abx_j^2\end{pmatrix}}{(1-abx_i^2)(1-abx_j^2)},\tag{5.5}$$

$$v_{ij}=\frac{abcx_ix_j\det\begin{pmatrix}x_i+ax_i^2 & 1-a(b+d)x_i^2-abdx_i^3\\ x_j+ax_j^2 & 1-a(b+d)x_j^2-abdx_j^3\end{pmatrix}}{(1-abx_i^2)(1-abx_j^2)(1-abcdx_i^2x_j^2)},\tag{5.6}$$

if $1\le i,j\le n$, and

$$\gamma_{0j}=1+\frac{ax_j(1+bx_j)}{1-abx_j^2}z\tag{5.7}$$

if $1\le j\le n$.

Especially, when $z=1$, we have

$$\xi(a,b,c,d;X_n)=\begin{cases}\mathrm{Pf}\left(\widetilde{\gamma}_{ij}\right)_{1\le i<j\le n}/\,\mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is even},\\ \mathrm{Pf}\left(\widetilde{\gamma}_{ij}\right)_{0\le i<j\le n}/\,\mathrm{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is odd},\end{cases}\tag{5.8}$$

where

$$\widetilde{\gamma}_{ij} = \begin{cases} \dfrac{1+ax_j}{1-abx_j^2} & \text{if } i=0, \\ \dfrac{x_i-x_j}{x_i+x_j} + \widetilde{v}_{ij} & \text{if } 1 \le i < j \le n, \end{cases} \quad \text{with} \tag{5.9}$$

$$\widetilde{v}_{ij} = \frac{a \det \begin{pmatrix} x_i + bx_i^2 & 1-b(a+c)x_i^2 - abcx_i^3 \\ x_j + bx_j^2 & 1-b(a+c)x_j^2 - abcx_j^3 \end{pmatrix}}{(1-abx_i^2)(1-abx_j^2)(1-abcdx_i^2x_j^2)}. \tag{5.10}$$

We can generalize this result in the following theorem (Theorem 5.2) using the generalized Vandermonde determinant used in [7]. Let n be a non-negative integer, and let $X = (x_1, \dots, x_{2n})$, $Y = (y_1, \dots, y_{2n})$, $A = (a_1, \dots, a_{2n})$ and $B = (b_1, \dots, b_{2n})$ be $2n$-tuples of variables. Let $V^n(X, Y, A)$ denote the $2n \times n$ matrix whose (i, j)th entry is $a_i x_i^{n-j} y_i^{j-1}$ for $1 \le i \le 2n$, $1 \le j \le n$, and let $U^n(X, Y; A, B)$ denote the $2n \times 2n$ matrix $\begin{pmatrix} V^n(X,Y,A) & V^n(X,Y,B) \end{pmatrix}$. For instance if $n = 2$ then $U^2(X, Y; A, B)$ is

$$\begin{pmatrix} a_1x_1 & a_1y_1 & b_1x_1 & b_1y_1 \\ a_2x_2 & a_2y_2 & b_2x_2 & b_2y_2 \\ a_3x_3 & a_3y_3 & b_3x_3 & b_3y_3 \\ a_4x_4 & a_4y_4 & b_4x_4 & b_4y_4 \end{pmatrix}.$$

Hereafter we use the following notation for n-tuples $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ of variables:

$$X + Y = (x_1 + y_1, \dots, x_n + y_n), \quad X \cdot Y = (x_1y_1, \dots, x_ny_n),$$

and, for integers k and l,

$$X^k = (x_1^k, \dots, x_n^k), \quad X^kY^l = (x_1^ky_1^l, \dots, x_n^ky_n^l).$$

Let $\mathbf{1}$ denote the n-tuple $(1, \dots, 1)$. For any subset $I = \{i_1, \dots, i_r\} \in \binom{[n]}{r}$, let X_I denote the r-tuple $(x_{i_1}, \dots, x_{i_r})$.

Theorem 5.2. *Let $q = abcd$. If n is an even integer, then we have*

$$\xi(a,b,c,d;X_n) = \sum_{r=0}^{n/2} \sum_{I \in \binom{[n]}{2r}} \frac{(-1)^{|I| - \binom{r+1}{2}} a^r q^{\binom{r}{2}}}{\prod_{i \in I}(1-abx_i^2)} \prod_{\substack{i,j \in I \\ i<j}} \frac{x_i + x_j}{(x_i - x_j)(1-qx_i^2x_j^2)} \times \det U^r(X_I^2, \mathbf{1} + qX_I^4, X_I + bX_I^2, \mathbf{1} - b(a+c)X_I^2 - abcX_I^3). \tag{5.11}$$

If n is an odd integer, then we have

$$\xi(a,b,c,d;X_n) = \sum_{m=1}^{n} \frac{1+ax_m}{1-abx_m^2} \sum_{r=0}^{(n-1)/2} \sum_{I \in \binom{[n]\setminus\{m\}}{2r}} \frac{(-1)^{|I| - \binom{r+1}{2}} a^r q^{\binom{r}{2}}}{\prod_{i \in I}(1-abx_i^2)} \prod_{i \in I} \frac{x_m + x_i}{x_m - x_i} \times \prod_{\substack{i,j \in I \\ i<j}} \frac{x_i + x_j}{(x_i - x_j)(1-qx_i^2x_j^2)} \cdot \det U^r(X_I^2, \mathbf{1} + qX_I^4, X_I + bX_I^2, \mathbf{1} - b(a+c)X_I^2 - abcX_I^3). \tag{5.12}$$

Theorem 5.3. *Let $q = abcd$. If n is an even integer, then $\zeta(a,b,c,d;z;X_n)$ is equal to*

$$\sum_{r=0}^{n/2} z^{2r} \sum_{I \in \binom{[n]}{2r}} \frac{(-1)^{|I| - \binom{r+1}{2}} (abc)^r q^{\binom{r}{2}} \prod_{i \in I} x_i}{\prod_{i \in I}(1-abx_i^2)} \prod_{\substack{i,j \in I \\ i<j}} \frac{x_i + x_j}{(x_i - x_j)(1-qx_i^2x_j^2)}$$

$$
\times \det U^r(X_I^2, \mathbf{1} + qX_I^4, X_I + aX_I^2, \mathbf{1} - a(b+d)X_I^2 - abdX_I^3)
+ \sum_{r=0}^{n/2} z^{2r-1} \sum_{I \in \binom{[n]}{2r}} \sum_{\substack{k<l\\ k,l\in I}} \frac{(-1)^{|I|-\binom{r}{2}-1} a^r b^{r-1} c^{r-1} q^{\binom{r-1}{2}} \{1 + b(x_k + x_l) + abx_kx_l\} \prod_{i\in I'} x_i}{\prod_{i\in I}(1 - abx_i^2)}
$$

$$
\times \frac{\prod_{\substack{i,j\in I\\ i<j}} (x_i + x_j) \cdot \det U^{r-1}(X_{I'}^2, \mathbf{1} + qX_{I'}^4, X_{I'} + aX_{I'}^2, \mathbf{1} - a(b+d)X_{I'}^2 - abdX_{I'}^3)}{\prod_{\substack{i,j\in I'\\ i<j}} (x_i - x_j)(1 - qx_i^2x_j^2)}, \tag{5.13}
$$

where $I' = I \setminus \{k, l\}$.

Note that we can obtain a similar formula when n is odd by expanding the Pfaffian in (5.3) along the first row/column.

To obtain the sum of this type we need a generalization of Lemma 3.7, in which the row/column indices always contain say the set $\{1, 2, \dots, n\}$, for some fixed n.

Lemma 5.4. *Let n and N be non-negative integers. Let $A = (a_{ij})$ and $B = (b_{ij})$ be skew-symmetric matrices of size $(n + N)$. We divide the set of row/column indices into two subsets, i.e. the first n indices $I_0 = [n]$ and the last N indices $I_1 = [n + 1, n + N]$. Then*

$$
\sum_{\substack{t\ge0\\ n+t \text{ even}}} z^{(n+t)/2} \sum_{I\in\binom{I_1}{t}} \gamma^{|I_0\uplus I|} \operatorname{Pf}\left(\Delta_{I_0\uplus I}^{I_0\uplus I}(A)\right) \operatorname{Pf}\left(\Delta_{I_0\uplus I}^{I_0\uplus I}(B)\right) = \operatorname{Pf}\begin{pmatrix} J_{n+N}\,{}^tA\,J_{n+N} & K_{n,N} \\ -{}^tK_{n,N} & C \end{pmatrix}, \tag{5.14}
$$

where $C = (C_{ij})_{1\le i,j,\le n+N}$ is given by $C_{ij} = \gamma^{i+j} b_{ij} z$ and $K_{n,N} = J_{n+N}\widetilde{E}_{n,N}$ with

$$
\widetilde{E}_{n,N} = \begin{pmatrix} O_n & O_{n,N} \\ O_{N,n} & E_N \end{pmatrix}.
$$

Proof. In general, if $P = \begin{pmatrix} P_{11} & P_{12} \\ -{}^tP_{12} & P_{22} \end{pmatrix}$ is a $2m \times 2m$ skew-symmetric matrix where P_{11}, P_{12} and P_{22} are $m \times m$ matrices, then $\operatorname{Pf} P$ is the sum (2.9) over all perfect matchings on the vertices $\{1, 2, \dots, m, m + 1, m + 2, \dots, 2m\}$. Meanwhile, one easily sees that $\operatorname{Pf}\begin{pmatrix} J_m P_{11} J_m & J_m P_{12} \\ -{}^tP_{12}J_m & P_{22} \end{pmatrix}$ is equal to a similar sum as in (2.9), but the sum should be taken over all perfect matchings on the vertices $\{m, m - 1, \dots, 1, m + 1, m + 2, \dots, 2m\}$.

Let $V = \{(n + N)^*, \dots, (n + 1)^*, n^*, \dots, 1^*, 1, \dots, n, n + 1, \dots, n + N\}$ be vertices arranged in this order on the x-axis. Put $V_0^* = \{n^*, \dots, 1^*\}$ and $V_1^* = \{(n + N)^*, \dots, (n + 1)^*\}$, $V_0 = \{1, \dots, n\}$ and $V_1 = \{n + 1, \dots, N\}$. A perfect matching $\sigma \in \mathcal{F}(V)$ on the vertices V is uniquely written as $\sigma = \sigma_1 \uplus \sigma_2 \uplus \sigma_3$ where σ_1 (resp. σ_3) is the set of arcs in σ connecting two vertices in $V_1^* \uplus V_0^*$ (resp. $V_0 \uplus V_1$) and σ_2 is the set of arcs in σ connecting a vertex in $V_1^* \uplus V_0^*$ and a vertex in $V_0 \uplus V_1$. Thus the Pfaffian in the right-hand side of (5.14) equals

$$
\sum_\sigma \operatorname{sgn}\sigma \prod_{(j^*,i^*)\in\sigma_1} a_{ij} \prod_{(i^*,j)\in\sigma_2} k_{ij} \prod_{(i,j)\in\sigma_3} C_{ij}
$$

summed over all perfect matchings $\sigma \in \mathcal{F}(V)$ on V. Here k_{ij} is the (i, j)th entry of $K_{n,N} = J_{n+N}\widetilde{E}_{n,N}$. From the definition of $\widetilde{E}_{n,N}$, $\prod_{(i^*,j)\in\sigma_2} k_{ij}$ vanishes unless σ_2 is a collection of arcs (i^*, i) $(i = n + 1, \dots, n + N)$. Thus we can assume that σ_1 is a perfect matching on $I^* \uplus V_0^*$ and σ_3 is a perfect matching on $V_0 \uplus I$ where I is a subset V_1. Here, if $I = \{i_1, \dots, i_t\} \in V_1$, then we write $I^* = \{i_t^*, \dots, i_1^*\}$ according to convention. Thus $n + t$ must be even, and $\prod_{(i,j)\in\sigma_3} C_{ij} = z^{(n+t)/2}\gamma^{|I_0\uplus I|} \prod_{(i,j)\in\sigma_3} b_{ij}$. Note that σ_2 is composed of arcs (i, i). This implies that $\operatorname{sgn}\sigma = \operatorname{sgn}\sigma_1 \operatorname{sgn}\sigma_3$ since the number of crossing between arcs in σ_1 and arcs in σ_2 equals the number of crossing between arcs in σ_1 and arcs in σ_2. Thus the above sum is equal to

$$
\sum_t z^{(t+n)/2} \sum_{I\in\binom{I_1}{t}} \gamma^{n+|I|} \sum_{(\sigma_1,\sigma_3)} \operatorname{sgn}\sigma_1 \operatorname{sgn}\sigma_3 \prod_{(i,j)\in\sigma_1} a_{ij} \prod_{(i,j)\in\sigma_3} b_{ij}.
$$

This is equal to the left-hand side of (5.14). $\square$

For a non-negative integer N, let $\mu^N = (N, \dots, 1, 0)$, and let Γ_{μ^N} denote the $n \times (N+1)$ matrix $\left(x_i^{N-j}\right)_{1\le i\le n, 0\le j\le N}$. Let

$$\mathcal{A}_{n,N} = \begin{pmatrix} A_n & \Gamma_{\mu^N} J_{N+1} \\ -J_{N+1}\,{}^t\Gamma_{\mu^N} & O_{N+1} \end{pmatrix}$$

which is a skew-symmetric matrix of size $n+N+1$. For example, if $n=4$ and $N=3$, then

$$\mathcal{A}_{4,3} = \begin{pmatrix} 0 & \frac{x_1-x_2}{x_1+x_2} & \frac{x_1-x_3}{x_1+x_3} & \frac{x_1-x_4}{x_1+x_4} & 1 & x_1 & {x_1}^2 & {x_1}^3 \\ \frac{x_2-x_1}{x_1+x_2} & 0 & \frac{x_2-x_3}{x_2+x_3} & \frac{x_2-x_4}{x_2+x_4} & 1 & x_2 & {x_2}^2 & {x_2}^3 \\ \frac{x_3-x_1}{x_1+x_3} & \frac{x_3-x_2}{x_2+x_3} & 0 & \frac{x_3-x_4}{x_3+x_4} & 1 & x_3 & {x_3}^2 & {x_3}^3 \\ \frac{x_4-x_1}{x_1+x_4} & \frac{x_4-x_2}{x_2+x_4} & \frac{x_4-x_3}{x_3+x_4} & 0 & 1 & x_4 & {x_4}^2 & {x_4}^3 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -x_1 & -x_2 & -x_3 & -x_4 & 0 & 0 & 0 & 0 \\ -{x_1}^2 & -{x_2}^2 & -{x_3}^2 & -{x_4}^2 & 0 & 0 & 0 & 0 \\ -{x_1}^3 & -{x_2}^3 & -{x_3}^3 & -{x_4}^3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let β_{ij} be as in (2.14). Let B_N denote the $(N+1)\times(N+1)$ matrix $(\beta_{ij})_{0\le i,j\le N}$ and let B'_N denote the $(N+2)\times(N+2)$ matrix $(\beta_{ij})_{-1\le i,j\le N}$.

Theorem 5.5. *Let n and N be integers such that $n \ge N \ge 0$. Then*

$$\zeta_N(a,b,c,d;z;X_n) = \operatorname{Pf}\left(\mathcal{C}_{n,N}\right) / \operatorname{Pf}_\emptyset(X_n), \tag{5.15}$$

where

$$\mathcal{C}_{n,N} = \begin{pmatrix} O_{N+1} & {}^t\Gamma_{\mu^N} J_n & J_{N+1} \\ -J_n\Gamma_{\mu^N} & J_n\,{}^tA_n\,J_n & O_{n,N+1} \\ -J_{N+1} & O_{N+1,n} & B_N \end{pmatrix}, \tag{5.16}$$

if n is even, and

$$\mathcal{C}_{n,N} = \begin{pmatrix} O_{N+1} & {}^t\Gamma_{\mu^N} J_n & J'_{N+1} \\ -J_n\Gamma_{\mu^N} & J_n\,{}^tA_n\,J_n & O_{n,N+2} \\ -{}^tJ'_{N+1} & O_{N+2,n} & B'_N \end{pmatrix} \tag{5.17}$$

where $J'_{N+1} = \begin{pmatrix} O_{N+1,1} & J_{N+1} \end{pmatrix}$ if n is odd.

Proof. Let $\mathcal{B}_{n,N}$ be the skew-symmetric matrix of size $(n+N+1)$ defined by

$$\mathcal{B}_{n,N} = \begin{pmatrix} S_n & O_{n,N+1} \\ O_{N+1.n} & B_N \end{pmatrix}$$

if n is even, and

$$\mathcal{B}_{n,N} = \begin{pmatrix} S_{n-1} & O_{n,N+2} \\ O_{N+2.n} & B'_N \end{pmatrix}$$

if n is odd. Fix a strict partition $\mu = (\mu_1, \dots, \mu_l)$ such that $\mu_1 > \cdots > \mu_l \ge 0$, and let $K_n(\mu) = \{n+\mu_l, \dots, n+\mu_1\}$. From the definition of $\mathcal{B}_{n,N}$ and Theorem 2.2, we have

$$\operatorname{Pf}\left(\Delta^{[n]\uplus K_n(\mu)}_{[n]\uplus K_n(\mu)}\left(\mathcal{B}_{n,N}\right)\right) = \omega(\mu)\, z^{\ell(\mu)}$$

if $n+l$ is even. Thus Lemma 5.4 immediately implies that $\mathrm{Pf}_{\emptyset}(X_n)\zeta_N(a,b,c,d;z;X_n)$ is equal to

$$\mathrm{Pf}\begin{pmatrix} J_{n+N+1}\,{}^t\mathcal{A}_{n,N}\,J_{n+N+1} & K_{n,N+1} \\ -\,{}^tK_{n,N+1} & \mathcal{B}_{n,N}\end{pmatrix}. \tag{5.18}$$

By simple elementary transformations on rows and columns, we obtain the desired results (5.16) and (5.17). □

Corollary 5.6. *Let n and N be integers such that $n \geq N \geq 0$. Then*

$$\zeta_N(a,b,c,d;z;X_n) = \mathrm{Pf}\left(\mathcal{D}_{n,N}\right)/\,\mathrm{Pf}_{\emptyset}(X_n), \tag{5.19}$$

where

$$\mathcal{D}_{n,N} = \left(\frac{x_i-x_j}{x_i+x_j} + \sum_{0\leq k,l\leq N}\beta_{kl}x_i^l x_j^k\right)_{1\leq i,j\leq n}, \tag{5.20}$$

if n is even, and

$$\mathcal{D}_{n,N} = \left(\begin{array}{c|c} 0 & \displaystyle\sum_{k=0}^{N}\beta_{-1,k}x_j^k \\ \hline \displaystyle\sum_{k=0}^{N}\beta_{k,-1}x_i^k & \displaystyle\frac{x_i-x_j}{x_i+x_j} + \sum_{0\leq k,l\leq N}\beta_{kl}x_i^l x_j^k \end{array}\right)_{0\leq i,j\leq n}, \tag{5.21}$$

if n is odd.

For instance, if $n=4$ and $N=2$, then $\mathcal{D}_{4,2}$ looks as follows:

$$\begin{pmatrix}
0 & 0 & 0 & x_4^2 & x_3^2 & x_2^2 & x_1^2 & 0 & 0 & 1 \\
0 & 0 & 0 & x_4 & x_3 & x_2 & x_1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
-x_4^2 & -x_4 & -1 & 0 & \frac{x_3-x_4}{x_3+x_4} & \frac{x_2-x_4}{x_2+x_4} & \frac{x_1-x_4}{x_1+x_4} & 0 & 0 & 0 \\
-x_3^2 & -x_3 & -1 & \frac{x_4-x_3}{x_4+x_3} & 0 & \frac{x_2-x_3}{x_2+x_3} & \frac{x_1-x_3}{x_1+x_3} & 0 & 0 & 0 \\
-x_2^2 & -x_2 & -1 & \frac{x_4-x_2}{x_4+x_2} & \frac{x_3-x_2}{x_3+x_2} & 0 & \frac{x_1-x_2}{x_1+x_2} & 0 & 0 & 0 \\
-x_1^2 & -x_1 & -1 & \frac{x_4-x_1}{x_4+x_1} & \frac{x_3-x_1}{x_3+x_1} & \frac{x_2-x_1}{x_2+x_1} & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & az & abz \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -az & 0 & abcz^2 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -abz & -abcz^2 & 0
\end{pmatrix}.$$

Proof of Corollary 5.6. When n is even, annihilate the entries in ${}^t\Gamma_{\mu^N}\,J_n$ of (5.16) by elementary transformation of columns, and annihilate the entries in $-J_n\Gamma_{\mu^N}$ of (5.16) by elementary transformation of columns. Then expand the Pfaffian $\mathrm{Pf}\left(\mathcal{C}_{n,N}\right)$ along the first $N+1$ rows. For the case when n is odd, perform the same operation on (5.17). □

Proof of Theorem 5.1. Perform the summations

$$\sum_{0\leq k<l}\beta_{kl}\det\begin{pmatrix} x_i^l & x_i^k \\ x_j^l & x_j^k\end{pmatrix}$$

and

$$\sum_{k=0}^{\infty}\beta_{-1,k}x_j^k,$$

and apply Corollary 5.6. The details are left to the reader (cf. Proof of Theorem 2.1 in [6]). □

To prove Theorems 5.2 and 5.3, we need to cite a lemma from [6]. (See Corollary 3.3 of [6] and Theorem 3.2 of [7].)

Lemma 5.7. *Let n be a non-negative integer. Let $X = (x_1, \ldots, x_{2n})$, $A = (a_1, \ldots, a_{2n})$, $B = (b_1, \ldots, b_{2n})$, $C = (c_1, \ldots, c_{2n})$ and $D = (d_1, \ldots, d_{2n})$ be $2n$-tuples of variables. Then*

$$\operatorname{Pf}\left[\frac{(a_i b_j - a_j b_i)(c_i d_j - c_j d_i)}{(x_i - x_j)(1 - t x_i x_j)}\right]_{1\le i<j\le 2n} = \frac{V^n(X, \mathbf{1} + tX^2; A, B) V^n(X, \mathbf{1} + tX^2; C, D)}{\prod_{1\le i<j\le 2n} (x_i - x_j)(1 - t x_i x_j)}, \tag{5.22}$$

where $\mathbf{1} + tX^2 = (1 + t x_1^2, \ldots, 1 + t x_n^2)$.

In particular, we have

$$\operatorname{Pf}\left[\frac{a_i b_j - a_j b_i}{1 - t x_i x_j}\right]_{1\le i<j\le 2n} = (-1)^{\binom{n}{2}} t^{\binom{n}{2}} \frac{V^n(X, \mathbf{1} + tX^2; A, B)}{\prod_{1\le i<j\le 2n} (1 - t x_i x_j)}. \quad \square \tag{5.23}$$

Proof of Theorem 5.2. First, assume that n is even. Using the formula

$$\operatorname{Pf}(A + B) = \sum_{r=0}^{\lfloor n/2\rfloor} \sum_{I\in\binom{[n]}{2r}} (-1)^{|I|-r} \operatorname{Pf}(A_I^I) \operatorname{Pf}(B_{\overline{I}}^{\overline{I}}), \tag{5.24}$$

where $\overline{I}$ denotes the complementary set of I, we see that $\xi(a, b, c, d; X_n)$ is equal to

$$\sum_{r=0}^{\lfloor n/2\rfloor} \sum_{I\in\binom{[n]}{2r}} (-1)^{|I|-r} \prod_{\substack{i,j\in I\\ i<j}} \frac{x_i + x_j}{x_i - x_j} \operatorname{Pf}(\widetilde{v}_{ij})_{i,j\in I}.$$

Apply Lemma 5.7 to obtain (5.11). When n is odd, first expand the Pfaffian along the first row/column and repeat the same argument. $\square$

Proof of Theorem 5.3. Note that the rank of the matrix $(u_{ij})_{1\le i,j\le n}$ is at most two. Thus we have

$$\operatorname{Pf}(u_{ij})_{1\le i,j\le n} = \begin{cases} \dfrac{a(x_1 - x_2)\{1 + b(x_1 + x_2) + abx_1x_2\}}{(1 - abx_1^2)(1 - abx_2^2)} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Using (5.24), we obtain

$$\begin{aligned}\operatorname{Pf}\left(\gamma_{ij}\right)_{1\le i,j\le n} &= \operatorname{Pf}\left(\frac{x_i - x_j}{x_i + x_j} + v_{ij} z^2\right)_{1\le i,j\le n} \\ &\quad + \sum_{1\le k<l\le n} (-1)^{k+l-1} \frac{az(x_k - x_l)\{1 + b(x_k + x_l) + abx_kx_l\}}{(1 - abx_k^2)(1 - abx_l^2)} \operatorname{Pf}\left(\frac{x_i - x_j}{x_i + x_j} + v_{ij} z^2\right)_{\substack{1\le i,j\le n\\ i,j\ne k,l}}.\end{aligned}$$

Use (5.24) again to see that $\zeta(a, b, c, d; z; X_n)$ is equal to

$$\begin{aligned}&\sum_{r=0}^{\lfloor n/2\rfloor} z^{2r} \sum_{I\in\binom{[n]}{2r}} (-1)^{|I|-r} \prod_{\substack{i,j\in I\\ i<j}} \frac{x_i + x_j}{x_i - x_j} \cdot \operatorname{Pf}(v_{ij})_{i,j\in I} \\ &\quad + \sum_{1\le k<l\le n} (-1)^{k+l-1} \frac{az(x_k - x_l)\{1 + b(x_k + x_l) + abx_kx_l\}}{(1 - abx_k^2)(1 - abx_l^2)} \\ &\quad \times \sum_{r=1}^{\lfloor n/2\rfloor} z^{2r-2} \sum_{I'\in\binom{[n]-\{k,l\}}{2r-2}} (-1)^{|I'|-r+1} \prod_{\substack{i,j\in I'\\ i<j}} \frac{x_i + x_j}{x_i - x_j} \cdot \operatorname{Pf}(v_{ij})_{i,j\in I'}.\end{aligned}$$

Put $I = I' \cup \{k, l\}$ and apply Lemma 5.7 to obtain (5.13). $\square$

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