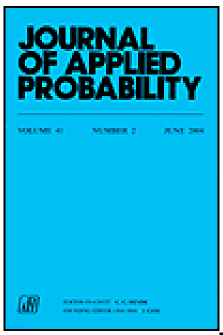

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ON A PROBABILISTIC ANALOGUE OF THE FIBONACCI SEQUENCE

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Abstract

One of the earliest population models to be studied gives rise to the Fibonacci sequence and has a history dating back more than 750 years. A stochastic version of the model is discussed in this paper, its basic defining property being $E(X_n | X_{n-1}, \cdots, X_0) = X_{n-1} + X_{n-2}$ a.s. The process $\{X_n\}$ mimics many of the standard properties of the Fibonacci sequence. In particular, under mild additional conditions, $X_{n+1}X_n^{-1} \to \alpha$ a.s. as $n \to \infty$ where α is the 'golden ratio' $\frac{1}{2}(1 + \sqrt{5})$.

POPULATION PROCESSES; GROWTH RATES; FIBONACCI NUMBERS; GOLDEN RATIO; GENERALIZED MARTINGALES

One of the oldest population growth models to be studied gives rise to the Fibonacci sequence. This is the following rabbit-breeding problem proposed by Leonardo of Pisa in his book *Liber Abacci* which was written in 1202 and has survived in its second edition of 1228. A closed population begins with one newborn pair of rabbits. Every month a pair of rabbits produces another pair and the rabbits begin to bear young 2 months after their birth. It is easily seen that the number of pairs u_n at time n is given by the recurrence relation

$$u_n = u_{n-1} + u_{n-2}$$

and the initial conditions $u_1 = u_2 = 1$. The sequence $\{u_n\}$ is known as the Fibonacci sequence (and the u_n as the Fibonacci numbers). It has been subjected to intensive scrutiny, partly through the Fibonacci Association (founded by mathematicians interested in the sequence) and in its journal, the *Fibonacci Quarterly*. The literature associated with the sequence is enormous.

Amongst the many interesting properties of the Fibonacci sequence is the result that

$$\lim_{n \to \infty} u_{n+1} u_n^{-1} = \alpha = \tfrac{1}{2}(1 + \sqrt{5}).$$

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One of the earliest population models to be studied gives rise to the Fibonacci sequence and has a history dating back more than 750 years. A stochastic version of the model is discussed in this paper, its basic defining property being E(Xn | Xn-1, ···, X0) = Xn-1+Xn-2 a.s. The process {Xn} mimics many of the standard properties of the Fibonacci sequence. In particular, under mild additional conditions, Xn+1Xn -1 → α a.s. as n→∞ where α is the 'golden ratio' 1/2(1+√5).