Research Article

Sequences of Non-Gegenbauer-Humbert Polynomials Meet the Generalized Gegenbauer-Humbert Polynomials

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Here, we present a connection between a sequence of polynomials generated by a linear recurrence relation of order 2 and sequences of the generalized Gegenbauer-Humbert polynomials. Many new and known transfer formulas between non-Gegenbauer-Humbert polynomials and generalized Gegenbauer-Humbert polynomials are given. The applications of the relationship to the construction of identities of polynomial sequences defined by linear recurrence relations are also discussed.

1. Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. A polynomial sequence $\{a_n(x)\}$ is called a sequence of order 2 if it satisfies the linear recurrence relation of order 2

$$a_n(x) = p(x)a_{n-1} + q(x)a_{n-2}(x), \quad n \geq 2, \tag{1.1}$$

for some coefficient $p(x) \not\equiv 0$ and $q(x) \not\equiv 0$ and initial conditions $a_0(x)$ and $a_1(x)$. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (see Comtet [1], Hsu [2], Strang [3], Wilf [4], etc.). In [5], the authors presented a new method to construct an explicit formula of $\{a_n(x)\}$ generated by (1.1). For the sake of the reader's convenience, we cite this result as follows.

Proposition 1.1. *Let $\{a_n(x)\}$ be a sequence of order 2 satisfying the linear recurrence relation (1.1), then*

$$a_n(x) = \begin{cases} \left(\dfrac{a_1(x) - \beta(x)a_0(x)}{\alpha(x) - \beta(x)}\right)\alpha^n(x) - \left(\dfrac{a_1(x) - \alpha(x)a_0(x)}{\alpha(x) - \beta(x)}\right)\beta^n(x) & \text{if } \alpha(x) \neq \beta(x), \\ na_1(x)\alpha^{n-1}(x) - (n-1)a_0(x)\alpha^n(x) & \text{if } \alpha(x) = \beta(x), \end{cases}$$

(1.2)

where $\alpha(x)$ and $\beta(x)$ are roots of $t^2 - p(x)t - q(x) = 0$, namely,

$$\alpha(x) = \frac{1}{2}\left(p(x) + \sqrt{p^2(x) + 4q(x)}\right), \qquad \beta(x) = \frac{1}{2}\left(p(x) - \sqrt{p^2(x) + 4q(x)}\right).$$

(1.3)

In [6], Aharonov et al. have proved that the solution of any sequence of numbers that satisfies a recurrence relation of order 2 with constant coefficients and initial conditions $a_0 = 0$ and $a_1 = 1$, called the primary solution, can be expressed in terms of Chebyshev polynomial values. For instance, the authors show $F_n = i^{-n}U_n(i/2)$ and $L_n = 2i^{-n}T_n(i/2)$, where F_n and L_n are, respectively, Fibonacci numbers and Lucas numbers, and $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of the first kind and the second kind, respectively. Some identities drawn from those relations were given by Beardon in [7]. Marr and Vineyard in [8] use the relationship to establish explicit expression of five-diagonal Toeplitz determinants. In [5], the authors presented a new method to construct an explicit formula of $\{a_n(x)\}$ generated by (1.1). Inspired with those results, in [9], The authors and Weng established a relationship between the number sequences defined by recurrence relation (1.1) and the generalized Gegenbauer-Humbert polynomial value sequences. The results are suitable for all such number sequences defined by (1.1) with arbitrary initial conditions a_0 and a_1, which includes the results in [6, 7] as the special cases. Many new and known formulas of Fibonacci, Lucas, Pell, and Jacobsthal numbers in terms of the generalized Gegenbauer-Humbert polynomial values were presented in [9]. In this paper, we will give an alternative form of (1.2) and find a relationship between all polynomial sequences defined by (1.1) and the generalized Gegenbauer-Humbert polynomial sequences.

A sequence of the generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$ is defined by the expansion (see, e.g., [1], Gould [10], and the authors with Hsu [11])

$$\Phi(t) \equiv \left(C - 2xt + yt^2\right)^{-\lambda} = \sum_{n\geq 0} P_n^{\lambda,y,C}(x)t^n,$$

(1.4)

where $\lambda > 0$, y and $C \neq 0$ are real numbers. As special cases of (1.4), we consider $P_n^{\lambda,y,C}(x)$ as follows (see [11]):

$P_n^{1,1,1}(x) = U_n(x)$, Chebyshev polynomial of the second kind,

$P_n^{1/2,1,1}(x) = \psi_n(x)$, Legendre polynomial,

$P_n^{1,-1,1}(x) = P_{n+1}(x)$, Pell polynomial,

$P_n^{1,-1,1}(x/2) = F_{n+1}(x)$, Fibonacci polynomial,

$P_n^{1,1,1}((x/2) + 1) = B_n(x)$, Morgan-Voyc polynomial, [12] by Koshy,

$P_n^{1,2,1}(x/2) = \Phi_{n+1}(x)$, Fermat polynomial of the first kind,

$P_n^{1,2a,2}(x) = D_n(x,a)$, Dickson polynomial of the second kind,

$a \neq 0$ (see, e.g., [13]) by Lidl et al.,

where a is a real parameter, and $F_n = F_n(1)$ is the Fibonacci number. In particular, if $y = C = 1$, the corresponding polynomials are called Gegenbauer polynomials (see [1]). More results on the Gegenbauer-type polynomials can be found in Hsu [14] and Hsu and Shiue [15], and so forth, it is interesting that for each generalized Gegenbauer-Humbert polynomial sequence there exists a nongeneralized Gegenbauer-Humbert polynomial sequence, for instance, corresponding to the Chebyshev polynomials of the second kind, Pell polynomials, Fibonacci polynomials, Fermat polynomials of the first kind, and the Dickson polynomials of the second kind, we have the Chebyshev polynomials of the first kind, Pell-Lucas polynomials (see [16] by Horadam and Mahon), Lucas polynomials, the Fermat polynomials of the second kind (see [17] by Horadam), and the Dickson polynomials of the first kind, respectively.

Similarly, for a class of the generalized Gegenbauer-Humbert polynomial sequences defined by

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x), \qquad (1.5)$$

for all $n \geq 2$ with initial conditions

$$\begin{aligned} P_0^{\lambda,y,C}(x) &= \Phi(0) = C^{-\lambda}, \\ P_1^{\lambda,y,C}(x) &= \Phi'(0) = 2\lambda x C^{-\lambda-1}, \end{aligned} \qquad (1.6)$$

the following theorem is obtained.

Theorem 1.2 (see [5]). *Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion* (1.4) *can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\frac{\left(x+\sqrt{x^2-Cy}\right)^{n+1} - \left(x-\sqrt{x^2-Cy}\right)^{n+1}}{2\sqrt{x^2-Cy}}. \qquad (1.7)$$

In next section, we will use an alternative form of (1.2) to establish a relationship between the polynomial sequences defined by recurrence relation (1.1) and the generalized Gegenbauer-Humbert polynomial sequences defined by (1.5). Many new and known formulas of polynomials in terms of the generalized Gegenbauer-Humbert polynomials and applications of the established relationship to the construction of identities of polynomial sequences will be presented in Section 3.

2. Main Results

We now modify the explicit formula of the polynomial sequences defined by linear recurrence relation (1.2) of order 2. If $\alpha(x) \neq \beta(x)$, the first formula in (1.2) can be written as

$$a_n(x) = \frac{a_1(x)\left((\alpha(x))^n - (\beta(x))^n\right) - a_0(x)\alpha(x)\beta(x)\left((\alpha(x))^{n-1} - (\beta(x))^{n-1}\right)}{\alpha(x) - \beta(x)}. \tag{2.1}$$

Noting that $-\alpha(x)\beta(x) = \alpha(x)(\alpha(x) - p(x)) = \beta(x)(\beta(x) - p(x))$, we may further write the above expression of $a_n(x)$ as

$$
\begin{aligned}
a_n(x) &= \frac{1}{\alpha(x) - \beta(x)}\Big[a_1(x)\left((\alpha(x))^n - (\beta(x))^n\right) + a_0(x)\alpha(x)(\alpha(x) - p(x)) \\
&\qquad \times (\alpha(x))^{n-1} - a_0(x)\beta(x)(\beta(x) - p(x))(\beta(x))^{n-1}\Big] \\
&= \frac{a_0(x)\left((\alpha(x))^{n+1} - (\beta(x))^{n+1}\right) + \left(a_1(x) - a_0(x)p(x)\right)\left((\alpha(x))^n - (\beta(x))^n\right)}{\alpha(x) - \beta(x)}.
\end{aligned}
\tag{2.2}
$$

Denote $r(x) = x + \sqrt{x^2 - Cy}$ and $s(x) = x - \sqrt{x^2 - Cy}$. To find a transfer formula between expressions (1.7) and (2.2), we set

$$\alpha(x) := \frac{r(x)}{k(x)}, \qquad \beta(x) := \frac{s(x)}{k(x)}, \tag{2.3}$$

for a nonzero real or complex-valued function $k(x)$, which are two roots of $t^2 - p(x)t - q(x) = 0$. Thus, adding and multiplying two equations of (2.3) side by side, we obtain

$$
\begin{aligned}
\alpha(x) + \beta(x) &= p(x) = \frac{2x}{k(x)}, \\
\alpha(x)\beta(x) &= -q(x) = \frac{Cy}{(k(x))^2}.
\end{aligned}
\tag{2.4}
$$

The above system implies

$$k(x) = \pm\sqrt{\frac{Cy}{-q(x)}}, \tag{2.5}$$

and at

$$x = \frac{p(x)k(x)}{2} = \pm\frac{p(x)}{2}\sqrt{\frac{Cy}{-q(x)}}, \tag{2.6}$$

$r(x)$ and $s(x)$ give expressions of $\alpha(x)$ and $\beta(x)$ as

$$\alpha(x) = \frac{r\left(\pm(p(x)/2)\sqrt{Cy/-q(x)}\right)}{\pm\sqrt{Cy/-q(x)}}, \qquad \beta(x) = \frac{s\left(\pm(p(x)/2)\sqrt{Cy/-q(x)}\right)}{\pm\sqrt{Cy/-q(x)}}. \qquad (2.7)$$

It is clear that $\alpha(x)$ and $\beta(x)$ satisfy $\alpha(x) + \beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$.

We first consider the case of $k(x) = \sqrt{-Cy/q(x)}$. Substituting the corresponding (2.7) with positive sign into (2.2), we have

$$a_n(x) = \frac{a_0(x)\left(r^{n+1}(x) - s^{n+1}(x)\right) + k(x)\left(a_1(x) - a_0(x)p(x)\right)\left(r^n(x) - s^n(x)\right)}{k^n(x)(r(x) - s(x))}$$

$$= a_0(x)C^{n+2}\left(\sqrt{\frac{-q(x)}{Cy}}\right)^n P_n^{1,y,C}\left(\frac{k(x)p(x)}{2}\right)$$

$$+ \left(a_1(x) - a_0(x)p(x)\right)C^{n+1}\left(\sqrt{\frac{-q(x)}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{k(x)p(x)}{2}\right) \qquad (2.8)$$

$$= a_0(x)C^{n+2}\left(\sqrt{\frac{-q(x)}{Cy}}\right)^n P_n^{1,y,C}\left(\frac{p(x)}{2}\sqrt{\frac{Cy}{-q(x)}}\right)$$

$$+ \left(a_1(x) - a_0(x)p(x)\right)C^{n+1}\left(\sqrt{\frac{-q(X)}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\frac{p(x)}{2}\sqrt{\frac{Cy}{-q(x)}}\right).$$

Similarly, for $k(x) = -\sqrt{-Cy/q(x)}$, we have

$$a_n(x) = a_0(x)C^{n+2}\left(-\sqrt{\frac{-q(x)}{Cy}}\right)^n P_n^{1,y,C}\left(-\frac{p(x)}{2}\sqrt{\frac{Cy}{-q(x)}}\right)$$

$$+ \left(a_1(x) - a_0(x)p(x)\right)C^{n+1}\left(-\sqrt{\frac{-q(x)}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(-\frac{p(x)}{2}\sqrt{\frac{Cy}{-q(x)}}\right). \qquad (2.9)$$

Therefore, we obtain our main result.

Theorem 2.1. *Let sequence $\{a_n(x)\}_{n\geq 0}$ be defined by $a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x)$ ($n \geq 2$) with initial conditions $a_0(x)$ and $a_1(x)$, then $a_n(x)$ can be presented as*

$$a_n(x) = a_0(x)C^{n+2}\left(\pm\sqrt{\frac{-q(x)}{Cy}}\right)^n P_n^{1,y,C}\left(\pm\frac{p(x)}{2}\sqrt{\frac{Cy}{-q(x)}}\right)$$
$$+ \left(a_1(x) - a_0(x)p(x)\right)C^{n+1}\left(\pm\sqrt{\frac{-q(X)}{Cy}}\right)^{n-1} P_{n-1}^{1,y,C}\left(\pm\frac{p(x)}{2}\sqrt{\frac{Cy}{-q(x)}}\right),$$

(2.10)

where $\{P_n^{1,y,c}\}$ is the sequence of any generalized Gegenbauer-Humbert polynomials with $\lambda = 1$. In particular, $a_n(x)$ can be expressed in terms of $\{P_n^{1,1,1} = U_n\}$, the sequence of the Chebyshev polynomials of the second kind,

$$a_n(x) = a_0(x)\left(\pm\sqrt{-q(x)}\right)^n U_n\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right)$$
$$+ \left(a_1(x) - a_0(x)p(x)\right)\left(\pm\sqrt{-q(x)}\right)^{n-1} U_{n-1}\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right),$$

(2.11)

which is a special case of (2.10) *for $(y, C) = (1, 1)$.*

Corollary 2.2. *For $(y, C) = (-1, 1), (1, 1), (2, 1), and\ (2a, 2)$ $(a \neq 0)$, respectively, from* (2.10)*, one has transfer formulas*

$$a_n(x) = a_0(x)\left(\pm\sqrt{q(x)}\right)^n P_{n+1}\left(\pm\frac{p(x)}{2\sqrt{q(x)}}\right)$$
$$+ \left(a_1(x) - a_0(x)p(x)\right)\left(\pm\sqrt{q(x)}\right)^{n-1} P_n\left(\pm\frac{p(x)}{2\sqrt{q(x)}}\right),$$

$$a_n(x) = a_0(x)\left(\pm\sqrt{q(x)}\right)^n F_{n+1}\left(\pm\frac{p(x)}{\sqrt{q(x)}}\right)$$
$$+ \left(a_1(x) - a_0(x)p(x)\right)\left(\pm\sqrt{q(x)}\right)^{n-1} F_n\left(\pm\frac{p(x)}{\sqrt{q(x)}}\right),$$

$$a_n(x) = a_0(x)\left(\pm\sqrt{-q(x)}\right)^n B_n\left(\pm\frac{p(x)}{\sqrt{-q(x)}} - 2\right)$$
$$+ \left(a_1(x) - a_0(x)p(x)\right)\left(\pm\sqrt{-q(x)}\right)^{n-1} B_{n-1}\left(\pm\frac{p(x)}{\sqrt{-q(x)}} - 2\right),$$

$$a_n(x) = a_0(x)\left(\pm\sqrt{\frac{-q(x)}{2}}\right)^n \Phi_{n+1}\left(\pm p(x)\sqrt{\frac{2}{-q(x)}}\right)$$

$$+ \left(a_1(x) - a_0(x)p(x)\right)\left(\pm\sqrt{\frac{-q(x)}{2}}\right)^{n-1}\Phi_n\left(\pm p(x)\sqrt{\frac{2}{-q(x)}}\right),$$

$$a_n(x) = 4a_0(x)\left(\pm\sqrt{\frac{-q(x)}{a}}\right)^n D_n\left(\pm p(x)\sqrt{\frac{a}{-q(x)}}, a\right)$$

$$+ 4\left(a_1(x) - a_0(x)p(x)\right)\left(\pm\sqrt{\frac{-q(x)}{a}}\right)^{n-1}D_{n-1}\left(\pm p(x)\sqrt{\frac{a}{-q(x)}}, a\right),$$

$$(2.12)$$

where $U_n(x)$, $P_n(x)$, $F_n(x)$, $\Phi_n(x)$, and $D_n(x, a)$ are the Chebyshev polynomials of the second order, Pell polynomials, Fibonacci polynomials, Fermat polynomials, and the Dickson polynomials of the second kind, respectively.

Example 2.3. As the first example, we consider the Chebyshev polynomials of the first kind $T_n(x) = \cos(n \arccos x)$ satisfying recurrence relation (1.1) with $p(x) = 2x$ and $q = -1$ and initial conditions $T_0(x) = 1$ and $T_1(x) = x$. From Corollary 2.2, we have

$$T_n(x) = U_n(x) - xU_{n-1}(x),$$

$$T_n(x) = (-1)^n(U_n(-x) + xU_{n-1}(x)),$$

$$T_n(x) = (\pm i)^n P_{n+1}(\mp xi) - x(\pm i)^{n-1}P_n(\mp xi),$$

$$T_n(x) = (\pm i)^n F_{n+1}(\mp 2xi) - x(\pm i)^{n-1}F_n(\mp 2xi),$$

$$T_n(x) = (\pm 1)^n B_n(\pm 2x - 2) - (\pm 1)^{n-1}xB_{n-1}(\pm 2x - 2),$$

$$T_n(x) = B_n(\pm 2x - 2) - xB_{n-1}(\pm 2x - 2),$$

$$T_n(x) = \left(\pm\frac{1}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\mp 2\sqrt{2}x\right) - x\left(\pm\frac{1}{\sqrt{2}}\right)^{n-1}\Phi_n\left(\mp 2\sqrt{2}x\right),$$

$$T_n(x) = \left(\pm\frac{1}{\sqrt{4a}}\right)^n D_n\left(\mp 2\sqrt{a}x, a\right) - x\left(\pm\frac{1}{\sqrt{4a}}\right)^{n-1}D_{n-1}\left(\mp 2\sqrt{a}x, a\right),$$

$$(2.13)$$

in which the first relation is equivalent to the well-known result $2T_n(x) = U_n(x) - U_{n-2}(x)$ due to

$$2T_n(x) = 2U_n(x) - 2xU_{n-1}(x) = U_n(x) + (2xU_{n-1}(x) - U_{n-2}(x)) - 2xU_{n-1}(x). \qquad (2.14)$$

For the special cases of $a_0(x)$ and $a_1(x)$, we have the following corollaries.

Corollary 2.4. *Let sequence* $\{a_n(x)\}_{n \geq 0}$ *be defined by* $a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x)$ $(n \geq 2)$ *with initial conditions* $a_0(x) = 0$ *and* $a_1(x) = d$. *Then*

$$a_n(x) = d\left(\pm\sqrt{-q(x)}\right)^{n-1} U_{n-1}\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right),$$

$$a_n(x) = d\left(\pm\sqrt{q(x)}\right)^{n-1} P_n\left(\pm\frac{p(x)}{2\sqrt{q(x)}}\right),$$

$$a_n(x) = d\left(\pm\sqrt{q(x)}\right)^{n-1} F_n\left(\pm\frac{p(x)}{\sqrt{q(x)}}\right),$$

$$a_n(x) = d\left(\pm\sqrt{-q(x)}\right)^{n-1} B_{n-1}\left(\pm\frac{p(x)}{\sqrt{-q(x)}} - 2\right), \tag{2.15}$$

$$a_n(x) = d\left(\pm\sqrt{\frac{-q(x)}{2}}\right)^{n-1} \Phi_n\left(\pm p(x)\sqrt{\frac{2}{-q(x)}}\right),$$

$$a_n(x) = 4d\left(\pm\sqrt{\frac{-q(x)}{a}}\right)^{n-1} D_{n-1}\left(\pm p(x)\sqrt{\frac{a}{-q(x)}}, a\right).$$

Corollary 2.5. *Let sequence* $\{a_n(x)\}_{n \geq 0}$ *be defined by* $a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x)$ $(n \geq 2)$ *with initial conditions* $a_0(x) = c$ *and* $a_1(x) = cp(x)$, *then*

$$a_n(x) = c\left(\pm\sqrt{-q(x)}\right)^n U_n\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right),$$

$$a_n(x) = c\left(\pm\sqrt{q(x)}\right)^n P_{n+1}\left(\pm\frac{p(x)}{2\sqrt{q(x)}}\right),$$

$$a_n(x) = c\left(\pm\sqrt{q(x)}\right)^n F_{n+1}\left(\pm\frac{p(x)}{\sqrt{q(x)}}\right),$$

$$a_n(x) = c\left(\pm\sqrt{-q(x)}\right)^n B_n\left(\pm\frac{p(x)}{\sqrt{-q(x)}} - 2\right), \tag{2.16}$$

$$a_n(x) = c\left(\pm\sqrt{\frac{-q(x)}{2}}\right)^n \Phi_{n+1}\left(\pm p(x)\sqrt{\frac{2}{-q(x)}}\right),$$

$$a_n(x) = 4c\left(\pm\sqrt{\frac{-q(x)}{a}}\right)^n D_n\left(\pm p(x)\sqrt{\frac{a}{-q(x)}}, a\right).$$

We now give another special case of Theorem 2.1 for the sequence defined by (1.1) with initial cases $a_0(x) = 2$ and $a_1(x) = p(x)$.

Corollary 2.6. *Let sequence* $\{a_n(x)\}_{n \geq 0}$ *be defined by* $a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x)$ $(n \geq 2)$ *with initial conditions* $a_0(x) = 2$ *and* $a_1(x) = p(x)$.

Then

$$
\begin{aligned}
a_n(x) &= 2\left(\pm\sqrt{-q(x)}\right)^n U_n\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right) \\
&\quad - p(x)\left(\pm\sqrt{-q(x)}\right)^{n-1} U_{n-1}\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_n(x) &= 2\left(\pm\sqrt{q(x)}\right)^n P_{n+1}\left(\pm\frac{p(x)}{2\sqrt{q(x)}}\right) \\
&\quad - p(x)\left(\pm\sqrt{q(x)}\right)^{n-1} P_n\left(\pm\frac{p(x)}{2\sqrt{q(x)}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_n(x) &= 2\left(\pm\sqrt{q(x)}\right)^n F_{n+1}\left(\pm\frac{p(x)}{\sqrt{q(x)}}\right) \\
&\quad - p(x)\left(\pm\sqrt{q(x)}\right)^{n-1} F_n\left(\pm\frac{p(x)}{\sqrt{q(x)}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_n(x) &= 2\left(\pm\sqrt{-q(x)}\right)^n B_n\left(\pm\frac{p(x)}{\sqrt{-q(x)}} - 2\right) \\
&\quad - p(x)\left(\pm\sqrt{-q(x)}\right)^{n-1} B_{n-1}\left(\pm\frac{p(x)}{\sqrt{-q(x)}} - 2\right),
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
a_n(x) &= 2\left(\pm\sqrt{\frac{-q(x)}{2}}\right)^n \Phi_{n+1}\left(\pm p(x)\sqrt{\frac{2}{-q(x)}}\right) \\
&\quad - p(x)\left(\pm\sqrt{\frac{-q(x)}{2}}\right)^{n-1} \Phi_n\left(\pm p(x)\sqrt{\frac{2}{-q(x)}}\right),
\end{aligned}
$$

$$
\begin{aligned}
a_n(x) &= 2^3\left(\pm\sqrt{\frac{-q(x)}{a}}\right)^n D_n\left(\pm p(x)\sqrt{\frac{a}{-q(x)}}, a\right) \\
&\quad - p(x)2^2\left(\pm\sqrt{\frac{-q(x)}{a}}\right)^{n-1} D_{n-1}\left(\pm p(x)\sqrt{\frac{a}{-q(x)}}, a\right).
\end{aligned}
$$

In addition, one has

$$
a_n(x) = 2\left(\pm\sqrt{-q(x)}\right)^n T_n\left(\pm\frac{p(x)}{2\sqrt{-q(x)}}\right), \tag{2.18}
$$

where $T_n(x)$ *are the Chebyshev polynomials of the first kind.*

Proof. It is sufficient to prove the positive case of (2.18). From the first formula shown in Corollary 2.6 and the recurrence relation $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, one easily sees

$$
\begin{aligned}
a_n(x) &= \left(\sqrt{-q(x)}\right)^n \left[2U_n\left(\frac{p(x)}{2\sqrt{-q(x)}}\right) - \frac{p(x)}{\sqrt{-q(x)}}U_{n-1}\left(\frac{p(x)}{2\sqrt{-q(x)}}\right)\right] \\
&= \left(\sqrt{-q(x)}\right)^n \left[2U_n\left(\frac{p(x)}{2\sqrt{-q(x)}}\right) - \left(U_n\left(\frac{p(x)}{2\sqrt{-q(x)}}\right) + U_{n-2}\left(\frac{p(x)}{2\sqrt{-q(x)}}\right)\right)\right] \\
&= \left(\sqrt{-q(x)}\right)^n \left[U_n\left(\frac{p(x)}{2\sqrt{-q(x)}}\right) - U_{n-2}\left(\frac{p(x)}{2\sqrt{-q(x)}}\right)\right].
\end{aligned}
$$

(2.19)

From the first formula of Example 2.3, the above last expression of $a_n(x)$ implies the positive case of (2.18). The negative case can be proved similarly. $\qquad\square$

Example 2.7. As an example, the Lucas polynomial sequence $\{L_n(x)\}$ defined by (1.1) with $p(x) = x$ and $q(x) = 1$ and initial conditions $L_0(x) = 2$ and $L_1(x) = x$ has an explicit formula for its general term

$$
L_n(x) = 2(\pm i)^n T_n\left(\mp\frac{xi}{2}\right).
$$

(2.20)

Using Corollary 2.6, we also have

$$
\begin{aligned}
&L_n(x) = 2(\pm i)^n U_n\left(\mp\frac{xi}{2}\right) - x(\pm i)^{n-1}U_{n-1}\left(\mp\frac{xi}{2}\right), \\
&L_n(x) = 2P_{n+1}\left(\pm\frac{x}{2}\right) - xP_n\left(\pm\frac{x}{2}\right), \\
&L_n(x) = 2F_{n+1}(\pm x) - xF_n(\pm x), \\
&L_n(x) = 2(\pm i)^n B_n(\mp xi - 2) - x(\pm i)^{n-1}B_{n-1}(\mp xi - 2), \\
&L_n(x) = 2\left(\pm\frac{i}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\mp\sqrt{2}xi\right) - x\left(\pm\frac{i}{\sqrt{2}}\right)^{n-1}\Phi_n\left(\mp\sqrt{2}xi\right), \\
&L_n(x) = 2^3\left(\pm\frac{i}{\sqrt{a}}\right)^n D_n(\mp\sqrt{a}xi, a) - x2^2\left(\pm\frac{i}{\sqrt{a}}\right)^{n-1}D_{n-1}(\mp\sqrt{a}xi, a).
\end{aligned}
$$

(2.21)

From Theorem 2.1, one may obtain transfer formulas between generalized Gegenbauer-Humbert polynomials.

3. Examples and Applications

We first give some examples of Theorem 2.1 for sequences $\{a_n(x)\}$ defined by (1.1).

Example 3.1. The Chebyshev polynomials of the third kind and fourth kind satisfy the same recurrence relationship as the Chebyshev polynomials of the first kind with the same constant initial term 1 and different linear initial terms, $2x - 1$ and $2x + 1$, respectively (see, e.g., Mason and Handscomb [18] and Rivlin [19]). Hence, the Chebyshev polynomials of the third kind, $T_n^{(3)}(x)$, and the Chebyshev polynomials of the fourth kind, $T_n^{(4)}(x)$, when $x^2 \neq 1$, have the following expressions using the argument shown in [5]:

$$T_n^{(3)}(x) = \frac{\sqrt{x^2-1}+x-1}{2\sqrt{x^2-1}}\left(x+\sqrt{x^2-1}\right)^n + \frac{\sqrt{x^2-1}-x+1}{2\sqrt{x^2-1}}\left(x-\sqrt{x^2-1}\right)^n,$$

$$T_n^{(4)}(x) = \frac{\sqrt{x^2-1}+x+1}{2\sqrt{x^2-1}}\left(x+\sqrt{x^2-1}\right)^n + \frac{\sqrt{x^2-1}-x-1}{2\sqrt{x^2-1}}\left(x-\sqrt{x^2-1}\right)^n. \tag{3.1}$$

Similarly to the Chebyshev polynomials of the first kind (see Example 2.3), we can transfer $T_n^{(3)}(x)$ and $T_n^{(4)}(x)$ to the generalized Gegenbauer-Humbert polynomials with $\lambda = 1$,

$$T_n^{(3)}(x) = U_n(x) - U_{n-1}(x),$$

$$T_n^{(3)}(x) = (-1)^n(U_n(-x) + U_{n-1}(x)),$$

$$T_n^{(3)}(x) = (\pm i)^n P_{n+1}(\mp xi) - (\pm i)^{n-1} P_n(\mp xi),$$

$$T_n^{(3)}(x) = (\pm i)^n F_{n+1}(\mp 2xi) - (\pm i)^{n-1} F_n(\mp 2xi),$$

$$T_n^{(3)}(x) = (\pm 1)^n B_n(\pm 2x - 2) - (\pm 1)^{n-1} B_{n-1}(\pm 2x - 2),$$

$$T_n^{(3)}(x) = \left(\pm\frac{1}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\mp 2\sqrt{2}x\right) - \left(\pm\frac{1}{\sqrt{2}}\right)^{n-1} \Phi_n\left(\mp 2\sqrt{2}x\right),$$

$$T_n^{(3)}(x) = \left(\pm\frac{1}{\sqrt{4a}}\right)^n D_n\left(\mp 2\sqrt{a}x, a\right) - \left(\pm\frac{1}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(\mp 2\sqrt{a}x, a\right),$$

$$T_n^{(4)}(x) = U_n(x) + U_{n-1}(x),$$

$$T_n^{(4)}(x) = (-1)^n(U_n(-x) - U_{n-1}(x)),$$

$$T_n^{(4)}(x) = (\pm i)^n P_{n+1}(\mp xi) + (\pm i)^{n-1} P_n(\mp xi),$$

$$T_n^{(4)}(x) = (\pm i)^n F_{n+1}(\mp 2xi) + (\pm i)^{n-1} F_n(\mp 2xi),$$

$$T_n^{(4)}(x) = (\pm 1)^n B_n(\pm 2x - 2) + (\pm 1)^{n-1} B_{n-1}(\pm 2x - 2),$$

$$T_n^{(4)}(x) = \left(\pm\frac{1}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\mp 2\sqrt{2}x\right) + \left(\pm\frac{1}{\sqrt{2}}\right)^{n-1} \Phi_n\left(\mp 2\sqrt{2}x\right),$$

$$T_n^{(3)}(x) = \left(\pm\frac{1}{\sqrt{4a}}\right)^n D_n\left(\mp 2\sqrt{a}x, a\right) + \left(\pm\frac{1}{\sqrt{4a}}\right)^{n-1} D_{n-1}\left(\mp 2\sqrt{a}x, a\right). \tag{3.2}$$

From the above formulas, one may obtain some identities between the Chebyshev polynomials of different kinds. For instance,

$$T_n^{(3)}(x) + T_n^{(4)}(x) = 2U_n(x),$$

$$T_n(x) + xT_n^{(4)}(x) = (1 + x)U_n(x),$$

$$T_n(x) - xT_n^{(3)}(x) = (1 - x)U_n(x).$$

(3.3)

Since $T_n(x) = \cos n\theta$, $U_n(x) = \sin(n + 1)\theta / \sin\theta$, $T_n^{(3)}(x) = \cos(n + 1/2)\theta / \cos(1/2)\theta$, and $T_n^{(4)}(x) = \sin(n + 1/2)\theta / \sin(1/2)\theta$, where $x = \cos\theta$, the above identities of Chebyshev polynomials also present the following identities of trigonometric functions, respectively,

$$\frac{\cos(n + 1/2)\theta}{\cos(1/2)\theta} + \frac{\sin(n + 1/2)\theta}{\sin(1/2)\theta} = 2\frac{\sin(n + 1)\theta}{\sin\theta},$$

$$\cos n\theta + \cos\theta\frac{\sin(n + 1/2)\theta}{\sin(1/2)\theta} = (1 + \cos\theta)\frac{\sin(n + 1)\theta}{\sin\theta},$$

$$\cos n\theta - \cos\theta\frac{\sin(n + 1/2)\theta}{\sin(1/2)\theta} = (1 - \cos\theta)\frac{\sin(n + 1)\theta}{\sin\theta}.$$

(3.4)

Example 3.2. Consider the Jacobsthal polynomials $\{J_n(x)\}$ defined by (1.1) with coefficients $p(x) = 1$ and $q(x) = 2x$ and initial conditions $J_0(x) = J_1(x) = 1$. One may use Corollary 2.5 to obtain transfer formulas

$$J_n(x) = \left(\pm\sqrt{-2x}\right)^n U_n\left(\pm\frac{1}{2\sqrt{-2x}}\right),$$

$$J_n(x) = \left(\pm\sqrt{2x}\right)^n P_{n+1}\left(\pm\frac{1}{2\sqrt{2x}}\right),$$

$$J_n(x) = \left(\pm\sqrt{2x}\right)^n F_{n+1}\left(\pm\frac{1}{\sqrt{2x}}\right),$$

$$J_n(x) = \left(\pm\sqrt{-2x}\right)^n B_n\left(\pm\frac{1}{\sqrt{-2x}} - 2\right),$$

$$J_n(x) = \left(\pm\sqrt{-x}\right)^n \Phi_{n+1}\left(\pm\frac{1}{\sqrt{-x}}\right),$$

$$J_n(x) = 2^2\left(\pm\sqrt{\frac{-2x}{a}}\right)^n D_n\left(\pm\sqrt{\frac{a}{-2x}}, a\right).$$

(3.5)

The first formula and its inverse (see the first formula below) were given on [20, page 76] by Riordan using a different method. The positive case of the third formula is easily to be transferred to the formula of Theorem 1 in [21], where they used a different recurrence relation with $p(x) = 1$ and $q(x) = x$ for constructing the Jacobsthal polynomials. Reference [20] also

gave the inverse formula to present $U_n(x)$ in terms of $J_n(x)$. Actually, we can easily have the inverse formulas of $U_n(x)$, $P_{n+1}(x)$, $F_{n+1}(x)$, $\Phi_{n+1}(x)$, and $D_n(x, a)$ in terms of $J_n(x)$ as follows:

$$
\begin{aligned}
U_n(x) &= (2x)^n J_n\left(-\frac{1}{8x^2}\right), \\[2mm]
P_{n+1}(x) &= (2x)^n J_n\left(\frac{1}{8x^2}\right), \\[2mm]
F_{n+1}(x) &= x^n J_n\left(\frac{1}{2x^2}\right), \\[2mm]
B_n(x) &= (x+2)^n J_n\left(-\frac{1}{2(x+2)^2}\right), \\[2mm]
\Phi_{n+1}(x) &= x^n J_n\left(-\frac{1}{x^2}\right), \\[2mm]
D_n(x, a) &= \frac{1}{4} x^n J_n\left(-\frac{a}{2x^2}\right).
\end{aligned}
\tag{3.6}
$$

Example 3.3. In Eu [22], the polynomial sequence $\{H_n(x)\}$ is defined by $S_n(x) = xS_{n-1}(x) - S_{n-2}(x)$ with initial conditions $S_0(x) = 1$ and $S_1(x) = x$. Using Corollary 2.5, we obtain

$$
\begin{aligned}
S_n(x) &= U_n\left(\pm\frac{x}{2}\right), \\[2mm]
S_n(x) &= (\pm i)^n P_{n+1}\left(\mp\frac{x}{2}i\right), \\[2mm]
S_n(x) &= (\pm i)^n F_{n+1}(\mp xi), \\[2mm]
S_n(x) &= (\pm 1)^n B_n(\pm x - 2), \\[2mm]
S_n(x) &= \left(\pm\frac{1}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\pm\sqrt{2}x\right), \\[2mm]
S_n(x) &= 4\left(\pm\frac{1}{\sqrt{a}}\right)^n D_n(\pm\sqrt{a}x, a),
\end{aligned}
\tag{3.7}
$$

in which the first formula was given in [22] using a different approach. Similar to the case of the Jacobsthal polynomial sequence shown in Example 3.2, we have the inverse formulas

$$
\begin{aligned}
U_n(x) &= S_n(\pm 2x), \\[2mm]
P_{n+1}(x) &= (\mp i)^n S_n(\pm 2xi), \\[2mm]
F_{n+1}(x) &= (\mp i)^n S_n(\pm xi), \\[2mm]
B_n(x) &= (\pm 1)^n S_n(\pm(x+2)),
\end{aligned}
$$

$$\Phi_{n+1}(x) = \left(\pm\sqrt{2}\right)^n S_n\left(\pm\frac{x}{\sqrt{2}}\right),$$

$$D_n(x, a) = \frac{1}{4}\left(\pm\sqrt{a}\right)^n S_n\left(\pm\frac{x}{\sqrt{a}}\right).$$

$$(3.8)$$

Another polynomial sequence $\{H_n(x)\}$ is defined by $H_n(x) = (1 - x)H_{n-1}(x) - x^2 H_{n-2}(x)$ with initial conditions $H_0(X) = 1$ and $H_1(x) = 1 - x$ [22]. Using Corollary 2.5, we obtain

$$H_n(x) = (\pm x)^n U_n\left(\pm\frac{1-x}{2x}\right),$$

$$H_n(x) = (\pm ix)^n P_{n+1}\left(\mp\frac{1-x}{2x}i\right),$$

$$H_n(x) = (\pm ix)^n F_{n+1}\left(\mp\frac{1-x}{x}i\right),$$

$$H_n(x) = (\pm x)^n B_n\left(\pm\frac{1-x}{x} - 2\right),$$ $$(3.9)$$

$$H_n(x) = \left(\pm\frac{x}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\pm\sqrt{2}\frac{1-x}{x}\right),$$

$$H_n(x) = 4\left(\pm\frac{x}{\sqrt{a}}\right)^n U_n\left(\pm\sqrt{a}\frac{1-x}{x}, a\right),$$

where the first formula has been established in [22] by using a different method. The inverse of the above formulas can be found similarly. For instance,

$$U_n(x) = (2x \pm 1)^n H_n\left(\frac{1}{1 \pm 2x}\right).$$ $$(3.10)$$

Example 3.4. In Riordan [23], the associate Legendre polynomial sequence $\{\rho_n(x)\}$ is defined by $\rho_n(x) = (2 + x)\rho_{n-1}(x) - \rho_{n-2}(x)$ with initial conditions $\rho_0(x) = 1$ and $\rho_1(x) = 1 + x$, then we use Theorem 2.1 and Corollary 2.2 to generate the following transfer formulas:

$$\rho_n(x) = U_n\left(\pm\left(1 + \frac{x}{2}\right)\right) - U_{n-1}\left(\pm\left(1 + \frac{x}{2}\right)\right),$$

$$\rho_n(x) = (\pm i)^n P_{n+1}\left(\mp i\left(1 + \frac{x}{2}\right)\right) - (\pm i)^{n-1} P_n\left(\mp i\left(1 + \frac{x}{2}\right)\right),$$

$$\rho_n(x) = (\pm i)^n F_{n+1}(\mp i(x + 2)) - (\pm i)^{n-1} F_n(\mp i(x + 2)),$$

$$\rho_n(x) = (\pm 1)^n B_n(\pm(x + 2) - 2) - (\pm 1)^{n-1} B_{n-1}(\pm(x + 2) - 2),$$

$$\rho_n(x) = \left(\pm\frac{1}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\pm\sqrt{2}(x+2)\right) - \left(\pm\frac{1}{\sqrt{2}}\right)^{n-1} \Phi_n\left(\pm\sqrt{2}(x+2)\right),$$

$$\rho_n(x) = 4\left(\pm\frac{1}{\sqrt{a}}\right)^n D_n\left(\pm\sqrt{a}(x+2), a\right) - 4\left(\pm\frac{1}{\sqrt{a}}\right)^{n-1} D_{n-1}\left(\pm\sqrt{a}(x+2), a\right),$$

(3.11)

where the first formula was given on [20, page 85] using a different method.

Example 3.5. In Chow and West [24], the polynomial sequence $\{p_n(x)\}$ is defined by $p_n(x) = -xp_{n-1}(x) - xp_{n-2}(x)$ with initial conditions $p_0(x) = 1 - x^{-1}$ and $p_1(x) = 2 - x$ $(x \neq 0)$. From Theorem 2.1 and Corollary 2.2, we obtain

$$p_n(x) = \left(1 - x^{-1}\right)(\pm\sqrt{x})^n U_n\left(\mp\frac{\sqrt{x}}{2}\right) + (\pm\sqrt{x})^{n-1} U_{n-1}\left(\mp\frac{\sqrt{x}}{2}\right),$$

$$p_n(x) = \left(1 - x^{-1}\right)(\pm\sqrt{xi})^n P_{n+1}\left(\pm\frac{\sqrt{xi}}{2}\right) + (\pm\sqrt{xi})^{n-1} P_n\left(\pm\frac{\sqrt{xi}}{2}\right),$$

$$p_n(x) = \left(1 - x^{-1}\right)(\pm\sqrt{xi})^n F_{n+1}(\pm\sqrt{xi}) + (\pm\sqrt{xi})^{n-1} F_n(\pm\sqrt{xi}),$$

$$p_n(x) = \left(1 - x^{-1}\right)(\pm\sqrt{x})^n B_n(\mp\sqrt{x} - 2) + (\pm\sqrt{x})^{n-1} B_{n-1}(\mp\sqrt{x} - 2),$$

(3.12)

$$p_n(x) = \left(1 - x^{-1}\right)\left(\pm\sqrt{\frac{x}{2}}\right)^n \Phi_{n+1}\left(\pm\sqrt{2xi}\right) + \left(\pm\sqrt{\frac{x}{2}}\right)^{n-1} \Phi_n\left(\pm\sqrt{2xi}\right),$$

$$p_n(x) = 4\left(1 - x^{-1}\right)\left(\pm\sqrt{\frac{x}{a}}\right)^n D_n\left(\pm\sqrt{axi}, a\right) + 4\left(\pm\sqrt{\frac{x}{a}}\right)^{n-1} D_{n-1}\left(\pm\sqrt{axi}, a\right).$$

Since $U_{n+1}(y) = 2yU_n(y) - U_{n-1}(y)$, we have

$$\begin{aligned}
U_{n+2}(y) &= 2yU_{n+1}(y) - U_n(y) \\
&= 2y(2yU_n(y) - U_{n-1}(y)) - U_n(y) \\
&= \left(4y^2 - 1\right)U_n(y) - 2yU_{n-1}(y).
\end{aligned}$$

(3.13)

Hence, from the last expression of U_{n+2} and the transfer formula of $p_n(x)$ in terms of $U_n(x)$ shown above, we obtain

$$p_n(x) = (\pm 1)^n x^{(n-2)/2} U_{n+2}\left(\mp\frac{\sqrt{x}}{2}\right),$$

(3.14)

in which the case of

$$p_n(x) = (-1)^n x^{(n-2)/2} U_{n+2}\left(\frac{\sqrt{x}}{2}\right)$$

(3.15)

was established in [24] using mathematical induction.

Equaling the right-hand expressions of the polynomials shown in each example, one may obtain various identities of generalized Gegenbauer-Humbert polynomials. For instance, from Example 2.3, we have

$$
\begin{aligned}
U_n(x) - xU_{n-1}(x) &= (-1)^n(U_n(-x) + xU_{n-1}(x)) \\
&= (\pm i)^n P_{n+1}(\mp xi) - x(\pm i)^{n-1} P_n(\mp xi) \\
&= (\pm i)^n F_{n+1}(\mp 2xi) - x(\pm i)^{n-1} F_n(\mp 2xi) \\
&= (\pm 1)^n B_n(\pm 2x - 2) - (\pm 1)^{n-1} xB_{n-1}(\pm 2x - 2) \\
&= \left(\pm\frac{1}{\sqrt{2}}\right)^n \Phi_{n+1}\left(\mp 2\sqrt{2}x\right) - x\left(\pm\frac{1}{\sqrt{2}}\right)^{n-1} \Phi_n\left(\mp 2\sqrt{2}x\right) \\
&= \left(\pm\frac{1}{\sqrt{4a}}\right)^n D_n(\mp 2\sqrt{a}x, a) - x\left(\pm\frac{1}{\sqrt{4a}}\right)^{n-1} D_{n-1}(\mp 2\sqrt{a}x, a).
\end{aligned}
\tag{3.16}
$$

Using the relationship established in Theorem 2.1 and Corollaries 2.2–2.6, we may obtain some identities of polynomial sequences from the generalized Gegenbauer-Humbert polynomial sequence identity described in [5]

$$
P_n^{1,y,C}(x) = \alpha(x)P_{n-1}^{1,y,C}(x) + C^{-2}(2x - \alpha(x)C)(\beta(x))^{n-1},
\tag{3.17}
$$

where $P_n^{1,y,C}(x)$ satisfies the recurrence relation of order 2, $P_n^{1,y,C}(x) = p(x)P_{n-1}^{1,y,C}(x) + q(x)P_{n-2}^{1,y,C}(x)$ with coefficients $p(x)$ and $q(x)$, and $\alpha(x) + \beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$. Clearly (see (2.41) in [5]),

$$
\begin{aligned}
\alpha(x) &= \frac{1}{C}\left\{x + \sqrt{x^2 - Cy}\right\}, \\
\beta(x) &= \frac{1}{C}\left\{x - \sqrt{x^2 - Cy}\right\}.
\end{aligned}
\tag{3.18}
$$

For $y = C = 1$, we have $P_n^{1,1,1}(x) = U_n(x)$, where $U_n(x)$ are the Chebyshev polynomials of the second kind, and we can write (3.17) as

$$
U_n(x) = \alpha(x)U_{n-1}(x) + (2x - \alpha(x))(\beta(x))^{n-1} = \alpha(x)U_{n-1}(x) + (\beta(x))^n,
\tag{3.19}
$$

where $\alpha(x) = x + \sqrt{x^2 - 1}$ and $\beta(x) = x - \sqrt{x^2 - 1}$. From the first formula of Example 3.2 and using transform $\pm 1/(2\sqrt{-2x}) \mapsto x$, we have

$$
U_n(x) = (2x)^n J_n\left(-\frac{1}{8x^2}\right).
\tag{3.20}
$$

Substituting the above expression to (3.19) yields the identity

$$(2x)^n J_n\left(-\frac{1}{8x^2}\right) = \left(x + \sqrt{x^2-1}\right)(2x)^{n-1}J_{n-1}\left(-\frac{1}{8x^2}\right) + \left(x - \sqrt{x^2-1}\right)^n. \tag{3.21}$$

Similarly, from Example 3.3, we obtain identities

$$S_n(\pm 2x) = \left(\pm x + \sqrt{x^2-1}\right)S_{n-1}(\pm 2x) + \left(\pm x - \sqrt{x^2-1}\right)^n,$$

$$(2x \pm 1)^n H_n\left(\frac{1}{1 \pm 2x}\right) = (2x \pm 1)^{n-1}\left(x + \sqrt{x^2-1}\right)H_{n-1}\left(\frac{1}{1 \pm 2x}\right) + \left(x - \sqrt{x^2-1}\right)^n. \tag{3.22}$$

One may also extend some well-known identities of a polynomial sequence to other polynomial sequences using the relationships we have established. For instance, from the Cassini-like formula for Fibonacci polynomials

$$F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n, \tag{3.23}$$

we use the relationship shown in Example 3.2 to obtain the Cassini-like formula for the Jacobsthal polynomials

$$J_n(x)J_{n-2}(x) - J_{n-1}^2(x) = (-2x)^n, \tag{3.24}$$

which can be transferred to the formula of Theorem 2 in [21] using the same argument in Example 3.2.

Similarly, from the transform

$$F_{n+1}(x) = (\pm i)^n U_n\left(\mp\frac{xi}{2}\right), \tag{3.25}$$

we have

$$U_n\left(\mp\frac{xi}{2}\right)U_{n-2}\left(\mp\frac{xi}{2}\right) - U_{n-1}^2\left(\mp\frac{xi}{2}\right) = (-1)^n. \tag{3.26}$$

To construct a transform relationship for the polynomials defined by recurrence relation with coefficients related to the order of polynomials is much more difficulty. One special example can be found on [25, page 240] by Andrews et al.. It seems there is no a general method applied to such polynomial sequences.

Acknowledgments

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