# Generalized Fibonacci Sequences 

V.K. Gupta ${ }^{1}$, Yashwant K. Panwar ${ }^{2}$ and Omprakash Sikhwal ${ }^{3}$


#### Abstract

The Fibonacci sequence is famous for possessing wonderful and amazing properties. In this paper, we introduce generalized Fibonacci sequences and related identities consisting even and odd terms. Also we present connection formulas for generalized Fibonacci sequences, Jacobsthal sequence and Jacobsthal-Lucas sequence.


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${ }^{1}$ Department of Mathematics, Govt. Madhav Science College, Ujjain, India, e-mail: dr_vkg61@yahoo.com
${ }^{2}$ Department of Mathematics, Mandsaur Institute of Technology, Mandsaur, India, e-mail: yashwantpanwar@gmail.com
${ }^{3}$ Department of Mathematics, Mandsaur Institute of Technology, Mandsaur, India, e-mail: opbhsikhwal@rediffmail.com ; opsikhwal@gmail.com

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## 1 Introduction

It is well-known that the Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences.
The Fibonacci sequence [8] is defined by the recurrence relation

$$
\begin{equation*}
F_{k}=F_{k-1}+F_{k-2}, k \geq 2 \text { with } F_{0}=0, F_{1}=1 \tag{1.1}
\end{equation*}
$$

The Lucas sequence [8] is defined by the recurrence relation
$L_{k}=L_{k-1}+L_{k-2}, k \geq 2$ with $L_{0}=2, L_{1}=1$
The Jacobsthal sequence [2] is defined by the recurrence relation

$$
\begin{equation*}
J_{k}=J_{k-1}+2 J_{k-2}, k \geq 2 \text { with } J_{0}=0, J_{1}=1 \tag{1.3}
\end{equation*}
$$

The Jacobsthal-Lucas sequence [2] is defined by the recurrence relation

$$
\begin{equation*}
j_{k}=j_{k-1}+2 j_{k-2}, k \geq 2 \text { with } j_{0}=2, j_{1}=1 \tag{1.4}
\end{equation*}
$$

B. Singh, O. Sikhwal and S. Bhatnagar [5] defined Fibonacci-Like Sequence

$$
\begin{equation*}
S_{k}=S_{k-1}+S_{k-2}, k \geq 2 \quad \text { with } \quad S_{0}=2, S_{1}=2 \tag{1.5}
\end{equation*}
$$

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [6] generalize the Fibonacci sequence by

$$
\begin{equation*}
F_{n}=a F_{n-1}+b F_{n-2}, n \geq 2 \quad \text { with } F_{0}=0, F_{1}=1 \tag{1.6}
\end{equation*}
$$

Horadam [3] defined generalized Fibonacci sequence $\left\{H_{n}\right\}$ by

$$
\begin{equation*}
H_{n}=H_{n-1}+H_{n-2}, n \geq 3 \text { with } H_{1}=p, H_{2}=p+q \tag{1.7}
\end{equation*}
$$

where p and q are arbitrary integers.
In this paper, we introduce generalized Fibonacci sequence and present identities consisting even and odd terms. Further we defined connection formulas for generalized Fibonacci sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence.

## 2 Generalized Fibonacci Sequence

We define generalized Fibonacci sequence as

$$
\begin{equation*}
F_{k}=p F_{k-1}+q F_{k-2}, k \geq 2 \quad \text { with } \quad F_{0}=a, F_{1}=b \tag{2.1}
\end{equation*}
$$

where $p, q, a$ and $b$ are positive integers
For different values of $p, q, a$ and $b$ many sequences can be determined.
We focus two cases of sequences $\left\{V_{k}\right\}_{k \geq 0}$ and $\left\{U_{k}\right\}_{k \geq 0}$ which generated in (2.1).
If $p=1, q=a=b=2$, we get
$V_{k}=V_{k-1}+2 V_{k-2}$, for $k \geq 2$ with $V_{0}=2, V_{1}=2$
The first few terms of $\left\{V_{k}\right\}_{k \geq 0}$ are $2,2,6,10,22,42$ and so on. Its Generating function is defined by

$$
\begin{equation*}
V_{k}=\frac{2}{1-x-2 x^{2}} \tag{2.3}
\end{equation*}
$$

Its Binet's formula is defined by

$$
\begin{equation*}
V_{k}=2 \frac{\mathfrak{R}_{1}^{k+1}-\Re_{2}^{k+1}}{\mathfrak{R}_{1}-\mathfrak{R}_{2}} \tag{2.4}
\end{equation*}
$$

where $\mathfrak{R}_{1}$ and $\mathfrak{R}_{2}$ are the roots of the characteristic equation $2 x^{2}+x-1=0$.
If $p=1, q=a=2, b=0$ we get

$$
\begin{equation*}
U_{k}=U_{k-1}+2 U_{k-2} \text { for } k \geq 2 \text { with } U_{0}=2, U_{1}=0 \tag{2.5}
\end{equation*}
$$

The first few terms of $\left\{U_{k}\right\}_{k \geq 0}$ are $2,0,4,4,12,20$ and so on. Its Generating function is defined by

$$
\begin{equation*}
U_{k}=\frac{2(1-x)}{1-x-2 x^{2}} \tag{2.6}
\end{equation*}
$$

Its Binet's formula is defined by

$$
\begin{equation*}
U_{k}=4 \frac{\mathfrak{R}_{1}^{k-1}-\mathfrak{R}_{2}^{k-1}}{\mathfrak{R}_{1}-\mathfrak{R}_{2}} \tag{2.7}
\end{equation*}
$$

where $\mathfrak{R}_{1}$ and $\mathfrak{R}_{2}$ are the roots of the characteristic equation $2 x^{2}+x-1=0$.

## 3 Identities of Generalized Fibonacci Sequence

Now we present identities consisting even and odd terms.

Theorem 3.1. If $\left\{V_{k}\right\}_{k \geq 0}$ and $\left\{U_{k}\right\}_{k \geq 0}$ are the generalized Fibonacci sequences, then

$$
V_{k} U_{k}=\frac{8}{9} j_{2 k}-(-2)^{k-1} \frac{8}{3}= \begin{cases}\frac{8}{9} j_{2 k}+2^{k-1} \frac{8}{3}, & k \text { even } \\ \frac{8}{9} j_{2 k}-2^{k-1} \frac{8}{3}, & k \text { odd }\end{cases}
$$

Proof. By Binet's formulas (2.4) and (2.7), we have

$$
\begin{aligned}
V_{k} U_{k} & =\frac{8}{9}\left\{\left(\mathfrak{R}_{1}^{k+1}-\Re_{2}^{k+1}\right)\left(\mathfrak{R}_{1}^{k-1}-\Re_{2}^{k-1}\right)\right\} \\
& =\frac{8}{9}\left\{\left(\mathfrak{R}_{1}^{2 k}+\mathfrak{R}_{2}^{2 k}\right)-\mathfrak{R}_{1}^{k} \Re_{2}^{k}\left(\Re_{1} \Re_{2}^{-1}-\Re_{2} \mathfrak{R}_{1}^{-1}\right)\right\} \\
& =\frac{8}{9}\left\{j_{2 k}-(-2)^{k}\left(\frac{\mathfrak{R}_{1}}{\mathfrak{R}_{2}}-\frac{\mathfrak{R}_{2}}{\mathfrak{R}_{1}}\right)\right\} \\
& =\frac{8}{9}\left\{j_{2 k}-(-2)^{k-1}\left(\Re_{1}^{2}-\mathfrak{R}_{2}^{2}\right)\right\} \\
& =\frac{8}{9} j_{2 k}-\frac{8}{3}(-2)^{k-1} \\
V_{k} U_{k}=\frac{8}{9} j_{2 k}-(-2)^{k-1} \frac{8}{3} & = \begin{cases}\frac{8}{9} j_{2 k}+2^{k-1} \frac{8}{3}, & k \text { even } \\
\frac{8}{9} j_{2 k}-2^{k-1} \frac{8}{3}, & k \text { odd }\end{cases}
\end{aligned}
$$

Theorem 3.2. If $\left\{V_{k}\right\}_{k \geq 0}$ is the generalized Fibonacci sequence, then
(i) $V_{k-1}^{2}+V_{k}^{2}=\frac{4^{k+1}}{3}+\frac{8}{9}\left\{j_{2 k}+(-2)^{k}\right\}$
(ii) $V_{k+1}^{2}-V_{k-1}^{2}= \begin{cases}\frac{5}{3}(4)^{k+1}-\frac{1}{3}(2)^{k+3}, & k \text { even } \\ \frac{5}{3}(4)^{k+1}+\frac{1}{3}(2)^{k+3}, & k \text { odd }\end{cases}$

Proof. (i): By Binet's formula (2.4), we have

$$
\begin{aligned}
& V_{k-1}^{2}+V_{k}^{2}=\frac{4}{9}\left\{\left(\Re_{1}^{k}-\Re_{2}^{k}\right)^{2}+\left(\Re_{1}^{k+1}-\Re_{2}^{k+1}\right)^{2}\right\} \\
&=\frac{4}{9}\left\{\Re_{1}^{2 k}+\Re_{1}^{2 k+2}+\Re_{2}^{k}+\Re_{2}^{2 k+2}-2\left(\Re_{1} \Re_{2}\right)^{k}\left(\Re_{1} \Re_{2}+1\right)\right\} \\
&=\frac{4}{9}\left\{3 \Re_{1}^{2 k}+2\left(\Re_{1}^{2 k}+\Re_{2}^{2 k}\right)+2(-2)^{k}\right\} \\
&=\frac{4}{9}\left\{3 \Re_{1}^{2 k}+2 j_{2 k}+2(-2)^{k}\right\} \\
& V_{k-1}^{2}+V_{k}^{2}=\frac{4^{k+1}}{3}+\frac{8}{9}\left\{j_{2 k}+(-2)^{k}\right\}
\end{aligned}
$$

(ii): By Binet's formula (2.4), we have

$$
\begin{gathered}
V_{k+1}^{2}-V_{k-1}^{2}=\frac{4}{9}\left\{\left(\mathfrak{R}_{1}^{k+2}-\mathfrak{R}_{2}^{k+2}\right)^{2}+\left(\mathfrak{R}_{1}^{k}-\Re_{2}^{k}\right)^{2}\right\} \\
=\frac{4}{9}\left[\mathfrak{R}_{1}^{2 k}\left(\Re_{1}^{4}-1\right)+\Re_{2}^{2 k}\left(\mathfrak{R}_{2}^{4}-1\right)-2\left(\Re_{1} \Re_{2}\right)^{k}\left\{\left(\Re_{1} \Re_{2}\right)^{k}-1\right\}\right] \\
V_{k+1}^{2}-V_{k-1}^{2}= \\
=\frac{4}{9}\left\{15 \Re_{1}^{2 k}-6(-2)^{k}\right\} \\
= \\
=\frac{5}{3}(4)^{k+1}-\frac{(-1)^{k}}{3}(2)^{k+3} \\
V_{k+1}^{2}-V_{k-1}^{2}= \begin{cases}\frac{5}{3}(4)^{k+1}-\frac{1}{3}(2)^{k+3}, & k \text { even } \\
\frac{5}{3}(4)^{k+1}+\frac{1}{3}(2)^{k+3}, & k \text { odd }\end{cases}
\end{gathered}
$$

Following theorems can be solved by Binet's formula (2.4) and (2.7)

Theorem 3.3. If $\left\{U_{k}\right\}_{k \geq 0}$ is the generalized Fibonacci sequence, then
(i) $U_{k+1}^{2}+U_{k+2}^{2}=\frac{4^{k+2}}{3}+\frac{32}{9}\left\{j_{2 k}+(-2)^{k}\right\}$
(ii) $U_{k+3}^{2}-U_{k+1}^{2}=\left\{\begin{array}{l}\frac{5}{3}(4)^{k+2}-\frac{1}{3}(2)^{k+5}, \text { k even } \\ \frac{5}{3}(4)^{k+2}+\frac{1}{3}(2)^{k+5}, \text { k odd }\end{array}\right.$

Theorem 3.4. Prove that
$V_{0}+V_{3}+V_{6}+\ldots+V_{3 k-3}=\sum_{n=1}^{k} V_{3 n-3}= \begin{cases}\frac{4}{7} J_{3 k} & , k \text { even } \\ \frac{2}{7}\left(2 J_{3 k}+1\right) & , k \text { odd }\end{cases}$

Corollary 3.5. $V_{0}+V_{3}+V_{6}+\ldots+V_{3 k-3}= \begin{cases}\frac{4}{21}\left(8^{k}-1\right), & k \text { even } \\ \frac{2}{21}\left(2^{3 k+1}+5\right), & k \text { odd }\end{cases}$

Theorem 3.6. Prove that

$$
U_{2}+U_{5}+U_{8}+\ldots+U_{3 k-1}= \begin{cases}\frac{8}{7} J_{3 k}, & k \text { even } \\ \frac{2}{7}\left(4 J_{3 k}+1\right), & k \text { odd }\end{cases}
$$

Corollary 3.7. $U_{2}+U_{5}+U_{8}+\ldots+U_{3 k-1}= \begin{cases}\frac{8}{21}\left(8^{k}-1\right), & k \text { even } \\ \frac{8^{k+1}+20}{21}, & k \text { odd }\end{cases}$

Theorem 3.8. Prove that
$V_{1}+V_{4}+V_{7}+\ldots+V_{3 k-2}=\sum_{n=1}^{k} V_{3 n-2}=\frac{1}{3} j_{3 k}+\frac{1}{21}(8)^{k}-\frac{5}{7}$

Corollary 3.9. $V_{1}+V_{4}+V_{7}+\ldots+V_{3 k-2}= \begin{cases}\frac{8}{21}\left(8^{k}-1\right), & k \text { even } \\ \frac{\left(8^{k+1}-20\right)}{21}, & k \text { odd }\end{cases}$

Theorem 3.10. Prove that

$$
U_{3}+U_{6}+U_{9}+\ldots+U_{3 k}=\sum_{n=1}^{k} U_{3 k}=\frac{2}{3} j_{3 k}+\frac{2}{21}(8)^{k}-\frac{10}{7}
$$

Corollary 3.11. $U_{3}+U_{6}+U_{9}+\ldots+U_{3 k}= \begin{cases}\frac{16}{21}\left(8^{k}-1\right), & k \text { even } \\ \frac{2}{21}\left(8^{k+1}-22\right), & k \text { odd }\end{cases}$

Theorem 3.12. Prove that
$U_{1}+U_{4}+U_{7}+\ldots+U_{3 k-2}= \begin{cases}\frac{4}{21}\left(j_{3 k}-2\right), & k \text { even } \\ \frac{4}{21}\left(j_{3 k}-7\right), & k \text { odd }\end{cases}$

Corollary 3.13. $U_{1}+U_{4}+U_{7}+\ldots+U_{3 k-2}=\left\{\begin{array}{l}\frac{4}{21}\left(8^{k}-1\right), k \text { even } \\ \frac{4}{21}\left(8^{k}-8\right), k \text { odd }\end{array}\right.$

## 4 Connection Formulas

Finally we present connection formulas for generalized Fibonacci Sequences, Jacobsthal sequence and Jacobsthal-Lucas sequence.

## Theorem 4.1. Prove that

(i) $2 V_{k+1}-V_{k}=2 j_{k+1}$
(ii) $V_{k}+4 V_{k-1}=2 j_{k+1}$

Proof. (i): For $k=0$,

$$
2 V_{0+1}-V_{0}=2 \times 2-2=2=2 \times 1=2 j_{1},
$$

which is true for $k=0$.
For $k=1$,

$$
2 V_{1+1}-V_{1}=2 \times 6-2=10=2 \times 5=2 j_{2},
$$

which is also true for $k=1$.
If result is true for $k=n$, then $2 V_{n+1}-V_{n}=2 j_{n+1}$. Now

$$
\begin{aligned}
2 V_{(n+1)+1}-V_{(n+1)} & =2 V_{n+2}-V_{n+1} \\
& =2 j_{n+1}+2\left(2 j_{n}\right) \quad \text { (By hypothesis) } \\
& =2\left(j_{n+1}+2 j_{n}\right)=2 j_{n+2}
\end{aligned} \quad \text { }
$$

Therefore, $2 V_{(n+1)+1}-V_{(n+1)}=2 j_{n+2}$, which is true for $k=n+1$.
This completes the proof.
(ii): By Binet's formula (2.4), we have
$V_{k}+4 V_{k-1}=2 \frac{\mathfrak{R}_{1}^{k+1}-\mathfrak{R}_{2}^{k+1}}{\mathfrak{R}_{1}-\mathfrak{R}_{2}}+8 \frac{\mathfrak{R}_{1}^{k}-\mathfrak{R}_{2}^{k}}{\mathfrak{R}_{1}-\mathfrak{R}_{2}}=2 j_{k+1}$
Following theorems can be solved with the help of Binet's formulas and as well as induction method.

Theorem 4.2. Prove that
(i) $U_{k+1}+4 U_{k}=4 j_{k}$
(ii) $2 U_{k+2}-U_{k+1}=4 j_{k}$

Theorem 4.3. Prove that
(i) $\quad j_{k+1}+4 j_{k}= \begin{cases}\frac{9}{2} V_{k}, & k \geq 0 \\ \frac{9}{4} U_{k+2}, & k \geq 0\end{cases}$
(ii) $\quad 2 j_{k+1}-j_{k}= \begin{cases}\frac{9}{2} V_{k-1}, & k \geq 1 \\ \frac{9}{4} U_{k+2}, & k \geq 0\end{cases}$

## 5 Conclusion

In this paper we have stated and derived many identities of generalized Fibonacci sequences consisting even and odd terms through Binet's formulas. Finally we presented some connection formulas and defined through induction method.

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