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# Matrix Displacement Decompositions and Applications to Toeplitz Linear Systems 

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#### Abstract

Using the approach of Bozzo, Di Fiore, and Zellini, new matrix displacement decomposition formulas are introduced. It is shown how an arbitrary square matrix $A$ can be expressed as sums of products of Hessenberg algebra matrices and high level (block) matrices whose submatrices are Hessenberg algebra matrices and have variable sizes. In most cases these block factors are block-diagonal matrices. Then these formulas are used in sequential and parallel solution of Toeplitz systems. (C) 1998 Elsevier Science Inc.


## 1. INTRODUCTION

The present paper considers some new matrix decompositions based on the concept of displacement rank [30]. The approach follows the main ideas developed in [21, 18], exploiting the notion of Hessenberg algebra, which
generalizes a class of hypergroups of matrices introduced by Bapat and Sunder [6].

In [21, 18, 17], many displacement-based decompositions of matrices known in the current literature, as well as new decompositions based on an algebra of $\tau$ type [21]-or on its variant $\tau_{\varepsilon, \varphi}$ [18]-were obtained as special cases of general formulas written in terms of whole classes of matrix Hessenberg algebras. Further developments, where noncommutative matrix algebras are exploited, are considered in [16, 17].

All the formulas introduced in $[21,18,16]$ can be specialized for the inverses of Toeplitz matrices and Toeplitz plus Hankel matrices, with and without symmetry.

The novelty of this paper is that a number of factors involved in the decompositions of a general $n \times n$ matrix $A$ are high level or block matrices, i.e. matrices whose elements are Hessenberg algebra matrices (in some cases null matrices) of variable sizes.

In Theorem 3.1, under the hypothesis of symmetry, the high level factors are two-block-diagonal matrices, where the blocks are Hessenberg algebra matrices of dimensions $(i-1) \times(i-1)$ and $(n-i) \times(n-i)$ (a zero element separates the two blocks) with $1 \leqslant i \leqslant n$. For $i=1, n$ we retrieve a result obtained in [21].

In Theorem 3.2, under the hypothesis of symmetry and persymmetry, we find a decomposition involving five-block matrices which can be reduced, in some particular cases, to three-block-diagonal matrices. As a special case we obtain a formula introduced in [21].

In Theorem 3.3 the block factors are two-block-diagonal, where the blocks are $\varepsilon$-circulant matrices of variable sizes.

For stating these theorems and all remarks about them, we need some preliminary results on Hessenberg algebras which are exposed in Section 2.

All theorems and their corollaries proved in Section 3 are exploited in Section 4 to write formulas for $A=T^{-1}$ and $A=(T+H)^{-1}$ where $T$ and $T+H$ are, respectively, nonsingular Toeplitz and Toeplitz plus Hankel matrices. In Section 4 two other representations of $T^{-1}$ are introduced when $T$ is symmetric and $n$ is even.

The interest of these results consists in the following reasons. The matrix-vector product $A \mathbf{b}$ is reduced, in part, to matrix-vector products of smaller sizes. This conforms to the general strategy of solving a problem by splitting it into smaller subproblems, in order to reduce computational complexity and possibly introduce parallel procedures. Here one can choose, with some restrictions, the dimensions of these subproblems (because the blocks involved in matrix decompositions have variable sizes), which is an advantage with respect to the more rigid formulas in [21] where only the dimensions $n, n-1, n-2$ were considered. In particular, for $n$ even, the
submatrices in the block factors of Theorems 3.2 and 3.3 can be of the same dimension $n / 2$. This implies, as consequence of Theorem 3.3, that one can avoid the use of matrices of different sizes as were considered in [21], and only fast transforms of dimension $n$ and $n / 2$ are calculated.

The computational interest of formulas involving factors in block-diagonal form is more evident in case a certain degree of parallelism is introduced. This aspect is briefly discussed in the concluding remarks for the computation of $T^{-1} \mathbf{f}, \mathbf{f} \in \mathbb{C}^{n}$.

## 2. HESSENBERG ALGEBRAS

Let $\mathbf{M}_{n}(\mathbb{C})$ be the space of $n \times n$ matrices over the complex field $\mathbb{C}$, and let $X \in \mathbf{M}_{n}(\mathbb{C})$. For $A \in \mathbf{M}_{n}(\mathbb{C})$ set

$$
\mathfrak{C}_{X}(A)=A X-X A .
$$

Let $X$ be the lower Hessenberg matrix

$$
X=\left(\begin{array}{ccccc}
r_{11} & b_{1} & 0 & \cdots & 0  \tag{2.1}\\
r_{21} & r_{22} & b_{2} & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
\vdots & & & \ddots & b_{n-1} \\
r_{n 1} & \cdots & \cdots & \cdots & r_{n n}
\end{array}\right), \quad b_{i} \neq 0 \quad \forall i
$$

and let $H_{X}$ denote the kernel of $\mathfrak{C}_{X}$.
As $X$ is nonderogatory, $H_{X}$ is the space of all polynomials in $X$ with coefficients in $\mathbb{C}$ and $\operatorname{dim} H_{X}=n$ [27, pp. 135-137]. Therefore $H_{X}$ is commutative and closed under multiplication. Moreover, if $A \in H_{X}$ and $\operatorname{det} A \neq 0$, then $A^{-1} \in H_{X}$.

We call $H_{X}$ a Hessenberg algebra, in conformity with $[21,18] . H_{X}$ can be represented as

$$
\begin{equation*}
H_{X}=\left\{\sum_{i=1}^{n} z_{i} X_{i}: z_{i} \in \mathbb{C}\right\}, \tag{2.2}
\end{equation*}
$$

where the matrices $X_{i}$ are defined as follows [21]:

$$
\begin{align*}
X_{1} & =I \\
X_{j+1} & =b_{j}^{-1}\left(X_{j} X-\sum_{m=1}^{j} r_{j m} X_{m}\right), \quad j=1, \ldots, n-1 . \tag{2.3}
\end{align*}
$$

The representation (2.2) is a consequence of the equalities $\mathbf{e}_{1}^{T} X_{i}=\mathbf{e}_{i}^{T}$, which can be easily proved by induction. It can be shown that

$$
\begin{equation*}
X_{i} X_{j}=\sum_{k=1}^{n}\left[X_{j}\right]_{i k} X_{k}, \quad 1 \leqslant i, j \leqslant n \tag{2.4}
\end{equation*}
$$

Moreover, from the commutativity of $H_{X}$, we have $\mathbf{e}_{1}^{T} X_{i} X_{j}=\mathbf{e}_{1}^{T} X_{j} X_{i}$ and then

$$
\begin{equation*}
\mathbf{e}_{j}^{T} X_{i}=\mathbf{e}_{i}^{T} X_{j}, \quad 1 \leqslant i, j \leqslant n . \tag{2.5}
\end{equation*}
$$

Let $H_{X}(\mathbf{z})$ denote the matrix $\sum_{i=1}^{n} z_{i} X_{i}$, i.e. the matrix of $H_{X}$ whose first row is $\mathbf{z}^{T}=\left[\begin{array}{llll}z_{1} & z_{2} & \cdots & z_{n}\end{array}\right]$ Let $J$ be the reversion matrix $\left(\delta_{n+1-i, j}\right)$ and $\hat{\mathbf{z}}=J \mathbf{z}$. A square matrix $A$ is persymmetric if $A^{T}=J A J$, and centrosymmetric if $A=J A J$.

In the following four propositions we state some properties of Hessenberg algebras which will be useful in the next section.

Proposition 2.1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n}$. Then
(i) $H_{X}\left(H_{X}(\mathbf{x})^{T} \mathbf{y}\right)=H_{X}(\mathbf{y}) H_{X}(\mathbf{x})$;
(ii) $\mathbf{x}^{T} H_{X}(\mathbf{y})=\mathbf{y}^{T} H_{X}(\mathbf{x})$;
(iii) if $X$ is persymmetric, then $H_{X}(\mathbf{x}) \mathbf{y}=H_{X}(\hat{\mathbf{y}}) \hat{\mathbf{x}}$.

Proof. (i): For $i=1, \ldots, n$,

$$
\begin{aligned}
\mathbf{e}_{i}^{T} H_{X}\left(H_{X}(\mathbf{x})^{T} \mathbf{y}\right) & =\mathbf{e}_{i}^{T} \sum_{k=1}^{n}\left[H_{X}(\mathbf{x})^{T} \mathbf{y}\right]_{k} X_{k} \\
& =\sum_{k=1}^{n}\left[H_{X}(\mathbf{x})^{T} \mathbf{y}\right]_{k} \mathbf{e}_{k}^{T} X_{i}=\mathbf{y}^{T} H_{X}(\mathbf{x}) X_{i} \\
& =\mathbf{y}^{T} X_{i} H_{X}(\mathbf{x})=\sum_{k=1}^{n} y_{k} \mathbf{e}_{k}^{T} X_{i} H_{X}(\mathbf{x}) \\
& =\mathbf{e}_{i}^{T} \sum_{k=1}^{n} y_{k} X_{k} H_{X}(\mathbf{x})=\mathbf{e}_{i}^{T} H_{X}(\mathbf{y}) H_{X}(\mathbf{x}) .
\end{aligned}
$$

(ii) is a consequence of (i) and of the commutativity of $H_{X}$, and (iii) follows from (ii).

Proposition 2.2. Let $A \in H_{X}$ ( $X$ in (2.1)). Then $A$ is invertible in $\mathbf{M}_{n}(\mathbb{C})$ if and only if there exists $\mathbf{z} \in \mathbb{C}^{n}$ such that $\mathbf{z}^{T} A=\mathbf{e}_{1}^{T}$. In this case $A^{-1}=H_{X}(\mathbf{z})$.

Proof. Let $\mathbf{z}$ be such that $\mathbf{z}^{T} A=\mathbf{e}_{1}^{T}$, and consider the matrix $H_{X}(\mathbf{z})$. Then observe that $\mathbf{e}_{i}^{T} H_{X}(\mathbf{z}) A=\mathbf{z}^{T} X_{i} A=\mathbf{z}^{T} A X_{i}=\mathbf{e}_{1}^{T} X_{i}=\mathbf{e}_{i}^{T}, i=1, \ldots, n$. Thus $H_{X}(\mathbf{z}) A=A H_{X}(\mathbf{z})=I$, that is, $A$ is invertible in $\mathbf{M}_{n}(\mathbb{C})$. The converse is obvious.

In the following two propositions we assume $X$ tridiagonal and, for the sake of simplicity, we set $a_{i}=r_{i i}, i=1, \ldots, n$, and $c_{i}=r_{i+1, i}, i=$ $1, \ldots, n-1$.

Proposition 2.3. Let $X$ in (2.1) be tridiagonal. Then
(i) $\operatorname{det} X_{n} \neq 0$ if and only if $c_{i} \neq 0$ for all $i$;
(ii) if $\operatorname{det} X_{n} \neq 0$ and $X$ is persymmetric, then $X_{n}^{-1}=\left(1 /\left[X_{n}\right]_{n 1}\right) X_{n}$;
(iii) $X$ is centrosymmetric if and only if $X_{n}=J$.

Proof. We refer the reader to [21] and [18]. However, point (iii) simply follows from the equivalences

$$
J X=X J \quad \Leftrightarrow \quad J \in H_{X} \quad \Leftrightarrow \quad J=H_{X}\left(\mathbf{e}_{n}\right)=X_{n}
$$

Proposition 2.4. Let $X$ in (2.1) be tridiagonal. Then
(i) if $n$ is even and $X$ is centrosymmetric (or persymmetric with $c_{i} \neq 0$, $i=1, \ldots, n / 2-1$ ), then $\operatorname{det} X_{n / 2} \neq 0$;
(ii) if $n$ is odd and $X$ is centrosymmetric (or persymmetric), then $\operatorname{det} X_{(n+1) / 2}=0$.

Proof. Let us prove assertions (i) and (ii) in the centrosymmetric case. For (ii) simply observe that point (iii) of Proposition 2.3 and the equalities (2.4) imply

$$
X_{(n+1) / 2} J=X_{(n-1) / 2} X_{n}=X_{(n+1) / 2} .
$$

As regards (i), by (2.4) we know that $X_{i} X_{2}=\sum_{k=1}^{n}\left[X_{2}\right]_{i k} X_{k}$. Exploiting this equality for $i=n / 2, n / 2-1, \ldots, 2$, and the equality $X_{n / 2+1}=X_{n / 2} J$, we obtain $X_{n / 2} Q_{n / 2+1-i}=X_{i-1}, i=n / 2, n / 2-1, \ldots, 2$, where the $Q_{i}$ 's are
polynomials in $X$ defined as follows:

$$
\begin{aligned}
& \quad Q_{0}=I, \quad Q_{1}=b_{n / 2+1}^{-1} b_{1}\left[X_{2}-b_{1}^{-1}\left(a_{n / 2}-a_{1}\right) I-b_{1}^{-1} b_{n / 2} J\right], \\
& Q_{i+1}=b_{n / 2+i+1}^{-1} b_{1}\left\{Q_{i}\left[X_{2}-b_{1}^{-1}\left(a_{n / 2-i}-a_{1}\right) I\right]-b_{1}^{-1} b_{n / 2-i} Q_{i-1}\right\}, \\
& i=1, \ldots, n / 2-2 . \text { In particular } X_{n / 2} Q_{n / 2-1}=X_{1}=I . \\
& \text { The proof of assertions (i) and (ii) in the persymmetric case is omitted. }
\end{aligned}
$$

The assertions of Proposition 2.4 are false in the symmetric case. Consider the $6 \times 6$ tridiagonal matrix $X$ with $b_{i}=c_{i}=1, i=1, \ldots, 5, a_{i}=0, i \neq 5$, and $a_{5}=1$. By using the definition (2.3), one can easily calculate the matrix $X_{3}$ and see that det $X_{3}=0$. Again, consider the $n \times n$ tridiagonal matrix $X$, $n$ odd, with $b_{i}=c_{i}=1, a_{i}=0, i=1, \ldots, n-1$, and $a_{n}=1$. It can be easily shown that a matrix $A=\left(a_{i j}\right) \in H_{X}$ (or equivalently $A X=X A$ ) if and only if its entries satisfy the condition

$$
\begin{equation*}
a_{i-1, j}+a_{i+1, j}=a_{i, j-1}+a_{i, j+1}, \quad 1 \leqslant i, j \leqslant n \tag{2.6}
\end{equation*}
$$

where $a_{0, i}=a_{i, 0}=0, a_{i, n+1}=a_{n+1, i}=a_{i, n}\left(=a_{n, i}\right), i=1, \ldots, n$. By using this fact and the identity $X_{(n+1) / 2}=H_{X}\left(\mathbf{e}_{(n+1) / 2}\right)$, we see that

Finally, the equalities

$$
\begin{aligned}
& \left(-\mathbf{e}_{(n-3) / 2}^{T}+\mathbf{e}_{(n-1) / 2}^{T}+\mathbf{e}_{(n+1) / 2}^{T}-\mathbf{e}_{(n+3) / 2}^{T}\right) X_{(n+1) / 2} \\
& =-\sum_{k=2}^{(n-1) / 2} \mathbf{e}_{2 k-1}^{T}+\sum_{k=1}^{(n-1) / 2} \mathbf{e}_{2 k}^{T}+\sum_{k=1}^{(n+1) / 2} \mathbf{e}_{2 k-1}^{T} \\
& \quad-\left(\sum_{k=1}^{(n-1) / 2} \mathbf{e}_{2 k}^{T}+\mathbf{e}_{n}^{T}\right)=\mathbf{e}_{1}^{T}
\end{aligned}
$$

and Proposition 2.2 prove that $\operatorname{det} X_{(n+1) / 2} \neq 0$.
Let $P_{\varepsilon}$ be the $n \times n$ matrix

$$
P_{\varepsilon}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0  \tag{2.8}\\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
0 & & & \ddots & 1 \\
\varepsilon & 0 & 0 & \cdots & 0
\end{array}\right), \quad \varepsilon \in \mathbb{C}
$$

and consider the persymmetric Hessenberg algebra $C_{\varepsilon}=H_{P_{\varepsilon}}$. The space $C_{\varepsilon}$, which is known as the algebra of $\varepsilon$-circulant matrices [19], is spanned by the matrices $\left(P_{\varepsilon}\right)_{i}=P_{\varepsilon}^{i-1}$ defined in (2.3). The matrices of $C_{\varepsilon}$ are simultaneously diagonalizable, that is, for $\mathbf{z} \in \mathbb{C}^{n}$,

$$
\begin{equation*}
F^{-1} D_{\mu}^{-1} C_{\varepsilon}(\mathbf{z}) D_{\mu} F=\sqrt{n} \operatorname{diag}\left(\mathbf{e}_{i}^{T} F D_{\mu} \mathbf{z}, i=1, \ldots, n\right) \tag{2.9}
\end{equation*}
$$

where $[F]_{i j}=(1 / \sqrt{n}) \omega^{(i-1)(j-1)}, i, j=1, \ldots, n, \omega=\exp (-\mathrm{i} 2 \pi / n), \quad \mathbf{i}=$ $\sqrt{-1}$, and $D_{\mu}=\operatorname{diag}\left(\mu^{i-1}, i=1, \ldots, n\right), \mu=\sqrt[n]{\varepsilon}$. The matrix $F$ is known as the Fourier matrix, and the linear transform $F \mathbf{z}, \mathbf{z} \in \mathbb{C}^{n}$, as the discrete Fourier transform (DFT) of $\mathbf{z}$. It is known that if $n$ is a highly composite integer, then the DFT of $\mathbf{z}$ can be efficiently computed with $O(n \log n)$ arithmetic operations (FFT) [32, 15].

The spaces $C_{1}(\varepsilon=1), C_{-1}(\varepsilon=-1)$, and $C_{0}(\varepsilon=0)$ are known as the algebras of circulant, skew-circulant, and upper triangular Toeplitz matrices, respectively. For the sake of simplicity we set $C=C_{1}, P=P_{1}, P_{i}=C\left(\mathbf{e}_{i}\right)=$ $P^{i-1}$, and $Z=P_{0}^{T}$.

Let $T_{\varepsilon, \varphi}$ be the $n \times n$ matrix

$$
T_{\varepsilon, \varphi}=\left(\begin{array}{ccccc}
\varepsilon & 1 & 0 & \cdots & 0  \tag{2.10}\\
1 & 0 & 1 & \ddots & \vdots \\
0 & 1 & & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & 1 \\
0 & \cdots & 0 & 1 & \varphi
\end{array}\right), \quad \varepsilon, \varphi \in \mathbb{C}
$$

and consider the symmetric Hessenberg algebra $\tau_{\varepsilon, \varphi}=H_{T_{e, \varphi}}$. The space $\tau_{\varepsilon, \varphi}$ is spanned by the matrices $\left(T_{\varepsilon, \varphi}\right)_{i}$ defined in (2.3). The matrices of $\tau_{\varepsilon, \varphi}$ are obviously simultaneously diagonalizable. Moreover, if $\varepsilon, \varphi \in\{1,-1\}, \tau_{\varepsilon, \varphi}$ can be reduced to diagonal form by means of fast discrete trigonometric transforms (DTT) [18, 36].

Consider, in particular, the space $\tau_{0,0}=\tau$. It is easily verified that a matrix $A \in \tau$ if and only if its entries $a_{i j}$ satisfy the condition (2.6) where $a_{0, i}=a_{n+1, i}=a_{i, 0}=a_{i, n+1}=0, i=1, \ldots, n$. The condition (2.6) can be used to investigate the structure of $T_{i}=\left(T_{0,0}\right)_{i}$, which is nothing but the matrix $\tau\left(\mathbf{e}_{i}\right)$ of $\tau$ whose first row is $\mathbf{e}_{i}^{T}$. Moreover, for $\mathbf{z} \in \mathbb{C}^{n}$,

$$
\begin{equation*}
S \tau(\mathbf{z}) S=\sqrt{\frac{n+1}{2}} \operatorname{diag}\left(\left(\sin \frac{i \pi}{n+1}\right)^{-1} \mathbf{e}_{i}^{T} S_{\mathbf{z}}, i=1, \ldots, n\right) \tag{2.11}
\end{equation*}
$$

where $[S]_{i j}=\sqrt{2 /(n+1)} \sin [i j \pi /(n+1)], \quad i, j=1, \ldots, n \quad\left(S^{-1}=S=\right.$ $S^{T}$ ). The linear transform $S \mathbf{z}, \mathbf{z} \in \mathbb{C}^{n}$, is known as the discrete sine transform of $\mathbf{z}$. It can be efficiently computed in $O(n \log n)$ arithmetic operations if $n+1$ is a highly composite integer [32].

For the sake of simplicity set $\tau_{1,1}=\tau_{++}, \tau_{-1,-1}=\tau_{--}, \tau_{-1,1}=\tau_{-+}$, and $\tau_{1,-1}=\tau_{+-}$.

The algebras $\tau$ and $C_{\varepsilon}$ are often used with similar techniques in many problems in numerical linear algebra. See for example [8-12, 39] and the references in [12]. Also, the matrices of the algebras $C_{\varepsilon}$ and $\tau_{\varepsilon, \varphi}$ have been involved in several known decomposition formulas [7, 13, 14, 16-18, 21-26, 30, 31]. The most significant decomposition formulas of this paper exploit these same algebras.

The following notation will be used in the next sections. Set $I_{k}=$ $\left(\delta_{i j}\right)_{i, j=1, \ldots, k}$ and $J_{k}=\left(\delta_{i, k+1-j}\right)_{i, j=1, \ldots, k}$. Let $I_{j}^{i}, 1 \leqslant i, j \leqslant n$, denote the $(|j-i|+1) \times n(0,1)$ matrix which maps a vector $\mathbf{z}=\left[\begin{array}{lll}z_{1} & \cdots & z_{n}\end{array}\right]^{T} \in \mathbb{C}^{n}$ to the vector $I_{j}^{i} \mathbf{z}=\left[\begin{array}{lll}z_{i} & \cdots & z_{j}\end{array}\right]^{T} \in \mathbb{C}^{|j-i|+1}\left(I=I_{n}=I_{n}^{1}, J=J_{n}=I_{1}^{n}\right)$. For the sake of simplicity we will often use the symbols $I$ and $J$ instead of $I_{k}$ and $J_{k}$, and $\hat{\mathbf{z}}$ for $\mathbf{z} \in \mathbb{C}^{k}$ even if $k \neq n$.

## 3. DECOMPOSITION FORMULAS

In this section new matrix displacement decomposition formulas are introduced. These formulas involve factors which are block matrices (with blocks belonging to Hessenberg algebras) and, in particular, block-diagonal matrices. Under some restrictions the dimensions of the blocks will be arbitrary, so to regain-for a suitable choice of these dimensions-some known formulas considered in [21, 24]. In the most significant case, the blocks will have the same dimension $n / 2$ ( $n$ even).

If $U, V \in M_{n}(\mathbb{C})$, then it is known that $\sum_{k=1}^{n}[U V-V U]_{k k}=0$. As a direct consequence of this fact and of Proposition 2.1(ii), we obtain the following

Lemma 3.1. Assume that for an $n \times n$ matrix A there exist $2 \alpha$ vectors $\mathbf{x}_{m}=\left[\begin{array}{llll}x_{1}^{(m)} & \cdots & x_{n}^{(m)}\end{array}\right]^{T}, \mathbf{y}_{m}=\left[\begin{array}{lll}y_{1}^{(m)} & \cdots & y_{n}^{(m)}\end{array}\right]^{T}, m=1, \ldots, \alpha$, such that $\mathscr{E}_{X}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}\left[X\right.$ in (2.1)]. Then $\sum_{m=1}^{\alpha} \mathbf{x}_{m}^{T} H_{X}\left(\mathbf{y}_{m}\right)^{T}=\mathbf{0}^{T}$.

In the theorems below, $A$ is always a generic $n \times n$ matrix of $\mathbf{M}_{n}(\mathbb{C})$.
Let $X$ be a symmetric tridiagonal matrix of order $n$ :

$$
X=\left(\begin{array}{ccccc}
a_{1} & b_{1} & 0 & \cdots & 0  \tag{3.I}\\
b_{1} & a_{2} & b_{2} & \ddots & \vdots \\
0 & b_{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & b_{n-1} \\
0 & \cdots & 0 & b_{n-1} & a_{n}
\end{array}\right), \quad b_{i} \neq 0 \quad \forall i .
$$

Denote by $X^{\prime}$ and $X^{\prime \prime}$ the upper left $(i-1) \times(i-1)$ submatrix of $X$ and the lower right $(n-i) \times(n-i)$ submatrix of $X$, respectively.

Theorem 3.1. Let $X_{i}\left(X_{i}=H_{X}\left(\mathbf{e}_{i}\right)\right)$ be invertible in $\mathbf{M}_{n}(\mathbb{C})$. Then the equality $\mathbb{S}_{X}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}$ implies

$$
\begin{align*}
A & =\sum_{m=1}^{\alpha} \chi_{i}\left(\mathbf{x}_{m}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right)+H_{X}\left(X_{i}^{-1} A^{T} \mathbf{e}_{i}\right)  \tag{3.2}\\
& =-\sum_{m=1}^{\alpha} H_{X}\left(X_{i}^{-1} \mathbf{x}_{m}\right) \chi_{i}\left(\mathbf{y}_{m}\right)+H_{X}\left(X_{i}^{-1} A \mathbf{e}_{i}\right) \tag{3.3}
\end{align*}
$$

where, for $\mathbf{z} \in \mathbb{C}^{n}$,

$$
\begin{equation*}
\chi_{i}(\mathbf{z})=\left(\right) \tag{3.4}
\end{equation*}
$$

Proof. Let $Q_{j}=\mathbf{e}_{j+1} \mathbf{e}_{j}^{T}+\mathbf{e}_{j} \mathbf{e}_{j+1}^{T} \quad\left(Q_{0}=Q_{n}=0\right)$. Then, taking into account Proposition 2.1 and Lemma 3.1,

$$
\begin{aligned}
\mathfrak{C}_{X}\left(\sum_{m=1}^{\alpha} \chi_{i}\left(\mathbf{x}_{m}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right)\right) & =\sum_{m=1}^{\alpha} \mathfrak{C}_{X}\left(\chi_{i}\left(\mathbf{x}_{m}\right)\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right) \\
& =\sum_{m=1}^{\alpha} \mathfrak{S}_{b_{i-1} Q_{i-1}+b_{i} Q_{i}}\left(X_{i}\left(\mathbf{x}_{m}\right)\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right) \\
& =\sum_{m=1}^{\alpha}\left(\mathbf{x}_{m} \mathbf{e}_{i}^{T}-\mathbf{e}_{i} \mathbf{x}_{m}^{T}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right) \\
& =\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}-\mathbf{e}_{i} \sum_{m=1}^{\alpha} \mathbf{x}_{m}^{T} H_{X}\left(\mathbf{y}_{m}\right) X_{i}^{-1} \\
& =\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}=\mathbb{C}_{X}(A)
\end{aligned}
$$

Thus $A-\sum_{m=1}^{\alpha} \chi_{i}\left(\mathbf{x}_{m}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right) \in \operatorname{Ker} \mathfrak{C}_{X}=H_{\chi}$. As $\mathbf{e}_{i}^{T} \chi_{i}\left(\mathbf{x}_{m}\right)=\mathbf{0}^{T}$, $m=1, \ldots, \boldsymbol{\alpha}$, we have (3.2). Exploiting (3.2) and the equality $\mathfrak{C}_{X}\left(A^{T}\right)=$ $-\mathfrak{C}_{X}(A)^{T}$, we have also (3.3).

As $X_{1}$ and $X_{n}$ are invertible in $\mathbf{M}_{n}(\mathbb{C})$ (see Proposition 2.3), the formulas (3.2) and (3.3) hold for $i=1$ and $i=n$, and we retrieve a result of [21]. If $X$ is also persymmetric, then the matrices $X_{n / 2}$ and $X_{n / 2+1}=J X_{n / 2}$ ( $n$ even) are nonsingular, and therefore (3.2) and (3.3) also hold for $i=n / 2$ and $i=n / 2+1$. On the contrary, they do not hold for $i=(n+1) / 2(n$ odd $)$, since $\operatorname{det} X_{(n+1) / 2}=0$. However, (3.2) and (3.3) always hold for $i=(n+$ 1)/2 ( $n$ odd) when $X=T_{2}+\mathbf{e}_{n} \mathbf{e}_{n}^{T}$. In fact, in this case, $\operatorname{det} X_{(n+1) / 2} \neq 0$.
(See Proposition 2.4 and related remarks.) Finally, notice that Proposition 2.1(i) allows us to take the matrix $X_{i}^{-1}$ out of the arguments in (3.2) and (3.3).

Let $X$ be a symmetric and persymmetric tridiagonal matrix of order $n$,

$$
X=\left(\begin{array}{ccccc}
a_{1} & b_{1} & 0 & \cdots & 0  \tag{3.5}\\
b_{1} & a_{2} & b_{2} & \ddots & \vdots \\
0 & b_{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & a_{2} & b_{1} \\
0 & \cdots & 0 & b_{1} & a_{1}
\end{array}\right), \quad b_{j} \neq 0 \quad \forall j,
$$

and fix a number $i$ such that

$$
\begin{array}{ll}
\text { for } n \text { even } & 1 \leqslant i \leqslant \frac{n}{2}+1 \\
\text { for } n \text { odd } & 1 \leqslant i \leqslant \frac{n+1}{2} . \tag{3.6}
\end{array}
$$

Denote by $X^{\prime}$ the lower right $(i-1) \times(i-1)$ submatrix of $X$, and by $X^{\prime \prime}$ the $(n-2 i) \times(n-2 i)$ submatrix of $X$ obtained by deleting its first $i$ rows and columns and its last $i$ rows and columns. (Observe that there are choices of $i$ for which either $X^{\prime}$ or $X^{\prime \prime}$ disappears.)

Theorem 3.2. Let $i$ satisfy (3.6), and let $X, X^{\prime}$, and $X^{\prime \prime}$ be defined as above. Let $X_{i}\left(X_{i}=H_{X}\left(\mathbf{e}_{i}\right)\right)$ be invertible in $\mathbf{M}_{n}(\mathbb{C})$. Then the equality $\mathbb{C}_{X}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}$ implies

$$
\begin{align*}
A+J A J & =\sum_{m=1}^{\alpha} \chi_{i}\left(\mathbf{x}_{m}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right)+H_{X}\left(X_{i}^{-1}(A+J A J)^{T} \mathbf{e}_{i}\right)  \tag{3.7}\\
& =-\sum_{m=1}^{\alpha} H_{X}\left(X_{i}^{-1} \mathbf{x}_{m}\right) \chi_{i}\left(\mathbf{y}_{m}\right)+H_{X}\left(X_{i}^{-1}(A+J A J) \mathbf{e}_{i}\right), \tag{3.8}
\end{align*}
$$

where, for $\mathbf{z} \in \mathbb{C}^{n}$,


Proof. Let $Q_{i}=\mathbf{e}_{i+1} \mathbf{e}_{i}^{T}+\mathbf{e}_{i} \mathbf{e}_{i+1}^{T} \quad\left(Q_{0}=0\right)$. For every choice of $i$ satisfying (3.6) $X$ can be rewritten as

$$
X=\left(\begin{array}{ccccccc} 
& & 0 & & 0 & & \\
& J X^{\prime} J & \vdots & 0 & \vdots & 0 & \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
& 0 & \vdots & X^{\prime \prime} & \vdots & 0 & \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
& 0 & \vdots & 0 & \vdots & X^{\prime} & \\
& & 0 & & 0 & &
\end{array}\right)+K
$$

where $K$ is defined as follows:
(1) $n$ even and $1 \leqslant i \leqslant n / 2-1$ or $n$ odd and $1 \leqslant i \leqslant(n-1) / 2 \Rightarrow$

$$
K=b_{i-1}\left(Q_{i-1}+J Q_{i-1} J\right)+a_{i}\left(\mathbf{e}_{i} \mathbf{e}_{i}^{T}+J \mathbf{e}_{i} \mathbf{e}_{\mathbf{i}}^{T} J\right)+b_{i}\left(Q_{i}+J Q_{i} J\right)
$$

(2) $n$ even and $i=n / 2 \Rightarrow$

$$
\begin{aligned}
K= & b_{n / 2-1}\left(Q_{n / 2-1}+J Q_{n / 2-1} J\right) \\
& +a_{n / 2}\left(\mathbf{e}_{n / 2} \mathbf{e}_{n / 2}^{T}+J \mathbf{e}_{n / 2} \mathbf{e}_{n / 2}^{T} J\right)+b_{n / 2} Q_{n / 2}
\end{aligned}
$$

(3) $n$ odd and $i=(n+1) / 2 \Rightarrow$

$$
K=b_{(n-1) / 2}\left(Q_{(n-1) / 2}+J Q_{(n-1) / 2} J\right)+a_{(n+1) / 2} \mathbf{e}_{(n+1) / 2} \mathbf{e}_{(n+1) / 2}^{T} ;
$$

(4) $n$ even and $i=n / 2+1 \Rightarrow K=b_{n / 2} Q_{n / 2}$.

By considering cases (1), (2), (3), and (4) one at a time, we realize that the identities

$$
\mathfrak{C}_{X}\left(\chi_{i}\left(\mathbf{x}_{m}\right)\right)=\mathfrak{C}_{K}\left(\chi_{i}\left(\mathbf{x}_{m}\right)\right)=\mathbf{x}_{m} \mathbf{e}_{i}^{T}-\mathbf{e}_{i} \mathbf{x}_{m}^{T}+J\left(\mathbf{x}_{m} \mathbf{e}_{i}^{T}-\mathbf{e}_{i} \mathbf{x}_{m}^{T}\right) J
$$

hold for all $i$ satisfying (3.6). Then, by using the assumption det $X_{i} \neq 0$, Proposition 2.1, and Lemma 3.1, we have

$$
\begin{aligned}
\mathfrak{S}_{X}\left(\sum_{m=1}^{\alpha} \chi_{i}\left(\mathbf{x}_{m}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right)\right) & =\sum_{m=1}^{\alpha}\left(5_{X}\left(\chi_{i}\left(\mathbf{x}_{m}\right)\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right)\right. \\
& =\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}+J \sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T} J \\
& =\mathfrak{S}_{X}(A+J A J) .
\end{aligned}
$$

Thus $A+J A J-\sum_{m=1}^{\alpha} \chi_{i}\left(\mathbf{x}_{m}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right) \in \operatorname{Ker} \mathbb{C}_{X}=H_{X} . A s$

$$
\mathbf{e}_{i}^{T} \sum_{m=1}^{\alpha} X_{i}\left(\mathbf{x}_{m}\right) H_{X}\left(X_{i}^{-1} \mathbf{y}_{m}\right)=\mathbf{0}^{T}
$$

for all $i$ satisfying (3.6), including the case $i=n / 2+1$, we have (3.7). Exploiting (3.7) and the equality $\mathbb{C}_{X}\left(A^{T}\right)=-\mathfrak{C}_{X}(A)^{T}$, we have also (3.8).

The formulas (3.7) and (3.8) obviously hold for $i=1$. In this case we retrieve a result of [21]. Moreover by Proposition 2.4 they hold for $i=n / 2$ and $i=n / 2+1$ ( $n$ even), and they do not hold for $i=(n+1) / 2(n$ odd $)$. As in Theorem 3.1, one can take the matrix $X_{i}^{-1}$ out of the arguments in the decomposition formulas (3.7) and (3.8). Finally observe that if $X^{\prime}=$ $J_{i-1} X^{\prime} J_{i-1}$, then
where

$$
I_{\#}=\left(\begin{array}{ccccccc} 
& & 0 & & 0 & &  \tag{3.10}\\
& \frac{1}{\sqrt{2}} I_{i-1} & \vdots & 0 & \vdots & \frac{1}{\sqrt{2}} I_{i-1} & \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
& 0 & \vdots & I_{n-2 i} & \vdots & 0 & \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
& \frac{1}{\sqrt{2}} I_{i-1} & \vdots & 0 & \vdots & \frac{-1}{\sqrt{2}} I_{i-1} & \\
& & 0 & & 0 & &
\end{array}\right) .
$$

Now we rewrite the Theorems 3.1 and 3.2 for $X=T_{0,0}=T_{2}$ (2.10). Some of the advantages of this choice are the following:
(1) It is easy to check the nonsingularity of $T_{i}$, and if $n$ is even, the $T_{i}$ 's are mostly invertible. Moreover, the matrix by vector product $T_{i}^{-1} \mathbf{z}$ can be calculated in some cases with only $O(n)$ additive operations.
(2) The corresponding decomposition formulas are always in terms of the same algebra $\tau$.
(3) The consequent formulas for the inverse of a Toeplitz or a Toeplitz plus Hankel matrix (see the next section) are computationally efficient because of the low computational cost of the product of a $\tau$ matrix by a vector [see (2.11)].

Corollary 3.1. If $\operatorname{det} T_{i} \neq 0$ and $\mathfrak{S}_{T_{2}}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}$, then

$$
\begin{align*}
A & =\sum_{m=1}^{\alpha} \tau_{\backslash}^{i}\left(\mathbf{x}_{m}\right) \tau\left(T_{i}^{-1} \mathbf{y}_{m}\right)+\tau\left(T_{i}^{-1} A^{T} \mathbf{e}_{i}\right)  \tag{3.11}\\
& =-\sum_{m=1}^{\alpha} \tau\left(T_{i}^{-1} \mathbf{x}_{m}\right) \tau_{\backslash}^{i}\left(\mathbf{y}_{m}\right)+\tau\left(T_{i}^{-1} A \mathbf{e}_{i}\right) \tag{3.12}
\end{align*}
$$

where, for $\mathbf{z} \in \mathbb{C}^{n}$,

$$
\tau^{i}(\mathbf{z})=\left(\begin{array}{ccccc} 
& & 0 & &  \tag{3.13}\\
& \tau\left(I_{1}^{i-1} \mathbf{z}\right) & \vdots & 0 & \\
0 & \ldots & 0 & \cdots & 0 \\
& 0 & \vdots & \tau\left(I_{n}^{i+1} \mathbf{z}\right) &
\end{array}\right)
$$

Corollary 3.2. Let i satisfy (3.6), and let $T_{i}$ be invertible in $\mathbf{M}_{n}(\mathbb{C})$. Then the equality $\mathfrak{C}_{T_{2}}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}$ implies

$$
\begin{align*}
A+J A J & =\sum_{m=1}^{\alpha} \tau_{\times}^{i}\left(\mathbf{x}_{m}\right) \tau\left(T_{i}^{-1} \mathbf{y}_{m}\right)+\tau\left(T_{i}^{-1}(A+J A J)^{T} \mathbf{e}_{i}\right)  \tag{3.14}\\
& =-\sum_{m=1}^{\alpha} \tau\left(T_{i}^{-1} \mathbf{x}_{m}\right) \tau_{\times}^{i}\left(\mathbf{y}_{m}\right)+\tau\left(T_{i}^{-1}(A+J A J) \mathbf{e}_{i}\right) \tag{3.15}
\end{align*}
$$

where, for $\mathbf{z} \in \mathbb{C}^{n}$,

$$
\tau_{\times}^{i}(\mathbf{z})=\left(\begin{array}{ccccccc} 
& & 0 & & 0 & &  \tag{3.16}\\
& \tau\left(I_{1}^{i-1} \mathbf{z}\right) & \vdots & 0 & \vdots & \tau\left(I_{n-i+2}^{n} \mathbf{z}\right) & \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
& 0 & \vdots & \tau\left(I_{n-i}^{i+1} \mathbf{z}\right) & \vdots & 0 & \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
& \tau\left(I_{n-i+2}^{n} \mathbf{z}\right) & \vdots & 0 & \vdots & \tau\left(I_{1}^{i-1} \mathbf{z}\right) & \\
& & 0 & & 0 & &
\end{array}\right)
$$

Moreover observe that $\tau_{\times}^{i}(\mathbf{z})=I_{\#} \tau_{\backslash}^{i}(\mathbf{z}) I_{\#}$, where $I_{\#}$ is defined in (3.10), and

$$
\tau_{\backslash( }^{i}(\mathbf{z})=\left(\begin{array}{ccccccc} 
& & 0 & & 0 & &  \tag{3.17}\\
& \tau\left(\left(I_{1}^{i-1}+I_{n-i+2}^{n}\right) \mathbf{z}\right) & \vdots & 0 & \vdots & 0 & \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
0 & 0 & \vdots & \tau\left(I_{n-1}^{i+1} \mathbf{z}\right) & \vdots & 0 & \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
& 0 & \vdots & 0 & \vdots & \tau\left(\left(I_{1}^{i-1}-I_{n-i+2}^{n}\right) \mathbf{z}\right) &
\end{array}\right) .
$$

Notice that if $n+1$ is a highly composite integer, it would be desirable to have a representation of $A$ in terms of size $n$ and size $(n-1) / 2 \tau$ matrices. Unfortunately, the choice $i=(n+1) / 2$ in Corollaries 3.1 and 3.2 , which would yield such formulas, is not possible, because $\operatorname{det} T_{(n+1) / 2}=0$.

In the next theorem we wish to state two decomposition formulas involving $\varepsilon$-circulant matrices. In particular, if $n$ is even, the sizes of the $\varepsilon$-circulant matrices involved can be $n$ and $n / 2$.

Let $\mathbf{p}$ and $\mathbf{q}$ be two arbitrary vectors of order $i$ and $n-i$, respectively, and let $\xi, \eta$, and $\delta$ be arbitrary complex numbers. Set $K=\mathbf{e}_{i} \mathbf{e}_{i+1}^{T}+$ $\delta \mathbf{e}_{n} \mathbf{e}_{1}^{T}-\xi \mathbf{e}_{i} \mathbf{e}_{1}^{T}-\eta \mathbf{e}_{n} \mathbf{e}_{i+1}^{T}$. Then

$$
\begin{align*}
\mathfrak{C}_{P_{\delta}}\left(\left(\begin{array}{cc}
C_{\xi}(\mathbf{p}) & 0 \\
0 & C_{\eta}(\mathbf{q})
\end{array}\right)\right)= & \mathfrak{C}_{K}\left(\left(\begin{array}{cc}
C_{\xi}(\mathbf{p}) & 0 \\
0 & C_{\eta}(\mathbf{q})
\end{array}\right)\right) \\
= & \binom{\hat{\mathbf{p}}}{-\eta \hat{\mathbf{q}}} \mathbf{e}_{i+1}^{T}+\binom{-\xi \hat{\mathbf{p}}}{\delta \hat{\mathbf{q}}} \mathbf{e}_{1}^{T} \\
& +\mathbf{e}_{i}\left(\xi \mathbf{p}^{T} \mid-\mathbf{q}^{T}\right)+\mathbf{e}_{n}\left(-\delta \mathbf{p}^{T} \mid \boldsymbol{\eta} \mathbf{q}^{T}\right) \tag{3.18}
\end{align*}
$$

For $\delta=\xi \eta$, (3.18) becomes

$$
\binom{\hat{\mathbf{p}}}{-\eta \hat{\mathbf{q}}}\left(\mathbf{e}_{i+1}^{T}-\xi \mathbf{e}_{1}^{T}\right)+\left(\mathbf{e}_{i}-\eta \mathbf{e}_{n}\right)\left(\xi \mathbf{p}^{T} \mid-\mathbf{q}^{T}\right)
$$

Moreover, for an arbitrary vector $\mathbf{r}$ of order $n$,

$$
\begin{align*}
\mathscr{C}_{P_{\xi \eta}}\left(\left(\begin{array}{cc}
C_{\xi}(\mathbf{p}) & 0 \\
0 & C_{\eta}(\mathbf{q})
\end{array}\right) C_{\xi \eta}(\mathbf{r})\right)= & \mathscr{C}_{P_{\xi \eta}}\left(\left(\begin{array}{cc}
C_{\xi}(\mathbf{p}) & 0 \\
0 & C_{\eta}(\mathbf{q})
\end{array}\right)\right) C_{\xi \eta}(\mathbf{r}) \\
= & \binom{\hat{\mathbf{p}}}{-\eta \hat{\mathbf{q}}} \mathbf{r}^{T}\left[\left(P_{\xi \eta}\right)_{i+1}-\xi I\right] \\
& +\left(\mathbf{e}_{i}-\eta \mathbf{e}_{n}\right)\left(\xi \mathbf{p}^{T} \mid-\mathbf{q}^{T}\right) C_{\xi \eta}(\mathbf{r}) \tag{3.19}
\end{align*}
$$

Assume $\mathfrak{C}_{P_{57}}(A)=\mathbf{x y}{ }^{T}$, where $\mathbf{x}$ and $\mathbf{y}$ are two vectors of order $n$, and choose $\mathbf{p}, \mathbf{q},{ }_{\text {fnd }} \mathbf{r}$ such that $\hat{\mathbf{p}}=I_{i}^{1} \mathbf{x},-\eta \hat{\mathbf{q}}=I_{n}^{i+1} \mathbf{x}$, and $\mathbf{r}^{T}\left[\left(P_{\xi \eta}\right)_{i+1}-\right.$ $\xi I]=\mathbf{y}^{T}$. If $\eta \neq 0$ and $\xi$ is not an eigenvalue of $\left(P_{\xi \eta}\right)_{i+1}$, then this choice is
possible and (3.19) becomes

$$
\begin{aligned}
\mathbf{x} \mathbf{y}^{T} & +\left(\mathbf{e}_{i}-\eta \mathbf{e}_{n}\right) \frac{1}{\eta} \mathbf{x}^{T}\left(P_{\xi \eta}\right)_{i+1}^{T} J C_{\xi \eta}\left(\left[\left(P_{\xi \eta}\right)_{i+1}^{T}-\xi I\right]^{-1} \mathbf{y}\right) \\
& =\mathbf{x y}^{T}+\left(\mathbf{e}_{i}-\eta \mathbf{e}_{n}\right) \frac{1}{\eta} \mathbf{x}^{T} C_{\xi \eta}(\mathbf{y})^{T}\left(P_{\xi \eta}\right)_{i+1}^{T}\left[\left(P_{\xi \eta}\right)_{i+1}^{T}-\xi I\right]^{-1} J=\mathbf{x y}^{T} .
\end{aligned}
$$

The last equality follows from Lemma 3.1.
Now we can state the following
Theorem 3.3. Let $\xi, \eta$ be complex numbers and $i \in\{0, \ldots, n-1\}$. Assume $\mathfrak{C}_{p_{\xi \eta}}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}$. If $\operatorname{det}\left[\left(P_{\xi \eta}\right)_{i+1}-\xi I\right] \neq 0$ and $\eta \neq 0$, then

$$
\begin{align*}
A= & \left(\sum_{m=1}^{\alpha} \phi_{i}\left(\mathbf{x}_{m}\right) C_{\xi \eta}\left(\mathbf{y}_{m}\right)+C_{\xi \eta}\left(A^{T}\left(\mathbf{e}_{i+1}-\xi \mathbf{e}_{1}\right)\right)\right)\left[\left(P_{\xi \eta}\right)_{i+1}-\xi I\right]^{-1}  \tag{3.20}\\
= & {\left[\left(P_{\xi \eta}\right)_{i+1}-\xi I\right]^{-1} } \\
& \times\left(-\sum_{m=1}^{\alpha} C_{\xi \eta}\left(\hat{\mathbf{x}}_{m}\right) J \phi_{i}\left(\hat{\mathbf{y}}_{m}\right)^{T} J+C_{\xi \eta}\left(J A\left(\mathbf{e}_{n-i}-\xi \mathbf{e}_{n}\right)\right)\right) \tag{3.21}
\end{align*}
$$

where, for $\mathbf{z} \in \mathbb{C}^{n}$,

$$
\phi_{i}(\mathbf{z})=\left(\begin{array}{cc}
C_{\xi}\left(I_{1}^{i} \mathbf{z}\right) & 0 \\
0 & -\frac{1}{\eta} C_{\eta}\left(I_{i+1}^{n} \mathbf{z}\right)
\end{array}\right)
$$

Proof. Let us prove (3.20). First observe that the previous argument can be extended to the case where $\mathfrak{C}_{P_{E_{7}}}(A)$ is the sum of more than one dyad. That is, under the assumption $\int_{P_{\xi \eta}}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}$, the matrix

$$
A-\sum_{m=1}^{\alpha} \phi_{i}\left(\mathbf{x}_{m}\right) C_{\xi \eta}\left(\left[\left(P_{\xi \eta}\right)_{i+1}^{T}-\xi I\right]^{-1} \mathbf{y}_{m}\right)
$$

commutes with $P_{\xi \eta}$ (the above proof does not include the case $i=0$, which is however easily verified). Thus it is an element of $C_{\xi \eta}$, say $C_{\xi \eta}(\mathbf{z})$. As $\left(\xi \mathbf{e}_{1}^{T}-\mathbf{e}_{i+1}^{T}\right) \sum_{m=1}^{\alpha} \phi_{i}\left(\mathbf{x}_{m}\right) C_{\xi \eta}\left(\left[\left(P_{\xi \eta}\right)_{i+1}^{T}-\xi I\right]^{-1} \mathbf{y}_{m}\right)=\mathbf{0}^{T}$ (see Proposition 2.1 and Lemma 3.1), $\mathbf{z}$ is necessarily the vector $\left(\left(P_{\xi \eta}\right)_{i+1}^{T}-\xi I\right)^{-1} A^{T}\left(\mathbf{e}_{i+1}-\right.$ $\xi \mathbf{e}_{1}$ ). By point (i) of Proposition 2.1, we have the thesis. The proof of (3.21) is left to the reader.

In the case $i=0$, if $\eta \neq 0$ and $\xi \neq 1$, Theorem 3.3 yields two formulas where all the $\varepsilon$-circulant matrices involved are $n \times n$ matrices. This type of formula is studied in [24] and retrieved in [21] as a consequence of more general decompositions. In the case $i=n / 2$ ( $n$ even), if $\xi \eta(\eta-\xi) \neq 0$, then the formulas (3.20) and (3.21) hold with $\left[\left(P_{\xi \eta}\right)_{n / 2+1}-\xi I\right]^{-1}=$ $[1 / \xi(\eta-\xi)]\left[\left(P_{\xi \eta}\right)_{n / 2+1}+\xi I\right]$. In other words, there exist displacement decomposition formulas where only $\varepsilon$-circulant matrices of order $n$ and $n / 2$ are used.

Now we rewrite Theorem 3.3 for $\xi=-1$ and $\eta=1$ (analogously we could rewrite it for $\xi=1$ and $\eta=-1$ ). In this case we can translate the hypothesis of Theorem 3.3 into a set of explicit values of $i$. In fact, the matrix $\left(P_{-1}\right)_{i+1}+I, i \in\{0, \ldots, n-1\}$, is singular if and only if -1 is an eigenvalue of $\left(P_{-1}\right)_{i+1}=P_{-1}^{i}$. As the eigenvalues of $P_{-1}$ are

$$
\exp \left(\mathrm{i} \frac{2 k+1}{n} \pi\right), \quad k=0, \ldots, n-1
$$

this happens if and only if there exist an integer $s$ and $k \in\{0, \ldots, n-1\}$ such that $i=n(2 s+1) /(2 k+1)$.

Corollary 3.3. Let $i \in\{0, \ldots, n-1\} \backslash\{n(2 s+1) /(2 k+1): k=$ $1, \ldots, n-1, s=0,1, \ldots\}$. If $\mathfrak{C}_{P_{-1}}(A)=\sum_{m=1}^{\alpha} \mathbf{x}_{m} \mathbf{y}_{m}^{T}$, then

$$
\begin{equation*}
A=\left(\sum_{m=1}^{\alpha} \phi_{i}\left(\mathbf{x}_{m}\right) C_{-1}\left(\mathbf{y}_{m}\right)+C_{-1}\left(A^{T}\left(\mathbf{e}_{i+1}+\mathbf{e}_{1}\right)\right)\right)\left[\left(P_{-1}\right)_{i+1}+I\right]^{-1} \tag{3.22}
\end{equation*}
$$

$$
\begin{align*}
= & {\left[\left(P_{-1}\right)_{i+1}+I\right]^{-1} } \\
& \times\left(-\sum_{m=1}^{\alpha} C_{-1}\left(\hat{\mathbf{x}}_{m}\right) J \phi_{i}\left(\hat{\mathbf{y}}_{m}\right)^{T} J+C_{-1}\left(J A\left(\mathbf{e}_{n-i}+\mathbf{e}_{n}\right)\right)\right), \tag{3.23}
\end{align*}
$$

where, for $\mathbf{z} \in \mathbb{C}^{n}$,

$$
\phi_{i}(\mathbf{z})=\left(\begin{array}{cc}
C_{-1}\left(I_{1}^{i} \mathbf{z}\right) & 0  \tag{3.24}\\
0 & -C\left(I_{i+1}^{n} \mathbf{z}\right)
\end{array}\right)
$$

As regards the Corollary 3.3 it is interesting to notice that if $n$ is a power of 2 there is no restriction on the choice of $i$. In other words, the dimensions of the submatrices in the block matrices of (3.22) and (3.23) are completely arbitrary. However notice that the decompositions (3.20), (3.21) in Theorem 3.3 hold for any value of $n$ and $i$ if we let $\xi, \eta$ assume, indifferently, the values $\xi=1, \eta=-1$ or $\xi=-1, \eta=1$. In the next section we will use Corollary 3.3 to find a well-known formula and to state a new, convenient formula for the inverse of a Toeplitz matrix (Theorem 4.1).

Observe that the dimensions of the matrices involved in the decomposition formulas of Theorems 3.1, 3.2, and 3.3 satisfy, respectively, the following equations ( $x, y, n \in \mathbb{N}$ ):

$$
\begin{align*}
x+y+1 & =n,  \tag{3.25}\\
2 x+y+2 & =n,  \tag{3.26}\\
x+y & =n . \tag{3.27}
\end{align*}
$$

In [34] it is shown that if $x, y, n$ are chosen in the set of integers of the form $p_{1}^{k_{1}} p_{2}^{k_{2}} \cdots p_{t}^{k_{t}}$ (where the $p_{i}^{\prime}$ 's are $t$ fixed prime numbers) and are pairwise relatively prime, then the equation (3.27) has a finite (depending on $\max _{1 \leqslant i \leqslant t} p_{i}$ ) number of solutions.

In the cases $\left\{t=3 ; p_{1}=2, p_{2}=3, p_{3}=5\right\}$ and $\left\{t=4 ; p_{1}=2, p_{2}=3\right.$, $p_{3}=5, p_{4}=7$ ), these solutions are listed in [3]. In [20] there is a method to derive the solutions (in the sense described above) of (3.27) in every case.

We recall that efficient algorithms for fast Fourier transforms were recently developed for dimensions that are products of powers of small prime integers [ $1,2,37,38,15$ ]. Then the equations (3.25)-(3.27) can be particularly interesting for the choice of the block dimensions in the previous theorems [especially (3.27) for Theorem 3.3].

## 4. APPLICATIONS: NUMERICAL SOLUTION OF TOEPLITZ SYSTEMS

Let $T=\left(t_{i-j}\right)$ and $H=\left(h_{i+j-2}\right), i, j=1, \ldots, n$, denote, respectively, a Toeplitz and a Hankel matrix of dimension $n \times n$ with complex elements, and assume that the Toeplitz plus Hankel matrix $T+H$ is not singular.

Notice that the direct triangular factorization of $(T+H)^{-1}$ may be considered as more fundamental than the problem of inverting $T+H$. In fact, the inverse problem can be obtained as the solution of a special triangular factorization problem. Some recent results in this directionleading to algorithms of complexity $O\left(n^{2}\right)$-can be found in [33] and [29] (see also the references in each of those papers).

Here we follow another approach where a distinction is emphasized between a preprocessing phase-where only operations on elements of $T+H$ are performed-and a successive phase of complexity $O(n \log n)$ where the linear system $(T+H) \mathbf{x}=\mathbf{f}, \mathbf{f} \in \mathbb{C}^{n}$, is solved.

In this section we find some representations for the inverse of $T+H$ as consequences of some decomposition formulas obtained in Section 3. Like analogous already known formulas [4, 5, 7, 13, 14, 16-18, 21, 24, 28], these representations can be exploited to calculate $(T+H)^{-1} \mathbf{f}$ by means of a constant number of discrete transforms. As the rank of the matrix $\left.{ }^{( } T_{T_{2}}(T+H)^{-1}\right)$ is 4 [26, p. 154], we can apply, for instance, the formulas (3.11) and (3.14) of Corollaries 3.1 and 3.2 and obtain the following representations of $(T+H)^{-1}$ :
(i) $(T+H)^{-1}=\left[\tau_{\backslash}^{i}\left(\mathbf{x}_{1}\right) \tau\left(\mathbf{v}_{1}\right)+\tau_{\backslash}^{i}\left(\mathbf{x}_{2}\right) \tau\left(\mathbf{v}_{n}\right)\right.$

$$
\begin{equation*}
\left.-\tau_{\backslash}^{i}\left(\mathbf{w}_{1}\right) \tau\left(\mathbf{x}_{3}\right)-\tau_{\backslash}^{i}\left(\mathbf{w}_{n}\right) \tau\left(\mathbf{x}_{4}\right)+\tau\left(\mathbf{v}_{i}\right)\right] T_{i}^{-1} ; \tag{4.1}
\end{equation*}
$$

(ii) If $T=T^{T}$ and $J H=H J$, then

$$
\begin{equation*}
(T+H)^{-1}=\left[\tau_{\times}^{i}\left(\mathbf{x}_{1}\right) \tau\left(\mathbf{w}_{1}\right)-\tau_{\times}^{i}\left(\mathbf{w}_{1}\right) \tau\left(\mathbf{x}_{1}\right)+\tau\left(\mathbf{w}_{i}\right)\right] T_{i}^{-1} . \tag{4.2}
\end{equation*}
$$

The vectors $\mathbf{x}_{i}, i=1,2,3,4$, are the solutions of particular Toeplitz plus Hankel systems with coefficients matrix $T+H$ (see [21] and [18]). The vectors $\mathbf{w}_{k}\left(\mathbf{v}_{k}^{T}\right), k=1, \ldots, n$, are the columns (rows) of $(T+H)^{-1}$. Obviously statements (i) and (ii) hold if $T_{i}$ is not singular. For their proof proceed as in [21] and [18].

The formulas (4.1), (4.2) generalize the representations of $(T+H)^{-1}$ (obtained for $i=1$ or $i=n$ ) introduced in [21], where only size $n$ and size $n-1$ (or $n-2$ ) $\tau$ matrices were used. However, they have mainly a theoretical interest. In fact the expressions of $(T+H)^{-1}$ found in [18] and in [16], where one exploits the algebras $\tau_{\varepsilon, \varphi}$ and the algebras $C+J C$ and $C_{-1}+J C_{-1}$, are more utilizable in the calculation of $(T+H)^{-1} \mathbf{f}, \mathbf{f} \in \mathbb{C}^{n}$, because they require the computation of discrete transforms all having order $n$.

Now assume $H=0$, and set $T^{-1}=S=\left(s_{i j}\right)$ and $\mathbf{s}_{k}=T^{-1} \mathbf{e}_{k}, k=$ $1, \ldots, n$. It is well known and easily verified that the rank of $\mathfrak{C}_{P_{-1}}\left(T^{-1}\right)$ is 2 [26, p. 16]. Thus, by using Corollary 3.3, we could obtain two simple expressions for the inverse of a general nonsingular Toeplitz matrix in terms of circulant and skew-circulant matrices. Here we calculate one of them in the particular case $s_{11}=\left[T^{-1}\right]_{11} \neq 0$, that is, we assume the $(n-1) \times$ ( $n-1$ ) upper left submatrix of $T$ to be nonsingular.

Proposition 4.1. Let $i \in\{0, \ldots, n-1\} \backslash\{n(2 j+1) /(2 k+1): k=$ $1, \ldots, n-1, j=0,1, \ldots\}$, and assume $s_{11}=\left[T^{-1}\right]_{11} \neq 0$. Then

$$
\begin{align*}
& T^{-1}=\frac{1}{2 s_{11}}\left\{\left[\left(\begin{array}{cc}
C_{-1}\left(I_{i}^{1} \mathbf{s}_{1}\right)^{T} & 0 \\
0 & C\left(I_{n}^{i+1} \mathbf{s}_{1}\right)^{T}
\end{array}\right)-s_{i+1,1} I\right] C_{-1}\left(\hat{\mathbf{s}}_{n}\right)\right. \\
& \left.-\left(\begin{array}{cc}
C_{-1}\left(I_{1}^{i} \mathbf{s}_{n}\right) & 0 \\
0 & -C\left(I_{i+1}^{n} \mathbf{s}_{n}\right)
\end{array}\right) C_{-1}\left(\mathbf{s}_{1}\right)^{T}+s_{11} C_{-1}\left(\hat{\mathbf{s}}_{n-i}\right)\right\} 2\left(P_{-1}^{i}+I\right)^{-1} \tag{4.3}
\end{align*}
$$

Proof. From the equality

$$
\begin{equation*}
\mathfrak{S}_{Z^{T}}\left(T^{-1}\right)=\frac{1}{s_{11}}\left[\mathbf{s}_{n}\left(Z \hat{\mathbf{s}}_{1}\right)^{T}-\left(J Z \hat{\mathbf{s}}_{1}\right) \hat{\mathbf{s}}_{n}^{T}\right] \tag{4.4}
\end{equation*}
$$

(see [21] or [18]), it follows that $\mathfrak{C}_{P_{-1}}\left(T^{-1}\right)=\left(1 / s_{11}\right)\left[s_{n}\left(P_{-1}^{T} \hat{\mathbf{s}}_{1}\right)^{T}-\right.$ $\left.\left(P_{-1} \mathbf{s}_{1}\right) \hat{\mathbf{s}}_{n}^{T}\right]$. Then apply the formula (3.22) of Corollary 3.3, taking into account Proposition 2.1(i) and the following identities:

$$
\begin{gathered}
C(\mathbf{z}) P=C(\hat{\mathbf{z}})^{T}, \quad C_{-1}(\mathbf{z}) P_{-1}=-C_{-1}(\hat{\mathbf{z}})^{T} \\
\phi_{i}\left(P_{-1} \mathbf{z}\right)=\left(\begin{array}{cc}
P_{-1}^{(i)} & 0 \\
0 & P^{(n-i)}
\end{array}\right) \phi_{i}(\mathbf{z})+\left(z_{i+1}+z_{1}\right) I
\end{gathered}
$$

$\left[\mathrm{z} \in \mathbb{C}^{n}\right.$, and $P_{-1}^{(k)}\left(P^{(k)}\right)$ is the matrix $P_{-1}(P)$ of order $\left.k\right]$.
Notice that (4.3) holds, in particular, for $i=0$ and $i=n / 2$ ( $n$ even). In the next theorem we rewrite (4.3) in these two cases under the further
assumption that $T$ is symmetric. For the sake of simplicity, if $n$ is even $(n=2 m)$, we set $\mathbf{a}=I_{m}^{1} \mathbf{s}_{1}, \mathbf{b}=I_{n}^{m+1} \mathbf{s}_{1}, \mathbf{c}=I_{m}^{1} \mathbf{s}_{m+1}, \mathbf{d}=I_{n}^{m+1} \mathbf{s}_{m+1}$, and $\lambda=s_{m+1,1}$.

Theorem 4.1. Assume $T=T^{T}$ and $s_{11}=\left[T^{-1}\right]_{11} \neq 0$. Then
(i) one has

$$
\begin{equation*}
T^{-1}=\frac{1}{2 s_{11}}\left\{C\left(\mathbf{s}_{1}\right) C_{-1}\left(\mathbf{s}_{1}\right)^{T}+C\left(\mathbf{s}_{1}\right)^{T} C_{-1}\left(\mathbf{s}_{1}\right)\right\} ; \tag{4.5}
\end{equation*}
$$

(ii) If $n$ is even $(n=2 m)$, then

$$
\begin{align*}
T^{-1} & =\frac{1}{2 s_{11}}\left\{\left[\left(\begin{array}{cc}
C_{-1}(\mathbf{a})^{T} & 0 \\
0 & C(\mathbf{b})^{T}
\end{array}\right)-\lambda I\right] C_{-1}\left(\mathbf{s}_{1}\right)\right. \\
& \left.-\left(\begin{array}{cc}
C_{-1}(\mathbf{b}) & 0 \\
0 & -C(\mathbf{a})
\end{array}\right) C_{-1}\left(\mathbf{s}_{1}\right)^{T}+s_{11} C_{-1}\left(\mathbf{s}_{m+1}\right)\right\}\left(I-P_{-1}^{m}\right) \tag{4.6}
\end{align*}
$$

The formula (4.5) is the well-known formula of Ammar and Gader [4]. If $\mathbf{s}_{1}$ is known, this formula allows one to solve the linear system $T \mathbf{x}=\mathbf{f}$, $\mathbf{f} \in \mathbb{C}^{n}$, with essentially eight order $n$ DFTs. The same result is obtained in the real case in [18] with a similar formula involving the algebras $\tau_{1,1}$ and $\tau_{-1,-1}$. The formula (4.6) is a new formula for $T^{-1}$. Notice that it requires, apparently, the extra calculation of $\boldsymbol{s}_{m+1}$. However, we will observe that, if parallel procedures are possible, the use of the representation (4.6) in the calculation of $T^{-1} \mathbf{f}$ may be more convenient than the use of (4.5).

From now on $n$ is always even $(n=2 m)$. Moreover, for the sake of simplicity, in all the structured matrices the dimension index is omitted. In fact the dimension will be always manifest in the context.

In the following theorem we state two other possible expressions for $T^{-1}$ in the case $n$ even. We essentially exploit the fact that the $m \times m$ matrices $S_{+}$and $S_{-}$in the equality

$$
\left(\begin{array}{rr}
I & J  \tag{4.7}\\
I & -J
\end{array}\right) T^{-1}\left(\begin{array}{rr}
I & I \\
J & -J
\end{array}\right)=\left(\begin{array}{cc}
S_{+} & 0 \\
0 & S_{-}
\end{array}\right)
$$

are the inverses of two particular Toeplitz plus Hankel matrices (see the remark at the end of the theorem).

Theorem 4.2. Assume $T=T^{T}$, $n$ even $(n=2 m)$, and $s_{11}=\left[T^{-1}\right]_{11} \neq$ 0. Then

$$
T^{-1}=\frac{1}{4}\left(\begin{array}{rr}
I & I \\
J & -J
\end{array}\right)\left(\begin{array}{cc}
S_{+} & 0 \\
0 & S_{-}
\end{array}\right)\left(\begin{array}{rr}
I & J \\
I & -J
\end{array}\right)
$$

where

$$
\begin{align*}
S_{ \pm}= & \frac{1}{s_{11}}[C(\mathbf{b}) \pm J C(\mathbf{a}) \pm \lambda J]\left[C_{-1}(\mathbf{b})^{T} \pm J C_{-1}(\mathbf{a})\right] \\
& +C\left(\mathbf{d}-\frac{\lambda}{s_{11}} \mathbf{b}\right)(I \pm J) \mp J\left[C_{-1}(\mathbf{d})^{T} \pm J C_{-1}(\mathbf{c})\right] \tag{4.8}
\end{align*}
$$

or

$$
\begin{align*}
S_{ \pm}=\frac{1}{s_{11}}[ & \pm \boldsymbol{\tau}_{\mp \pm}(\mathbf{a} \pm \hat{\mathbf{b}}) \tau_{ \pm \pm}\left(I_{m+1}^{2} \mathbf{s}_{1} \pm Z \hat{\mathbf{b}} \pm s_{11} \mathbf{e}_{1}\right) \\
& \left.\mp \boldsymbol{\tau}_{\mp \pm}\left(I_{m+1}^{2} \mathbf{s}_{1} \pm Z \hat{\mathbf{b}} \mp s_{11} \mathbf{e}_{1}\right) \tau_{ \pm \pm}(\mathbf{a} \pm \hat{\mathbf{b}})\right] \tag{4.9}
\end{align*}
$$

Proof. Before proving the theorem we need to recall a result of [18]. Assume that for an $m \times m$ matrix $A$ there exist $2 \alpha$ vectors $\mathbf{x}_{k}, \mathbf{y}_{k} \in \mathbb{C}^{m}$ such that $\mathfrak{C}_{T_{ \pm 1 . \pm 1}}(A)=\sum_{k=1}^{\alpha} \mathbf{x}_{k} \mathbf{y}_{k}^{T}$. Then

$$
\begin{equation*}
2 A= \pm \sum_{k=1}^{\alpha} \tau_{\mp \pm}\left(\mathbf{x}_{k}\right) \tau_{ \pm \pm}\left(\mathbf{y}_{k}\right)+2 \tau_{ \pm \pm}\left(A^{T} \mathbf{e}_{1}\right) . \tag{4.10}
\end{equation*}
$$

Let $S_{U L}, S_{U R}, S_{D L}$ and $S_{D R}$ be the upper left, upper right, lower left, and lower right $m \times m$ submatrices of $S=T^{-1}$, respectively. Observe that $S_{D L}=J S_{U R} J=S_{U R}^{T}, S_{U L}=J S_{D R} J=S_{U L}^{T}$, and therefore

$$
\left(\begin{array}{rr}
I & J  \tag{4.11}\\
I & -J
\end{array}\right) T^{-1}\left(\begin{array}{rr}
I & I \\
J & -J
\end{array}\right)=2\left(\begin{array}{cc}
S_{U L}+S_{U R} J & 0 \\
0 & S_{U L}-S_{U R} J
\end{array}\right)
$$

From the equality (4.4), rewritten for $T=T^{T}$ with the new notation,

$$
\begin{aligned}
& \left(\begin{array}{cc}
S_{U L} & S_{U R} \\
S_{D L} & S_{D R}
\end{array}\right)\left(\begin{array}{cc}
Z^{T} & \mathbf{e}_{m} \mathbf{e}_{1}^{T} \\
0 & Z^{T}
\end{array}\right)-\left(\begin{array}{cc}
\mathrm{Z}^{T} & \mathbf{e}_{m} \mathbf{e}_{1}^{T} \\
0 & Z^{T}
\end{array}\right)\left(\begin{array}{cc}
S_{U L} & S_{U R} \\
S_{D L} & S_{D R}
\end{array}\right) \\
& \quad=\frac{1}{s_{1.1}}\left[\binom{\hat{\mathbf{b}}}{\hat{\mathbf{a}}}\left(\hat{\mathbf{b}}^{T} \mid \hat{\mathbf{a}}^{T}\right)\left(\begin{array}{cc}
\mathrm{Z}^{T} & \mathbf{e}_{m} \mathbf{e}_{1}^{T} \\
0 & Z^{T}
\end{array}\right)-\left(\begin{array}{cc}
\mathrm{Z}^{T} & \mathbf{e}_{m} \mathbf{e}_{1}^{T} \\
0 & \mathrm{Z}^{T}
\end{array}\right)\binom{\mathbf{a}}{\mathbf{b}}\left(\mathbf{a}^{T} \mid \mathbf{b}^{T}\right)\right],
\end{aligned}
$$

it follows that

$$
\begin{align*}
& \mathfrak{C}_{Z^{T}}\left(S_{U L}\right)=\mathbf{e}_{m} \mathbf{c}^{T}+\frac{1}{s_{11}}\left[\hat{\mathbf{b}}(Z \hat{\mathbf{b}})^{T}-\left(I_{m+1}^{2} \mathbf{s}_{1}\right) \mathbf{a}^{T}\right]  \tag{4.12}\\
& \mathfrak{C}_{\mathrm{Z}}\left(S_{U L}\right)=-\mathfrak{C}_{\mathrm{Z}^{T}}\left(S_{U L}\right)^{T}=-\mathbf{c e}_{m}^{T}-\frac{1}{s_{11}}\left[(Z \hat{\mathbf{b}}) \hat{\mathbf{b}}^{T}-\mathbf{a}\left(I_{m+1}^{2} \mathbf{s}_{1}\right)^{T}\right] \tag{4.13}
\end{align*}
$$

$$
\begin{align*}
\mathfrak{§}_{Z^{r}}\left(S_{U R}\right) & =\mathbf{e}_{m} \mathbf{d}^{T}-\hat{\mathbf{d}} \mathbf{e}_{1}^{T}+\frac{1}{s_{11}}\left[\hat{\mathbf{b}}\left(I_{2}^{m+1} \mathbf{s}_{1}\right)^{T}-\left(I_{m+1}^{2} \mathbf{s}_{1}\right) \mathbf{b}^{T}\right],  \tag{4.14}\\
\mathfrak{C}_{Z}\left(S_{U R}\right) & =-\mathfrak{C}_{Z^{T}}\left(S_{D L}\right)^{T}=\frac{1}{s_{11}}\left[\mathbf{a}\left(Z^{T} \mathbf{b}\right)^{T}-(Z \hat{\mathbf{b}}) \hat{\mathbf{a}}^{T}\right] . \tag{4.15}
\end{align*}
$$

By adding the identities (4.12) and (4.13) and the identities (4.14) and (4.15) and then exploiting the equalities $\mathfrak{C}_{T_{2}}\left(S_{U L} \pm S_{U R} J\right)=\mathfrak{C}_{T_{2}}\left(S_{U L}\right) \pm$ $\mathfrak{C}_{T_{2}}\left(S_{U R}\right) J$, we have

$$
\begin{align*}
& \mathfrak{S}_{r_{2}}\left(S_{U L} \pm S_{U R} J\right)=\mathbf{e}_{m}(\mathbf{c} \pm \hat{\mathbf{d}})^{T}-(\mathbf{c} \pm \hat{\mathbf{d}}) \mathbf{e}_{m}^{T} \\
& \quad+\frac{1}{s_{11}}\left\{(\mathbf{a} \pm \hat{\mathbf{b}})\left(I_{m+1}^{2} \mathbf{s}_{1} \pm Z \hat{\mathbf{b}}\right)^{T}-\left(I_{m+1}^{2} \mathbf{s}_{1} \pm Z \hat{\mathbf{b}}\right)(\mathbf{a} \pm \hat{\mathbf{b}})^{T}\right\} \tag{4.16}
\end{align*}
$$

Finally, as consequence of (4.16), we easily obtain

$$
\begin{align*}
\mathfrak{S}_{T_{ \pm 1, \pm 1}}\left(S_{U L} \pm S_{U R} J\right)=\frac{1}{s_{11}} & {\left[(\mathbf{a} \pm \hat{\mathbf{b}})\left(I_{m+1}^{2} \mathbf{s}_{1} \pm Z \hat{\mathbf{b}} \pm s_{11} \mathbf{e}_{1}\right)^{T}\right.} \\
& \left.-\left(I_{m+1}^{2} \mathbf{s}_{1} \pm Z \hat{\mathbf{b}} \pm s_{11} \mathbf{e}_{1}\right)(\mathbf{a} \pm \hat{\mathbf{b}})^{T}\right] \tag{4.17}
\end{align*}
$$

The formula (4.9) follows from (4.10) and (4.17). Moreover, by exploiting the equalities (4.12) and (4.14), we have

$$
\begin{align*}
\mathfrak{S}_{P_{-1}}\left(S_{U L}\right)= & \mathbf{e}_{m} \mathbf{c}^{T}-\hat{\mathbf{d}} \mathbf{e}_{1}^{T}+\frac{1}{s_{11}}\left[\hat{\mathbf{b}}(Z \hat{\mathbf{b}})^{T}-\left(I_{m+1}^{2} \mathbf{s}_{1}-s_{11} \mathbf{e}_{m}\right) \mathbf{a}^{T}\right]  \tag{4.18}\\
\mathscr{S}_{P_{-1}}\left(S_{U R}\right)= & \mathbf{e}_{m} \mathbf{d}^{T}-\hat{\mathbf{d}} \mathbf{e}_{1}^{T} \\
& +\frac{1}{s_{11}}\left[\hat{\mathbf{b}}\left(I_{2}^{m+1} \mathbf{s}_{1}-s_{11} \mathbf{e}_{1}\right)^{T}-\left(I_{m+1}^{2} \mathbf{s}_{1}-s_{11} \mathbf{e}_{m}\right) \mathbf{b}^{T}\right] \tag{4.19}
\end{align*}
$$

respectively. The formula (3.22) of Corollary 3.3 for $i=0$ and the identities (4.18) and (4.19) imply

$$
\begin{align*}
2 S_{U L}= & C(\mathbf{d})-C_{-1}(\mathbf{c}) \\
& +\frac{1}{s_{11}}\left\{C(\mathbf{b})\left[C_{-1}(\mathbf{b})^{T}-\lambda I\right]+\left[C(\mathbf{a})^{T}+\lambda I\right] C_{-1}(\mathbf{a})\right\},  \tag{4.20}\\
2 S_{U R}= & -C_{-1}(\mathbf{d})+C(\mathbf{d}) \\
& +\frac{1}{s_{11}}\left\{C(\mathbf{b})\left[C_{-1}(\mathbf{a})^{T}-\lambda I\right]+\left[C(\mathbf{a})^{T}+\lambda I\right] C_{-1}(\mathbf{b})\right\} . \tag{4.21}
\end{align*}
$$

Finally, write the matrices $2\left(S_{U L}+S_{U R} J\right)$ and $2\left(S_{U L}-S_{U R} J\right)$ to obtain (4.8).

Remark. If we denote by $T_{U L}, T_{U R}, T_{D L}$, and $T_{D R}$ the upper left, upper right, lower left, and lower right $m \times m$ submatrices of $T$ respectively, then

$$
\left(\begin{array}{rr}
I & J \\
I & -J
\end{array}\right) T\left(\begin{array}{cc}
I & I \\
J & -J
\end{array}\right)=2\left(\begin{array}{cc}
T_{U L}+T_{U R} J & 0 \\
0 & T_{U L}-T_{U R} J
\end{array}\right)
$$

By inverting this equality we obtain a new equality which, compared with (4.11), yields

$$
S_{U L} \pm S_{U R} J=\left(T_{U L} \pm T_{U R} J\right)^{-1}
$$

For the sake of completeness observe that the vector $s_{m+1}$ and its subvectors $\mathbf{c}$ and $\mathbf{d}$, used in the decompositions (4.6) and (4.8), can be easily expressed in terms of $\mathbf{s}_{1}$, $\mathbf{a}$ and $\mathbf{b}$, by the following identities:

$$
\begin{aligned}
& \text { (i) } \mathbf{d}+\mathbf{c}=\frac{1}{s_{11}}[C(\mathbf{a}) \mathbf{a}-C(\mathbf{b}) \mathbf{b}+\lambda(\mathbf{b}+\mathbf{a})] ; \\
& \text { (ii) } \mathbf{d}-\mathbf{c}=\frac{1}{s_{11}}\left[C_{-1}(\mathbf{a}) \mathbf{a}-C_{-1}(\mathbf{b}) \mathbf{b}+\lambda(\mathbf{b}-\mathbf{a})\right] ; \\
& \text { (iii) } \mathbf{d}-J P \mathbf{c}=\frac{\lambda}{s_{11}}(\mathbf{b}-J P \mathbf{a})+\left(s_{m+1, m+1}-\frac{\lambda^{2}}{s_{11}}\right) \mathbf{e}_{1} ; \\
& \text { (iv) } \mathbf{s}_{m+1}=\frac{1}{2 s_{11}}\left[C_{-1}\left(\mathbf{s}_{1}\right) P_{m+1} \mathbf{s}_{1}+C_{-1}\left(\mathbf{s}_{1}\right)^{T} P_{m} \hat{\mathbf{s}}_{1}\right] \text { or } \mathbf{s}_{m+1}= \\
& \frac{1}{s_{11}}\left[C_{-1}\left(\mathbf{s}_{1}\right)\binom{\mathbf{0}}{\mathbf{a}}+C_{-1}\left(\mathbf{s}_{1}\right)^{T}\binom{\mathbf{0}}{Z \hat{\mathbf{b}}}\right] .
\end{aligned}
$$

(i), (ii), (iii) are obtained by calculating the first column of $2 S_{U L}$, the first row of $2 S_{U L}$, and the first column of $2 S_{U R}$ through the formulas (4.20), (4.20), and (4.21), respectively. (iv) is obtained by exploiting Proposition 2.1 to calculate $\mathbf{e}_{m+1}^{T} T^{-1}$ with $T^{-1}$ expressed either by (4.5) or by (4.6) and by observing that $\mathbf{s}_{m+1}=\left(\mathbf{e}_{m+1}^{T} T^{-1}\right)^{T}$.

## 5. CONCLUDING REMARKS

The displacement decompositions considered in this paper involve Hessenberg algebras and block matrices whose blocks belong to Hessenberg algebras and have variable sizes. They turn out to be more useful (flexible) than those considered in [21]. In fact, in (3.2), (3.3), (3.7), (3.8), (3.20), (3.21), one can choose the dimensions of the blocks according to the particular problem features and in such a way that the orders of the discrete transforms involved are all highly composite integers (typically powers of 2 ).

As special instances, new decompositions of the inverse $T^{-1}$ of a Toeplitz matrix are obtained [see (4.6), (4.8), and (4.9)]. These formulas, like the Ammar-Gader formula (4.5), can be used to solve linear systems $T \mathbf{x}=\mathbf{f}$ via the computation of $T^{-1} \mathbf{f}$ by means of a constant number of discrete transforms ( $s_{1}$ is assumed to be known). The Ammar-Gader formula (4.5) reaches the best known sequential time; in particular (4.5) allows one to compute $T^{-1} \mathbf{f}$ with essentially eight order $n$ DFTs. The formulas (4.6), (4.8) and (4.9) may be more efficient than (4.5), in particular in a parallel computation where different discrete transforms are performed simultaneously, either in
parallel or sequentially [one may conceive of a reduction of parallel time, with low redundancy, by a factor $\frac{1}{4}$ with (4.6), and by a factor $\frac{1}{2}$ with (4.8) or (4.9)].

Moreover, if the preprocessing phase includes the computation of the discrete transforms of vectors not depending on $\mathbf{f}$, the formulas (4.8) and (4.9) require an amount of computation of twelve order $n / 2$ (six order $n$ ) discrete transforms plus $O(n)$ arithmetic operations, the same amount required by (4.5). The distinction of a preprocessing phase in the solution of the system $T \mathbf{x}=\mathbf{f}$ is significant when several linear systems with the same $T$ have to be solved.

Finally observe that if $m(m=n / 2)$ is even, the twelve order $m$ DFTs required by (4.8) can be reduced to eleven (ten of order $m$ plus two of order $m / 2$ ). In fact two of them are DFTs of vectors of the type $(I+J) \mathbf{z}$ and ( $I-J$ ) $\mathbf{z}, \mathbf{z} \in \mathbb{C}^{m}$, possessing special symmetries. In [35] such symmetries are called QE and QO symmetries, respectively, and an order $m$ DFT of a QE (QO) symmetric vector is shown to cost the same as an order $m / 2$ DFT.

The previous results are obtained by using the known splits of matrices of the type involved in (4.5), (4.6), (4.8), and (4.9) (see (2.9), [16], [18]), and by exploiting the above relations (i)-(iv) between $\mathbf{s}_{m+1}$ and $\mathbf{s}_{1}$.

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