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q-COMTET AND GENERALIZED q-HARMONIC NUMBERS

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Abstract

In this paper, we derive some generalizations of q-Stirling and q-harmonic numbers. Recurrence relations, generating functions, explicit formulae, and a connection between these numbers are given. Moreover, some important special cases and new combinatorial identities are obtained. Finally, matrix representation using Maple is given.

1. Introduction

The generalized Stirling numbers of first and second kind, respectively, were introduced by Comtet [6] with

$$(t;\,\overline{\alpha})_{\underline{n}} = \sum_{k=0}^{n} s_{\overline{\alpha}}(n,\,k)t^{k}, \tag{1.1}$$

where $s_{\overline{\alpha}}(0,\,0) = 1$ and $s_{\overline{\alpha}}(n,\,k) = 0$ for $k > n$, and

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$$t^n = \sum_{k=0}^{n} S_{\overline{\alpha}}(n, k)(t; \overline{\alpha})_{\underline{k}}, \tag{1.2}$$

where $S_{\overline{\alpha}}(0, 0) = 1$, $S_{\overline{\alpha}}(n, k) = 0$ for $k > n$, $\overline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$, and $(t; \overline{\alpha})_{\underline{n}} = (t - \alpha_0)(t - \alpha_1)\cdots(t - \alpha_{n-1})$.

Through this article, we use the following notations, see [4], [5], and [9].

Let $0 < q < 1$, t be a real number and n be a positive integer. The q-number is defined by $[t]_q = \frac{1 - q^t}{1 - q}$ and q-factorial of t is given by

$$[t]_q! = [t]_q[t-1]_q, \cdots, [1]_q, \quad t = 1, 2, \cdots. \tag{1.3}$$

The falling and rising factorial of the q-number $[t]_q$ of order n are defined, respectively, by

$$[t]_{\underline{n}, q} = [t]_q[t-1]_q \cdots [t - n + 1]_q, \tag{1.4}$$

and

$$[t]_{\overline{n}, q} = [t]_q[t+1]_q \cdots [t + n - 1]_q. \tag{1.5}$$

Generally, we have the following definition:

Definition 1.1. Let the generalized falling and rising factorial of q-number $[t]_q$ of order n, associated with the sequence $\overline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ be defined, respectively, by

$$[t; \overline{\alpha}]_{\underline{n}, q} = [t - \alpha_0]_q[t - \alpha_1]_q \cdots [t - \alpha_{n-1}]_q,$$

and

$$[t; \overline{\alpha}]_{\overline{n}, q} = [t + \alpha_0]_q[t + \alpha_1]_q \cdots [t + \alpha_{n-1}]_q.$$

The q-Vandermond's formula may be expressed as

$$[u + t]_{\underline{n}, q} = \sum_{k=0}^{n} q^{-k(n-k-u)}[u]_{\underline{n-k}, q}[t]_{\underline{k}, q}. \tag{1.6}$$

In Section 2, we derive generalization of some results given in [3], [10], and [5], and study the generalized q-Stirling numbers, may be called *q-Comtet numbers*, of first and second kind and its relationships with other types of Stirling numbers. Also, in Section 3, we define generalized non-central q-Stirling numbers of first and second kind. Moreover, we give a generalization of the q-harmonic numbers and obtain some of their connections with the generalized q-Stirling numbers. Furthermore, some special cases, explicit formula of these numbers and some combinatorial identities are derived. In Section 4, algorithms matrix representation of these numbers are given by using Maple program.

2. The Generalized q-Stirling Numbers of First and Second Kind

Since

$$\begin{aligned}[t;\,\overline{\alpha}]_{\underline{n},q} &= [t-\alpha_0]_q[t-\alpha_1]_q\cdots[t-\alpha_{n-1}]_q \\ &= q^{-\alpha_0}([t]_q-[\alpha_0]_q)q^{-\alpha_1}([t]_q-[\alpha_1]_q)\cdots q^{-\alpha_{n-1}}([t]_q-[\alpha_{n-1}]_q) \\ &= q^{-\sum_{i=0}^{n-1}\alpha_i}\prod_{i=0}^{n-1}([t]_q-[\alpha_i]_q).\end{aligned}$$

This is a polynomial of degree n of the q-number $[t]_q$ and its coefficients is given by $[\alpha_i]_q,\ i = 0, 1, \cdots, n-1$. Thus, we have the following definition:

Definition 2.1. Let $s_{q,\overline{\alpha}}(n,\,k)$ and $S_{q,\overline{\alpha}}(n,\,k)$ be the generalized q-Stirling numbers of first and second kind, associated with sequence $\overline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$, which we call *q-Comtet numbers*, are defined, respectively, by

$$[t;\,\overline{\alpha}]_{\underline{n},q} = q^{-\sum_{i=0}^{n-1}\alpha_i}\sum_{k=0}^{n}s_{q,\overline{\alpha}}(n,\,k)[t]_q^k, \tag{2.1}$$

where $s_{q,\overline{\alpha}}(0,\,0) = 1$, $s_{q,\overline{\alpha}}(n,\,k) = 0$ for $k > n$, and

$$[t]_q^n = \sum_{k=0}^{n} q^{\sum_{i=0}^{k-1}\alpha_i} S_{q,\overline{\alpha}}(n, k)[t; \overline{\alpha}]_{\underline{k},q}, \tag{2.2}$$

where $S_{q,\overline{\alpha}}(0, 0) = 1$ and $S_{q,\overline{\alpha}}(n, k) = 0$ for $k > n$.

Theorem 2.1. *The numbers* $s_{q,\overline{\alpha}}(n, k)$ *and* $S_{q,\overline{\alpha}}(n, k)$ *satisfy the recurrence relations*

$$s_{q,\overline{\alpha}}(n+1, k) = s_{q,\overline{\alpha}}(n, k-1) - [\alpha_n]_q s_{q,\overline{\alpha}}(n, k), \tag{2.3}$$

and

$$S_{q,\overline{\alpha}}(n+1, k) = S_{q,\overline{\alpha}}(n, k-1) + [\alpha_k]_q S_{q,\overline{\alpha}}(n, k). \tag{2.4}$$

Proof. Since $[t; \overline{\alpha}]_{\underline{n+1},q} = [t; \overline{\alpha}]_{\underline{n},q}[t-\alpha_n]_q = q^{-\alpha_n}([t]_q - [\alpha_n]_q)[t; \overline{\alpha}]_{\underline{n},q}$,

then using (2.1)

$$q^{-\sum_{i=0}^{n}\alpha_i} \sum_{k=0}^{n+1} s_{q,\overline{\alpha}}(n+1, k)[t]_q^k = q^{-\alpha_n}([t]_q - [\alpha_n]_q) q^{-\sum_{i=0}^{n-1}\alpha_i} \sum_{k=0}^{n} s_{q,\overline{\alpha}}(n, k)[t]_q^k$$

$$= q^{\sum_{i=0}^{n}\alpha_i} \sum_{k=0}^{n+1} (s_{q,\overline{\alpha}}(n, k-1) - [\alpha_n]_q s_{q,\overline{\alpha}}(n, k))[t]_q^k.$$

Equating the coefficients of $[t]_q^k$ on both sides yields (2.3).

Similarly, the proof of (2.4) easily can be given. □

We discuss the following interesting special cases:

(i) If $\alpha_i = i, \bar{i} = (0, 1, \cdots, n-1)$, then (2.1) and (2.2), respectively, give

$$s_{q,\bar{i}}(n, k) = s_q(n, k), \tag{2.5}$$

where $s_q(n, k)$ are q-Stirling numbers of first kind, see [4], and

$$S_{q,\bar{i}}(n, k) = S_q(n, k), \tag{2.6}$$

where $S_q(n, k)$ are q-Stirling numbers of second kind, see [4].

(ii) If $\alpha_i = r + \bar{i}, \bar{i} = (0, 1, \cdots, n-1)$, then (2.1) and (2.2), respectively, give

$$s_{q, r+\bar{i}}(n, k) = \tilde{s}_q(n, k; r), \tag{2.7}$$

and

$$S_{q, r+\bar{i}}(n, k) = \tilde{S}_q(n, k; r), \tag{2.8}$$

where $\tilde{s}_q(n, k; r)$ and $\tilde{S}_q(n, k; r)$ are non-central q-Stirling numbers of first and second kind, see [5], are defined, respectively, by

$$[t-r]_{\underline{n}, q} = q^{-\binom{n}{2} - rn} \sum_{k=0}^{n} \tilde{s}_q(n, k; r)[t]_q^k,$$

and

$$[t]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2} + rk} \tilde{S}_q(n, k; r)[t-r]_{\underline{k}, q}.$$

(iii) If $\alpha_i = \lambda\bar{i}, \lambda\bar{i} = (0, \lambda, \cdots, \lambda(n-1))$, then (2.1) and (2.2), respectively, give

$$s_{q, \lambda\bar{i}}(n, \ell) = \sum_{k=\ell}^{n} R_q(n, k; \lambda) s_q(k, \ell), \tag{2.9}$$

and

$$S_q(n, \ell) = \sum_{k=\ell}^{n} R_q(k, \ell; \lambda) s_{q, \lambda\bar{i}}(n, k), \tag{2.10}$$

where $R_q(n, k; \lambda)$ are the generalized q-factorial coefficients, see [4], defined by

$$[t|\lambda]_{\underline{n}, q} = q^{-\lambda\binom{n}{2}} \sum_{k=0}^{n} q^{\binom{k}{2}} R_q(n, k; \lambda)[t]_{\underline{k}, q}.$$

In (2.1), replacing q by q^{-1} and t by $-t$ and notice that $[-1]_q^n[-t;\ \overline{\alpha}]_{\underline{n},\, q^{-1}} = [t;\ \overline{\alpha}]_{\overline{n},\, q}$, we obtain

$$
\begin{aligned}
[t;\ \overline{\alpha}]_{\overline{n},\, q} &= [-1]_q^n q^{\sum_{i=0}^{n-1}\alpha_i} \sum_{k=0}^{n} s_{q^{-1},\, \overline{\alpha}}(n,\ k)[-t]_{q^{-1}}^k \\
&= q^{\sum_{i=0}^{n-1}\alpha_i} \sum_{k=0}^{n} s_{q^{-1},\, \overline{\alpha}}(n,\ k)[-1]_q^{n-k}[t]_q^k \\
&= q^{\sum_{i=0}^{n-1}\alpha_i} \sum_{k=0}^{n} |s_{q^{-1},\, \overline{\alpha}}(n,\ k)|[t]_q^k,
\end{aligned}
$$

where $|s_{q^{-1},\, \overline{\alpha}}(n,\ k)| = [-1]_q^{n-k} s_{q^{-1},\, \overline{\alpha}}(n,\ k)$, which are called the *signless generalized q-Stirling numbers* of first kind.

Let

$$
s\overline{\alpha} = (s\alpha_0,\ s\alpha_1,\ \cdots,\ s\alpha_{n-1}),
$$

and since $[s]_q^n[t;\ \overline{\alpha}]_{\underline{n},\, q^s} = [st;\ s\overline{\alpha}]_{\underline{n},\, q}$, using (2.1), we have

$$
\begin{aligned}
[s]_q^n q^{-s\sum_{i=0}^{n-1}\alpha_i} \sum_{k=0}^{n} s_{q^s,\, \overline{\alpha}}(n,\ k)[t]_{q^s}^k &= q^{-s\sum_{i=0}^{n-1}\alpha_i} \sum_{k=0}^{n} s_{q,\, s\overline{\alpha}}(n,\ k)[st]_q^k \\
&= q^{-s\sum_{i=0}^{n-1}\alpha_i} \sum_{k=0}^{n} s_{q,\, s\overline{\alpha}}(n,\ k)[s]_q^k[t]_{q^s}^k.
\end{aligned}
$$

Equating the coefficients of $[t]_{q^s}^k$ on both sides yields

$$
s_{q,\, s\overline{\alpha}}(n,\ k) = [s]_q^{n-k} s_{q^s,\, \overline{\alpha}}(n,\ k). \tag{2.11}
$$

Theorem 2.2. *The numbers* $s_{q,\overline{\alpha}}(n, k)$ *and* $S_{q,\overline{\alpha}}(n, k)$, *respectively, have the explicit formulas*

$$s_{q,\overline{\alpha}}(n, k) = \sum_{\sigma_n = n-k} (-1)^{n-k} \begin{bmatrix} \alpha_0 \\ i_0 \end{bmatrix}_q \begin{bmatrix} \alpha_1 \\ i_1 \end{bmatrix}_q \cdots \begin{bmatrix} \alpha_{n-1} \\ i_{n-1} \end{bmatrix}_q, \tag{2.12}$$

and

$$S_{q,\overline{\alpha}}(n, k) = \sum_{\sigma_n = k} \begin{bmatrix} \alpha_{i_0} \\ 1 - i_0 \end{bmatrix}_q \begin{bmatrix} \alpha_{i_0 + i_1} \\ 1 - i_1 \end{bmatrix}_q \cdots \begin{bmatrix} \alpha_{i_0 + i_1 + \cdots + i_{n-1}} \\ 1 - i_{n-1} \end{bmatrix}_q, \tag{2.13}$$

where $\sigma_n = i_0 + i_1 + \cdots + i_{n-1}$, $i_\ell \in \{0, 1\}$, *and* $\ell = 0, 1, \cdots, n-1$.

Proof. Setting $k = 0$ in (2.12),

$$s_{q,\overline{\alpha}}(n, 0) = (-1)^n [\alpha_0]_q [\alpha_1]_q \cdots [\alpha_{n-1}]_q,$$

that is easily verified by using (2.3).

If $i_{n-1} \in \{0, 1\}$, then

$$s_{q,\overline{\alpha}}(n, k) = \sum_{\sigma_{n-1} = (n-1)-(k-1)} (-1)^{(n-1)-(k-1)} \begin{bmatrix} \alpha_0 \\ i_0 \end{bmatrix}_q \begin{bmatrix} \alpha_1 \\ i_1 \end{bmatrix}_q \cdots \begin{bmatrix} \alpha_{n-2} \\ i_{n-2} \end{bmatrix}_q$$

$$- [\alpha_{n-1}]_q \sum_{\sigma_{n-1} = (n-1)-k} (-1)^{(n-1)-k} \begin{bmatrix} \alpha_0 \\ i_0 \end{bmatrix}_q \begin{bmatrix} \alpha_1 \\ i_1 \end{bmatrix}_q \cdots \begin{bmatrix} \alpha_{n-2} \\ i_{n-2} \end{bmatrix}_q,$$

hence

$$s_{q,\overline{\alpha}}(n, k) = s_{q,\overline{\alpha}}(n-1, k-1) - [\alpha_{n-1}]_q s_{q,\overline{\alpha}}(n-1, k),$$

this by virtue of (2.3) completes the proof of (2.12).

Also, the proof of (2.13) is given as follows:

Setting $k = 0$ in (2.13),

$$S_{q,\overline{\alpha}}(n, 0) = \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix}_q \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix}_q \cdots \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix}_q = [\alpha_0]_q^n,$$

that is easily verified by using (2.4).

If $i_{n-1} \in \{0, 1\}$, then

$$S_{q,\overline{\alpha}}(n, k) = \sum_{\sigma_{n-1}=k} [\alpha_k]_q \begin{bmatrix} \alpha_{i_0} \\ 1-i_0 \end{bmatrix}_q \begin{bmatrix} \alpha_{i_0+i_1} \\ 1-i_1 \end{bmatrix}_q \cdots \begin{bmatrix} \alpha_{i_0+i_1+\cdots+i_{n-2}} \\ 1-i_{n-2} \end{bmatrix}_q$$

$$+ \sum_{\sigma_{n-1}=k-1} \begin{bmatrix} \alpha_{i_0} \\ 1-i_0 \end{bmatrix}_q \begin{bmatrix} \alpha_{i_0+i_1} \\ 1-i_1 \end{bmatrix}_q \cdots \begin{bmatrix} \alpha_{i_0+i_1+\cdots+i_{n-2}} \\ 1-i_{n-2} \end{bmatrix}_q,$$

hence

$$S_{q,\overline{\alpha}}(n, k) = S_{q,\overline{\alpha}}(n-1, k-1) + [\alpha_k]_q S_{q,\overline{\alpha}}(n-1, k),$$

this by virtue of (2.4) completes the proof of (2.13). □

3. The Generalized Non-Central *q*-Stirling Numbers of First and Second Kind

Definition 3.1. Let $s_q(n, k, r; \overline{\alpha})$ and $S_q(n, k, r; \overline{\alpha})$ be the generalized non-central q-Stirling numbers of first and second kind, associated with the sequence $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$, are defined, respectively, by

$$[t-r; \overline{\alpha}]_{\underline{n},q} = q^{-\sum_{i=0}^{n-1}\alpha_i - rn} \sum_{k=0}^{n} s_q(n, k, r; \overline{\alpha})[t]_q^k, \tag{3.1}$$

where $s_q(0, 0, r; \overline{\alpha}) = 1$ and $s_q(n, k, r; \overline{\alpha}) = 0$ for $n < k$, and

$$[t]_q^n = \sum_{k=0}^{n} q^{rk+\sum_{i=0}^{k-1}\alpha_i} S_q(n, k, r; \overline{\alpha})[t-r; \overline{\alpha}]_{\underline{k},q}, \tag{3.2}$$

where $S_q(0, 0, r; \overline{\alpha}) = 1$ and $S_q(n, k, r; \overline{\alpha}) = 0$ for $n < k$.

Remark 3.1. If $r \neq 0$, the generalized non-central q-Stirling numbers of first kind can be represented in terms of the generalized q-Stirling numbers of the first kind as the follows: Using (2.1) and (3.1), we have

$$[t-r;\ \overline{\alpha}]_{\underline{n},q} = q^{-\sum_{i=0}^{n-1}\alpha_i}\sum_{k=0}^{n} s_{q,\overline{\alpha}}(n,\ k)[t-r]_q^k$$

$$= q^{-\sum_{i=0}^{n-1}\alpha_i}\sum_{k=0}^{n} s_{q,\overline{\alpha}}(n,\ k)q^{-rk}([t]_q - [r]_q)^k$$

$$= \sum_{k=0}^{n} q^{-\sum_{i=0}^{n-1}\alpha_i - rk} s_{q,\overline{\alpha}}(n,\ k)\sum_{j=0}^{k}((-1)^{k-j}\binom{k}{j}[t]_q^j[r]_q^{k-j}),$$

thus,

$$q^{-\sum_{i=0}^{n-1}\alpha_i - rn}\sum_{j=0}^{n} s_q(n,\ j,\ r;\ \overline{\alpha})[t]_q^j = \sum_{j=0}^{n}(\sum_{k=j}^{n} q^{-\sum_{i=0}^{n-1}\alpha_i - rk}(-1)^{k-j}\binom{k}{j}[r]_q^{k-j})[t]_q^j,$$

hence, we get

$$s_q(n,\ j,\ r;\ \overline{\alpha}) = \sum_{k=j}^{n} s_{q,\overline{\alpha}}(n,\ k)(-1)^{k-j}q^{r(n-k)}\binom{k}{j}[r]_q^{k-j}. \tag{3.3}$$

Lemma 3.1. *The numbers* $s_q(n, k, r; \overline{\alpha})$ *and* $S_q(n, k, r; \overline{\alpha})$ *satisfy the recurrence relations*

$$s_q(n+1,\ k,\ r;\ \overline{\alpha}) = s_q(n,\ k-1,\ r;\ \overline{\alpha}) - [r+\alpha_n]_q s_q(n,\ k,\ r;\ \overline{\alpha}), \tag{3.4}$$

where $s_q(n,\ 0,\ r;\ \overline{\alpha}) = (-1)^n[r+\alpha_0]_q[r+\alpha_1]_q\cdots[r+\alpha_{n-1}]_q$, *and*

$$S_q(n+1,\ k,\ r;\ \overline{\alpha}) = S_q(n,\ k-1,\ r;\ \overline{\alpha}) + [r+\alpha_k]_q S_q(n,\ k,\ r;\ \overline{\alpha}). \tag{3.5}$$

The proof is left.

Theorem 3.1. *The numbers* $s_q(n,\ k,\ r;\ \overline{\alpha})$ *have the explicit formula*

$$s_q(n,\ k,\ r;\ \overline{\alpha}) = \sum_{\ell_1+\ell_2+\cdots+\ell_n = n-k}(-1)^{n-k}[r+\alpha_0]_q^{\ell_1}[r+\alpha_1]_q^{\ell_2}\cdots[r+\alpha_{n-1}]_q^{\ell_n}, \tag{3.6}$$

where $\ell_i \in \{0,\ 1\}$, $i = 0,\ 1,\ \cdots,\ n-1$.

Proof. Setting $k = 0$ in (3.6), then

$$s_q(n, 0, r; \overline{\alpha}) = \sum_{\ell_1+\ell_2+\cdots+\ell_n=n} (-1)^n [r+\alpha_0]_q [r+\alpha_1]_q \cdots [r+\alpha_{n-1}]_q$$

$$= (-1)^n [r+\alpha_0]_q [r+\alpha_1]_q \cdots [r+\alpha_{n-1}]_q,$$

that is verified by successive application of (3.4).

For $\ell_n \in \{0, 1\}$, we get

$$s_q(n, k, r; \overline{\alpha}) = \sum_{\ell_1+\ell_2+\cdots+\ell_{n-1}=(n-1)-(k-1)}$$

$$\times (-1)^{(n-1)-(k-1)} [r+\alpha_0]_q^{\ell_1} [r+\alpha_1]_q^{\ell_2} \cdots [r+\alpha_{n-2}]_q^{\ell_{n-1}}$$

$$+ (-1)[r+\alpha_{n-1}]_q \sum_{\ell_1+\ell_2+\cdots+\ell_{n-1}=(n-1)-k}$$

$$\times (-1)^{(n-1)-k} [r+\alpha_0]_q^{\ell_1} [r+\alpha_1]_q^{\ell_2} \cdots [r+\alpha_{n-2}]_q^{\ell_{n-1}},$$

this leads to

$$s_q(n, k, r; \overline{\alpha}) = s_q(n-1, k-1, r; \overline{\alpha}) - [r+\alpha_{n-1}]_q s_q(n-1, k, r; \overline{\alpha}).$$

By virtue of (3.5), this completes the proof. □

Definition 3.2. Let the generalized q-harmonic numbers of order k, associated with sequence $\overline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$, or briefly are called multiparameter q-harmonic numbers be defined by

$$H_{n,q}^{(k)}(r; \overline{\alpha}) = \sum_{j=0}^{n-1} \frac{1}{[r+\alpha_j]_q^k}. \tag{3.7}$$

Theorem 3.2. *The numbers* $s_q(n, k, r; \overline{\alpha})$ *can be expressed as*

$$s_q(n, k, r; \overline{\alpha}) = (-1)^n [r; \overline{\alpha}]_{\overline{n}, q} \sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k}$$

$$\times \frac{(-1)^{\ell_1+\ell_2+\cdots+\ell_k}}{\ell_1!\,\ell_2!\cdots\ell_k!} \prod_{i=1}^{k} \left(\frac{H_{n,q}^{(i)}(r; \overline{\alpha})}{i} \right)^{\ell_i}. \tag{3.8}$$

Proof. Since

$$[t-r;\ \overline{\alpha}]_{\underline{n},q} = [t-r-\alpha_0]_q[t-r-\alpha_1]_q\cdots[t-r-\alpha_{n-1}]_q$$

$$= q^{-(r+\alpha_0)}([t]_q-[r+\alpha_0]_q)q^{-(r+\alpha_1)}$$

$$\times([t]_q-[r+\alpha_1]_q)\cdots q^{-(r+\alpha_{n-1})}([t]_q-[r+\alpha_{n-1}]_q)$$

$$= (-1)^n[r;\ \overline{\alpha}]_{\overline{n},q}q^{-rn-\sum_{i=0}^{n-1}\alpha_i}\prod_{j=0}^{n-1}\Big(1-\frac{[t]_q}{[r+\alpha_j]_q}\Big)$$

$$= (-1)^n[r;\ \overline{\alpha}]_{\overline{\alpha}n,q}q^{-rn-\sum_{i=0}^{n-1}\alpha_i}\exp\Big(\sum_{j=0}^{n-1}\log\Big(1-\frac{[t]_q}{[r+\alpha_j]_q}\Big)\Big)$$

$$= (-1)^n[r;\ \overline{\alpha}]_{\overline{n},q}q^{-rn-\sum_{i=0}^{n-1}\alpha_i}\exp\Big(-\sum_{m=1}^{\infty}\sum_{j=0}^{n-1}\frac{1}{[r+\alpha_j]_q^m}\frac{[t]_q^m}{m}\Big)$$

$$= (-1)^n[r;\ \overline{\alpha}]_{\overline{n},q}q^{-rn-\sum_{i=0}^{n-1}\alpha_i}\exp\Big(-\sum_{m=1}^{\infty}H_{n,q}^{(m)}(r;\ \overline{\alpha})\frac{[t]_q^m}{m}\Big),$$

then

$$[t-r;\ \overline{\alpha}]_{\underline{n},q} = (-1)^n[r;\ \overline{\alpha}]_{\overline{n},q}\ q^{-rn-\sum_{i=0}^{n-1}\alpha_i}\sum_{k=0}^{\infty}\sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k}$$

$$\times\frac{(-1)^{\ell_1+\ell_2+\cdots+\ell_k}}{\ell_1!\,\ell_2!\cdots\ell_k!}\prod_{i=1}^{k}\left(\frac{H_{n,q}^{(i)}(r;\ \overline{\alpha})}{i}\right)^{\ell_i}[t]_q^k,$$

from (2.11), this leads to

$$\sum_{k=0}^{n}s_q(n,k,r;\ \overline{\alpha})[t]_q^k = \sum_{k=0}^{\infty}\sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k}\frac{(-1)^{\ell_1+\ell_2+\cdots+\ell_k}}{\ell_1!\,\ell_2!\cdots\ell_k!}\prod_{i=1}^{k}\left(\frac{H_{n,q}^{(i)}(r;\ \overline{\alpha})}{i}\right)^{\ell_i}[t]_q^k.$$

Equating the coefficients of $[t]_q^k$ on both sides, yields (3.8). □

Special cases:

(i) If putting $\alpha_j = j,\ \bar{j} = (0, 1, \cdots, n-1)$ in (3.8) gives

$$s_q(n, k; r, \bar{i}) = \tilde{s}_q(n, k; r)$$

$$= (-1)^n [r]_{\bar{n},q} \sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k} \frac{(-1)^{\ell_1+\ell_2+\cdots+\ell_k}}{\ell_1!\ell_2!\cdots\ell_k!} \prod_{i=1}^{k} \left(\frac{H_{n,q}^{(i)}(r; \bar{j})}{i} \right)^{\ell_i}, \tag{3.9}$$

where $H_{n,q}^{(i)}(r; \bar{j}) = \sum_{j=0}^{n-1} \frac{1}{[r+j]_q^i}$ and $\tilde{s}_q(n, k; r)$ are the non-central q-Stirling numbers of first kind.

(ii) If putting $\alpha_j = j, \bar{j} = (0, 1, \cdots, n-1)$, and $r = 1$ in (3.8) gives

$$s_q(n, k; 1, \bar{j}) = s_q(n+1, k+1)$$

$$= (-1)^n [n]_q! \sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k} \frac{(-1)^{\ell_1+\ell_2+\cdots+\ell_k}}{\ell_1!\ell_2!\cdots\ell_k!} \prod_{i=1}^{k} \left(\frac{H_{n,q}^{(i)}(1; \bar{j})}{i} \right)^{\ell_i}, \tag{3.10}$$

where $H_{n,q}^{(i)}(1; \bar{j}) = \sum_{j=0}^{n-1} \frac{1}{[1+j]_q^i} = \sum_{j=1}^{n} \frac{1}{[j]_q^i} = H_n^{(i)}(q)$ are q-harmonic numbers of order i, see [8] and $s_q(n, k+1)$ are q-Stirling numbers of first kind.

For the particular case, $n = 3$ and $k = 1$ in (3.9),

$$s_q(3, 1; r, \bar{j}) = [r]_q[r+1]_q[r+2]_q H_q^{(1)}(r; \bar{j})$$

$$= [r]_q[r+1]_q[r+2]_q \left(\frac{1}{[r]_q} + \frac{1}{[r+1]_q} + \frac{1}{[r+2]_q} \right)$$

$$= [r+1]_q[r+2]_q + [r]_q[r+2]_q + [r]_q[r+1]_q.$$

Also, when $n = 3$ and $k = 2$ in (3.10), we have

$$s_q(3,\ 2;\ 1,\ \bar{j}) = -[1]_q[2]_q[3]_q\left(-\frac{H_q^{(2)}(1;\ \bar{j})}{2} + \frac{(H_q^{(1)}(1;\ \bar{j}))^2}{2}\right)$$

$$= -([1]_q + [2]_q + [3]_q) = s_q(4,\ 3).$$

Remark 3.2. Notice that if $q \to 1$ in (3.8), we have

$$s(n,\ k,\ r;\ \overline{\alpha})$$

$$= (-1)^n (r;\ \overline{\alpha})_{\overline{n}} \sum_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = k} \frac{(-1)^{\ell_1 + \ell_2 + \cdots + \ell_k}}{\ell_1!\,\ell_2! \cdots \ell_k!} \prod_{i=1}^{k}\left(\frac{H_n^{(i)}(r;\ \overline{\alpha})}{i}\right)^{\ell_i}$$

$$= (-1)^{n-k} (r;\ \overline{\alpha})_{\overline{n}} \sum_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = k} \frac{1}{\ell_1!\,\ell_2! \cdots \ell_k!} \prod_{i=1}^{k}\left(\frac{(-1)^{i-1} H_n^{(i)}(r;\ \overline{\alpha})}{i}\right)^{\ell_i},$$

then

$$|s(n,\ k,\ r;\ \overline{\alpha})|$$

$$= (r;\ \overline{\alpha})_{\overline{n}} \sum_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = k} \frac{1}{\ell_1!\,\ell_2! \cdots \ell_k!} \prod_{i=1}^{k}\left(\frac{(-1)^{i-1} H_n^{(i)}(r;\ \overline{\alpha})}{i}\right)^{\ell_i}$$

$$= \frac{(r;\ \overline{\alpha})_{\overline{n}}}{k!} Y_k(H^{(1)}(r;\ \overline{\alpha}),\ -1!\,H^{(2)}(r;\ \overline{\alpha}),\ \cdots,\ (-1)^{k-1}(k-1)!\,H^{(k)}(r;\ \overline{\alpha})). \tag{3.11}$$

When $r = 1$ and $\alpha_j = j$ in (3.11) gives

$$|s(n,\ k,\ 1;\ \bar{j})| = |s(n+1,\ k+1)|$$

$$= [n]_q! \sum_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = k} \frac{1}{\ell_1!\,\ell_2! \cdots \ell_k!} \prod_{i=1}^{k}\left(\frac{(-1)^{i-1} H_n^{(i)}(1;\ \bar{j})}{i}\right)^{\ell_i}$$

$$= \frac{[n]_q!}{k!} Y_k(H_n^{(1)}(1;\ \bar{j}),\ -1!\,H_n^{(2)}(1;\ \bar{j}),\ \cdots,\ (-1)^{k-1}(k-1)!\,H_n^{(k)}(1;\ \bar{j})), \tag{3.12}$$

which agrees with [7, Equation (7b)].

Setting $r = 0$ in (3.11), gives Cakić's result [2].

Corollary 3.1. *The numbers* $s_q(n, k, r; \overline{\alpha})$ *have the explicit formula*

$$s_q(n, k, r; \overline{\alpha}) = (-1)^n [r; \overline{\alpha}]_{\overline{n}, q} \sum_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = k} \frac{(-1)^{\ell_1 + \ell_2 + \cdots + \ell_k}}{\ell_1! \, \ell_2! \cdots \ell_k!}$$

$$\times \prod_{i=1}^{k} \Big(\frac{1}{i} \sum_{j=0}^{n-1} \sum_{\ell=0}^{\infty} (-1)^{\ell} q^{\ell \alpha_j} \binom{\ell + k - 1}{\ell} [r]_q^{\ell} [\alpha_j]_q^{-k-\ell} \Big)^{\ell_i}. \quad (3.13)$$

Proof. Since

$$H_{n,q}^{(i)}(r; \overline{\alpha}) = \sum_{j=0}^{n-1} [r + \alpha_j]_q^{-i} = \sum_{j=0}^{n-1} \sum_{\ell=0}^{\infty} (-1)^{\ell} q^{\ell \alpha_j} \binom{\ell + i - 1}{\ell} [r]_q^{\ell} [\alpha_j]_q^{-i-\ell}.$$

Substituting in (3.8) yields (3.13). □

Also, if $q \to 1$ in (3.13), it worth noting that

$s(n, k, r; \overline{\alpha}) =$

$$(-1)^n (r; \overline{\alpha})_{\overline{n}} \sum_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = k} \prod_{i=1}^{k} \Big(\frac{1}{i} \sum_{j=0}^{n-1} \sum_{\ell=0}^{\infty} (-1)^{\ell} \binom{\ell + k - 1}{\ell} (r)^{\ell} (\alpha_j)^{-k-\ell} \Big)^{\ell_i}.$$

Theorem 3.3. *The signless generalized q-Stirling numbers of first kind can be expressed as*

$$|s_{q^{-1}, \overline{\alpha}}(n, k)| = q^{-\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} [\alpha_i]_q \sum_{r=0}^{k} \frac{1}{r!} \sum_{\ell_1 + \ell_2 + \cdots + \ell_r = k}$$

$$\times \frac{\prod_{i=1}^{r} \mathcal{H}^{(\ell_i)}{}_{n,q}(\overline{\alpha})}{\ell_1 \ell_2 \cdots \ell_r} (-1)^{k-r} (1-q)^k, \quad (3.14)$$

where $\mathcal{H}_{n,q}^{(k)}(\overline{\alpha}) = \sum_{j=1}^{n} \frac{q^{k\alpha_j}}{(1 - q^{\alpha_j})^k}$, *which gives new extension of the q-harmonic numbers, see* [13] *and* [11], *and* $\overline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Proof. Since

$$[t;\ \overline{\alpha}]_{\overline{n},\, q} = q^{\sum_{i=1}^{n} \alpha_i} \sum_{k=0}^{n} |s_{q^{-1},\, \overline{\alpha}}(n,\ k)| [t]_q^k. \tag{3.15}$$

$$\begin{aligned}
&[t + \alpha_1]_q [t + \alpha_2]_q \cdots [t + \alpha_n]_q \\
&= ([\alpha_1]_q + q^{\alpha_1}[t]_q)([\alpha_2]_q + q^{\alpha_2}[t]_q) \cdots ([\alpha_n]_q + q^{\alpha_n}[t]_q) \\
&= [\alpha_1]_q [\alpha_2]_q \cdots [\alpha_n]_q \prod_{j=1}^{n} \left(1 + \frac{q^{\alpha_j}[t]_q}{[\alpha_j]_q}\right) \\
&= \prod_{i=1}^{n} [\alpha_i]_q \exp\left(\sum_{k=1}^{\infty} \sum_{j=1}^{n} \left(\frac{q^{\alpha_j}}{[\alpha_j]_q}\right)^k (-1)^{k-1} \frac{[t]_q^k}{k} (1-q)^k\right) \\
&= \prod_{i=1}^{n} [\alpha_i]_q \exp\left(\sum_{k=1}^{\infty} \mathcal{H}_{n,q}^{(k)}(\overline{\alpha})(-1)^{k-1}(1-q)^k \frac{[t]_q^k}{k}\right).
\end{aligned}$$

Then

$$[t + \alpha_1]_q [t + \alpha_2]_q \cdots [t + \alpha_n]_q = \prod_{i=1}^{n} [\alpha_i]_q \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{k=0}^{\infty} \left(\mathcal{H}_{n,q}^{(k)}(\overline{\alpha})(-1)^{k-1}(1-q)^k \frac{[t]_q^k}{k}\right)^r,$$

using Cauchy rule of product of series, this leads to

$$\begin{aligned}
[t;\ \overline{\alpha}]_{\overline{n},\, q} &= \prod_{i=1}^{n} [\alpha_i]_q \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^{k-r}}{r!} \sum_{\ell_1 + \ell_2 + \cdots + \ell_r = k} \\
&\quad \times \frac{\mathcal{H}_{n,q}^{(\ell_1)}(\overline{\alpha}) \mathcal{H}_{n,q}^{(\ell_2)}(\overline{\alpha}) \cdots \mathcal{H}_{n,q}^{(\ell_r)}(\overline{\alpha})}{\ell_1 \ell_2 \cdots \ell_r} (1-q)^k [t]_q^k.
\end{aligned}$$

Equating the coefficients of $[t]_q^k$ on both sides of (3.15) and last equation, yields (3.14). □

When $k = 1$ and $\alpha_j = j$ in $\mathcal{H}_{n,q}^{(k)}(\overline{\alpha}) = \sum_{j=1}^{n} \frac{q^{k\alpha_j}}{(1 - q^{\alpha_j})^k}$, we obtain

$\mathcal{H}_{n,q}^{(1)} = \sum_{j=1}^{n} \frac{q^j}{(1-q^j)} = \mathcal{H}_n$, the q-harmonic numbers, see [13] and [11].

If putting $\alpha_j = j$ and $\bar{j} = (1, 2, \cdots, n)$ in (3.14), gives

$$|s_{q^{-1},\bar{j}}(n, k)| = |s_{q^{-1}}(n+1, k+1)|$$

$$= q^{-\binom{n+1}{2}}[n]_q!(1-q)^k \sum_{r=0}^{k} \frac{(-1)^{k-r}}{r!} \sum_{\ell_1+\ell_2+\cdots+\ell_r=k} \frac{\prod_{i=1}^{r} \mathcal{H}_{n,q}^{(\ell_i)}(\bar{j})}{\ell_1\ell_2\cdots\ell_r}, \tag{3.16}$$

where $|s_{q^{-1}}(n, k)|$ are the signless q-Stirling numbers of first kind.

For particular case, when $n = 3$ and $k = 2$ in (3.16), gives

$$|s_{q^{-1},\bar{j}}(3, 2)|$$

$$= q^{-6}[1]_q[2]_q[3]_q(1-q)^2\left(\frac{-\mathcal{H}_{3,q}^{(2)}(\bar{j})}{2} + \frac{H_q^3 H_q^3}{2}\right)$$

$$= q^{-6}[1]_q[2]_q[3]_q(1-q)^2\left(\frac{q^3}{(1-q)(1-q^2)} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^5}{(1-q^2)(1-q^3)}\right)$$

$$= q^{-6}[3]_q + q^{-2}[2]_q + q^{-1} = |s_{q^{-1}}(4, 3)|.$$

Theorem 3.4. *The generating function of the generalized non-central q-Stirling numbers of second kind,* $S_q(n, k, r; \overline{\alpha})$ *is given by*

$$\phi_{k,q}(t, r; \overline{\alpha}) = [t]_q^k \prod_{j=0}^{k} (1 - [r+\alpha_j]_q[t]_q)^{-1}. \tag{3.17}$$

The proof is left.

Corollary 3.2. *The numbers* $S_q(n, k, r; \overline{\alpha})$ *have the explicit formula*

$$S_q(n, k, r; \overline{\alpha}) = \sum [r+\alpha_0]_q^{\ell_0}[r+\alpha_1]_q^{\ell_1}\cdots[r+\alpha_k]_q^{\ell_k}, \tag{3.18}$$

where the summation is extended over all integer $\ell_j \geq 0,\ j = 0, 1, \cdots, k,$ *such that* $\ell_0 + \ell_1 + \cdots + \ell_k = n - k.$

Proof. Using (3.17), we can prove (3.18). □

Theorem 3.5. *The numbers* $s_q(n, k; r, \overline{\alpha})$ *and* $S_q(n, k; r, \overline{\alpha})$, *respectively, have the explicit formulas*

$$s_q(n, k; r, \overline{\alpha}) = \sum_{\sigma_n = n-k} (-1)^{n-k} \begin{bmatrix} r + \alpha_0 \\ i_0 \end{bmatrix}_q \begin{bmatrix} r + \alpha_1 \\ i_1 \end{bmatrix}_q \cdots \begin{bmatrix} r + \alpha_{n-1} \\ i_{n-1} \end{bmatrix}_q, \tag{3.19}$$

and

$$S_q(n, k; r, \overline{\alpha}) = \sum_{\sigma_n = k} \begin{bmatrix} r + \alpha_{i_0} \\ 1 - i_0 \end{bmatrix}_q \begin{bmatrix} r + \alpha_{i_0 + i_1} \\ 1 - i_1 \end{bmatrix}_q \cdots \begin{bmatrix} r + \alpha_{i_0 + i_1 + \cdots + i_{n-1}} \\ 1 - i_{n-1} \end{bmatrix}_q, \tag{3.20}$$

where $\sigma_n = i_0 + i_1 + \cdots + i_{n-1}$ *and* $i_\ell \in \{0, 1\}$.

Proof. Replacing α_i by $\alpha_i + r$ in (2.12) and (2.13), respectively, yields (3.19) and (3.20). □

From (3.18) and (3.19), we have the combinatorial identity

$$\sum [r + \alpha_0]_q^{\ell_0} [r + \alpha_1]_q^{\ell_1} \cdots [r + \alpha_k]_q^{\ell_k}$$

$$= \sum_{\sigma_n = k} \begin{bmatrix} r + \alpha_{i_0} \\ 1 - i_0 \end{bmatrix}_q \begin{bmatrix} r + \alpha_{i_0 + i_1} \\ 1 - i_1 \end{bmatrix}_q \cdots \begin{bmatrix} r + \alpha_{i_0 + i_1 + \cdots + i_{n-1}} \\ 1 - i_{n-1} \end{bmatrix}_q, \tag{3.21}$$

where the summation in the left hand side as given in (3.18).

4. Matrix Representation

Let $\mathbf{s_q}$, $\mathbf{S_q}$; $\mathbf{s}_{\mathbf{q}, \overline{\alpha}}$, $\mathbf{S}_{\mathbf{q}, \overline{\alpha}}$; $\mathbf{s}_{\mathbf{q}, \overline{\alpha}}(\mathbf{r})$, $\mathbf{S}_{\mathbf{q}, \overline{\alpha}}(\mathbf{r})$; $\tilde{\mathbf{s}}_{\mathbf{q}}(\mathbf{r})$; and $\mathbf{R_q}(\alpha)$ be $n \times n$ lower triangular matrices. Where $\mathbf{s_q}$ and $\mathbf{S_q}$ are matrices, whose entries are the q-Stirling numbers of first and second kind (i.e., $\mathbf{s_q} = [(\mathbf{s_q})_{\mathbf{ij}}]$ and $\mathbf{S_q} = [(\mathbf{S_q})_{\mathbf{ij}}]$); $\mathbf{s}_{\mathbf{q}, \overline{\alpha}}$ and $\mathbf{S}_{\mathbf{q}, \overline{\alpha}}$ are matrices, whose entries are generalized q-Stirling numbers of first and second kind (i.e.,

$\mathbf{s}_{\mathbf{q},\bar{\alpha}} = [(\mathbf{s}_{\mathbf{q},\bar{\alpha}})_{\mathbf{ij}}]$ and $\mathbf{S}_{\mathbf{q},\bar{\alpha}} = [(\mathbf{S}_{\mathbf{q},\bar{\alpha}})_{\mathbf{ij}}]$); $\mathbf{s}_{\mathbf{q},\bar{\alpha}}(\mathbf{r})$ and $\mathbf{S}_{\mathbf{q},\bar{\alpha}}(\mathbf{r})$ are matrices, whose entries are generalized non-central q-Stirling numbers of first and second kind (i.e., $\mathbf{s}_{\mathbf{q},\bar{\alpha}}(\mathbf{r}) = [(\mathbf{s}_{\mathbf{q},\bar{\alpha}}(\mathbf{r}))_{\mathbf{ij}}]$ and $\mathbf{S}_{\mathbf{q},\bar{\alpha}}(\mathbf{r}) = [(\mathbf{S}_{\mathbf{q},\bar{\alpha}}(\mathbf{r}))_{\mathbf{ij}}]$); $\tilde{\mathbf{s}}_{\mathbf{q}}(\mathbf{r})$ is matrix, whose entries are non-central q-Stirling numbers of first kind (i.e., $\tilde{\mathbf{s}}_{\mathbf{q}}(\mathbf{r}) = [(\tilde{\mathbf{s}}_{\mathbf{q}}(\mathbf{r}))_{\mathbf{ij}}]$), and $\mathbf{R}_{\mathbf{q}}(\alpha)$ is matrix, whose entries are coefficient of the generalized q-factorials (i.e., $\mathbf{R}_{\mathbf{q}}(\alpha) = [(\mathbf{R}_{\mathbf{q}}(\alpha))_{\mathbf{ij}}]$.

An algorithm to determine the matrices of generalized q-Stirling numbers of first and second kind, are denoted, respectively, by $\mathbf{s}_{\mathbf{q},\bar{\alpha}}$ and $\mathbf{S}_{\mathbf{q},\bar{\alpha}}$ are given as follows:

For a non negative integer $n > 0$, the elements of the n by n lower triangular matrix of $\mathbf{s}_{\mathbf{q},\bar{\alpha}}$ may be calculated as follows:

Algorithm

Set $\mathbf{s}_{\mathbf{q},\bar{\alpha}}^{\mathbf{1},\mathbf{1}} = 1$

For i = 2 to n do

Set $\mathbf{s}_{\mathbf{q},\bar{\alpha}}^{\mathbf{i},\mathbf{i}} = 1$

Calculate

$$\mathbf{s}_{\mathbf{q},\bar{\alpha}}^{\mathbf{i},\mathbf{1}} = (-1)^{i-1}\prod_{r=0}^{i-2}[\alpha_r]_q - [\alpha_{i-1}]_q \mathbf{s}_{\mathbf{q},\bar{\alpha}}^{\mathbf{i-1},\mathbf{1}}$$

Next i

For i = 3 to n do

for j = 2 to $i-1$ do calculate

$$\mathbf{s}_{\mathbf{q},\bar{\alpha}}^{\mathbf{i},\mathbf{j}} = \mathbf{s}_{\mathbf{q},\bar{\alpha}}^{\mathbf{i-1},\mathbf{j-1}} - [\alpha_{i-1}]_q \mathbf{s}_{\mathbf{q},\bar{\alpha}}^{\mathbf{i-1},\mathbf{j}}$$

Next j

Next i.

For a non negative integer $n > 0$, the elements of the n by n lower triangular matrix of $\mathbf{S}_{\mathbf{q},\overline{\alpha}}$ may be calculated as follows:

A computer program is written by using Maple program and executed for calculating. Let $\mathbf{s}_{\mathbf{q},\overline{\alpha}}$ and $\mathbf{S}_{\mathbf{q},\overline{\alpha}}$ are matrices, whose entries are generalized q-Stirling numbers of first and second kind (i. e., $\mathbf{s}_{\mathbf{q},\overline{\alpha}} = [(\mathbf{s}_{\mathbf{q},\overline{\alpha}})_{\mathbf{ij}}]$ and $\mathbf{S}_{\mathbf{q},\overline{\alpha}} = [(\mathbf{S}_{\mathbf{q},\overline{\alpha}})_{\mathbf{ij}}]$). For example, if $n = 3$, then

$$\mathbf{s}_{\mathbf{q},\overline{\alpha}} = \begin{pmatrix} 1 & 0 & 0 \\ -[\alpha_0]-[\alpha_1] & 1 & 0 \\ [\alpha_0][\alpha_1]+[\alpha_2][\alpha_0]+[\alpha_2][\alpha_1] & -[\alpha_0]-[\alpha_1]-[\alpha_2] & 1 \end{pmatrix},$$

and

$$\mathbf{S}_{\mathbf{q},\overline{\alpha}} = \begin{pmatrix} 1 & 0 & 0 \\ [\alpha_0]+[\alpha_1] & 1 & 0 \\ [\alpha_0]^2+[\alpha_0][\alpha_1]+[\alpha_1]^2 & [\alpha_0]+[\alpha_1]+[\alpha_2] & 1 \end{pmatrix}.$$

Equation (2.9) can be written in matrix form

$$\mathbf{s}_{\mathbf{q},\alpha\bar{\mathbf{i}}} = \mathbf{R}_{\mathbf{q}}(\alpha)\mathbf{s}_{\mathbf{q}}. \tag{4.1}$$

For example, if $n = 3$, then

$$\begin{pmatrix} 1 & 0 & 0 \\ -[\alpha] & 1 & 0 \\ [2\alpha][\alpha] & -[\alpha]-[2\alpha] & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1-[\alpha] & 1 & 0 \\ 1-[\alpha]-[2\alpha]+[2\alpha][\alpha] & 1-[\alpha]+[2]-[2\alpha] & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ [2] & -1-[2] & 1 \end{pmatrix}.$$

Equation (2.10) can be written in matrix form

$$\mathbf{S}_{\mathbf{q}} = \mathbf{R}_{\mathbf{q}}(\alpha)\mathbf{S}_{\mathbf{q},\alpha\bar{\mathbf{i}}}. \tag{4.2}$$

For example, if $n = 3$, then

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & [2]+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1-[\alpha] & 1 & 0 \\ 1-[\alpha]-[2\alpha]+[2\alpha][\alpha] & 1-[\alpha]+[2]-[2\alpha] & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ [\alpha] & 1 & 0 \\ [\alpha_2] & [\alpha]+[2\alpha] & 1 \end{pmatrix}.$$

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