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Decomposition and Group Theoretic Characterization of Pairs of Inverse Relations of the Riordan Type *

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Abstract. A new solution to Riordan's problem of combinatorial identities classification is presented. An algebgraic characterization of pairs of inverse relations of the Riordan type is given. The use of the integral representation approach for generating new types of combinatorial identities is demonstrated.

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1. Introduction

In 1968 Riordan [23] posed the problem of the characterization of known pairs of inverse relations of the form

$$a_m = \sum_{k=0}^{\infty} c_{mk} b_k, \qquad b_m = \sum_{k=0}^{\infty} d_{mk} a_k, \quad m = 0, 1, 2, \ldots, \tag{1}$$

where $C = (c_{mk})$ is an invertible infinite lower triangular matrix whose general term is a linear combination of known combinatorial numbers, and $D = (d_{mk})$ is its inverse. An interpretation of such relations with the help of generating function technique is given in [22–24]. Each pair of relations of this form generates a combinatorial identity

$$\sum_k c_{nk} d_{km} = \delta_{mn}, \quad n, m = 0, 1, 2, \ldots,$$

where δ_{mn} is the Kronecker symbol.

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A large part of Riordan's monograph [23] on combinatorial identities is concerned with pairs of inverse relations with binomial coefficients in the one-dimensional case.

The first complete solution to the Riordan problem was given in [5–7] by studying properties of a special type of matrices, defined by certain integral construction and a certain 5-tuple $\mathbb{F} = (\{\alpha_m\}, \{\beta_k\}, \varphi(x), f(x), \psi(x))$, where $\{\alpha_m\}_{m=0,1,2,\dots}$, $\{\beta_k\}_{k=0,1,2,\dots}$ are sequences of the nonzero numbers and $\varphi(x)$, $f(x)$, $\psi(x)$ – the Laurent formal power series over the field $\mathbb{C}$. In this paper, we extend the algebraic results of [5–7] and demonstrate how the integral representation can be used in a unified approach for generating new types of combinatorial identities. In Section 2, we give necessary technical preliminaries. In Section 3, we present a new solution to Riordan's problem. In Section 4, we compare this results with other known classification approaches.

2. Preliminaries

In this section we briefly recall the properties of the **res** operator. A detailed description can be found in [5]. Here we shall explore univariate series only, although the **res** concept can be also used for multivariate series. Let $\mathbb{L}$ be the set of a Laurent formal power series over the field $\mathbb{C}$ containing only finitely many terms with negative powers. The *order* of a monomial $c_k w^k$ is k. The *order* of a series $C(w) = \sum_k c_k w^k$ from $\mathbb{L}$ is the minimal order of monomials with nonzero coefficient. Let $\mathbb{L}_k$ denote a set of series of order k, $\mathbb{L} = \bigcup_{k=-\infty}^{\infty} \mathbb{L}_k$. Two series $A(w) = \sum_k a_k w^k$ and $B(w) = \sum_k b_k w^k$ from $\mathbb{L}$ are equal if and only if $a_k = b_k$ for all k. We can introduce in $\mathbb{L}$ operations of addition, multiplication, substitution, inversion and differentiation (see [12, 3]).*

For $C(w) \in \mathbb{L}$ define the *formal residue* as $\mathbf{res}_w\, C(w) = c_{-1}$.

Let $f(w), \psi(w) \in \mathbb{L}_0$. Further we will use the following notations:

- $h(w) = wf(w) \in \mathbb{L}_1$;
- $l(w) = w/\psi(w) \in \mathbb{L}_1$;
- $z'(w) = \mathrm{d}/\mathrm{d}w z(w)$;
- $\overline{h} = \overline{h}(z) \in \mathbb{L}_1$ – inverse of series $z = h(w) \in \mathbb{L}_1$.

Let $A(w) = \sum_k a_k w^k$ be a generating function for sequence $\{a_k\}$. Then

$$a_k = \mathbf{res}_w\, A(w) w^{-k-1}, \quad k = 0, 1, 2, \dots. \tag{2}$$

For example, one of the possible representations of the binomial coefficient is

$$\binom{n}{k} = \mathbf{res}_w (1 + w)^n w^{-k-1}, \quad k = 0, 1, \dots, n. \tag{3}$$

* In combinatorial literature 'Cauchy algebra of formal power series' is often used for the same purpose [23].

There are several properties (rewriting rules) for the **res** operator which immediately follow from its definition and properties of operations on Laurent formal power series over field $\mathbb{C}$. We list only a few of them which will be used in this paper. Let $A(w) = \sum_k a_k w^k$ and $B(w) = \sum_k b_k w^k$ be a generating functions from $\mathbb{L}$.

Rule 1 (**res** removal).

$$\mathbf{res}_w\, A(w)w^{-k-1} = \mathbf{res}_w\, B(w)w^{-k-1} \quad \text{for all } k \text{ if and only if } A(w) = B(w). \tag{4}$$

Rule 2 (linearity). For any α, β from $\mathbb{C}$

$$\alpha\mathbf{res}_w\, A(w)w^{-k-1} + \beta\mathbf{res}_w\, B(w)w^{-k-1} = \mathbf{res}_w((\alpha A(w) + \beta B(w))w^{-k-1}). \tag{5}$$

Rule 3 (substitution). (a) For $f(w) \in \mathbb{L}_k$ ($k \geqslant 1$) and $A(w)$ – arbitrary element of $\mathbb{L}$, or (b) for $A(w)$ polynomial and $f(w)$ – arbitrary element of $\mathbb{L}$ including a constant

$$\sum_k f^k(w)\mathbf{res}_z\big(A(z)z^{-k-1}\big) = A(f(w)).$$

Rule 4 (inversion). For $f(w)$ from $\mathbb{L}_0$

$$\sum_k z^k\mathbf{res}_w\big(A(w)f^k(w)w^{-k-1}\big) = [A(w)/f(w)h'(w)]_{w=\bar{h}(z)}. \tag{6}$$

Rule 5 (change of variable). If $f(w) \in \mathbb{L}_0$, then

$$\mathbf{res}_w\big(A(w)f^k(w)w^{-k-1}\big) = \mathbf{res}_z([A(w)/f(w)h'(w)]_{w=\bar{h}(z)}z^{-k-1}).$$

3. Decomposition and Algebraic Characterization of Invertible Pairs of Relations of Riordan Type

Let $A(w) = \sum_k a_k w^k$ be a generating function for sequence $\{a_k\}$.

DEFINITION. We say that matrix $C = (c_{mk})_{m,k=0,1,2,\ldots}$ in (1) is of type R or $R^q(\alpha_m, \beta_k; \varphi, f, \psi)$, if its general term is defined by the formula

$$c_{mk} = \frac{\beta_k}{\alpha_m}\mathbf{res}_z(\varphi(z)f^k(z)\psi^m(z)z^{-m+qk-1}), \tag{7}$$

where q is a positive integer, $\alpha_m, \beta_k \neq 0$, and $\varphi(z), f(z), \psi(z) \in \mathbb{L}_0$. In particular, for $q = 1$, the matrix (c_{mk}) is infinite lower triangular with the general term

$$c_{mk} = \frac{\beta_k}{\alpha_m}\mathbf{res}_z(\varphi(z)f^k(z)\psi^m(z)z^{-m+k-1}). \tag{8}$$

Relations (1) are completely defined by the matrix $C = (c_{mk})_{m,k=0,1,2,\dots}$. That is why we attach the type of this matrix to the relation, and use terms as 'a relation of type R' when necessary. Some time we will omit superscript in the type (for example, when $q = 1$ or when it is not important in context).

LEMMA 1. *Matrix (c_{mk}) of the type $R(\alpha_m, \beta_k; \varphi(z), f(z), \psi(z))$ can be uniquely represented as a matrix of type*

$$\overline{R} = R(\alpha_m, \beta_k; z\overline{h}'(z)\varphi(\overline{h}(z))/\overline{h}(z), 1, \psi(\overline{h}(z))\overline{h}(z)/z),$$

or as a matrix of the type

$$\underline{R} = R(\alpha_m, \beta_k; z\bar{l}'(z)\varphi(\bar{l}(z))/\bar{l}(z), f(\bar{l}(z))\bar{l}(z)/z, 1).$$

Proof. It suffies to make a change of variable $w = h(z) = zf(z)$ or respecitvely $w = l(z) = z/\psi(z)$ in (7) under the **res** sign using Rule 5. □

The result of this lemma means that each matrix $A = (a_{mk})_{m,k=0,1,2,\dots}$ of the type $R(\alpha_m, \beta_k; \varphi, f, \psi)$ possesses two canonical representations – of the type $\overline{R}$ and of the type $\underline{R}$.

EXAMPLE 1. The binomial coefficients $\binom{n}{k}$, $n, k = 0, 1, 2, \dots$, admit integral representations of following types:

(a) of the type $\overline{R} = R(1, 1; 1, 1, (1 + w))$:

$$\binom{n}{k} = \mathbf{res}_w (1 + w)^n w^{-n+k-1}; \tag{9}$$

(b) of the type $\underline{R} = R(1, 1; (1 - w)^{-1}, (1 - w)^{-1}, 1)$:

$$\binom{n}{k} = \mathbf{res}_w (1 - w)^{-k-1} w^{-n+k-1};$$

(c) ordinary:

$$\binom{n}{k} = \mathbf{res}_w (1 + w)^n w^{-k-1}, \qquad \binom{n}{k} = \mathbf{res}_w (1 - w)^{-n+k-1} w^{-k-1}.$$

THEOREM 2 (on inverse [5]). (a) *Relations of the type R^1 are equivalent to the functional relations between generating functions $\widetilde{A}(w) = \sum_{m\geqslant 0} \alpha_m a_m w^m$, $\widetilde{B}(w) = \sum_{k\geqslant 0} \beta_k b_k w^k$:*

$$\widetilde{A}(l(w))l'(w)\psi(w) = \varphi(w)\widetilde{B}(h(w)), \tag{10}$$

where $h(w) = wf(w)$, $l(w) = w/\psi(w)$.

(b) *Matrix* (d_{mk}), *the inverse of the matrix* (c_{mk}) *of the type* R^1 *exists, unique, is of the type* R^1, *and has the following general term:*

$$d_{mk} = \frac{\alpha_k}{\beta_m} \mathbf{res}_w \big\{ \varphi^{-1}(w) l'(w) h'(w) \psi^{-k+1}(w) f^{-m-1}(w) w^{-m+k-1} \big\}.$$

Proof (Proof is given here for completeness). (a) We have

$$\begin{aligned} a_m &= \sum_{k=0}^{m} c_{mk} b_k = \sum_{k=0}^{m} b_k \frac{\beta_k}{\alpha_m} \mathbf{res}_z (\varphi(z) f^k(z) \psi^m(z) z^{-m+k-1}) = \sum_{k=0}^{\infty} \ldots^{\star} \\ &= \frac{1}{\alpha_m} \mathbf{res}_z (\varphi(z) \psi^m(z) z^{-m-1} \left(\sum_{k=0}^{\infty} \beta_k b_k (zf)^k \right) \\ &= \frac{1}{\alpha_m} \mathbf{res}_z \varphi(z) \psi^m(z) \widetilde{B}(zf) z^{-m-1}, \quad m = 0, 1, 2, \ldots, \Leftrightarrow \end{aligned}$$

$$\begin{aligned} &\alpha_m a_m = \mathbf{res}_w\, \widetilde{A}(w) w^{-m-1} = \mathbf{res}_z\, \varphi(z) \psi^m(z) \widetilde{B}(zf) z^{-m-1}, \\ &\quad m = 0, 1, 2, \ldots. \end{aligned}$$

By the change of variable $w = l(z) = z/\psi(z) \in L_1$ under the $\mathbf{res}_z$ sign and the Rule 1, we get formula (10).

(b) From (10) we have $\widetilde{B}(h(w)) = \widetilde{A}(l(w)) l'(w) \psi(w)/\varphi(w)$, and by change of variable $z = \overline{h}(w) \in L_1$ we get

$$\widetilde{B}(z) = \widetilde{A}(l(\overline{h})) l'(\overline{h}) \psi(\overline{h}) / \varphi(\overline{h}) \Rightarrow$$

$$\begin{aligned} b_m &= \frac{1}{\beta_m} \mathbf{res}_z \widetilde{B}(z) z^{-m-1} \\ &= \frac{1}{\beta_m} \mathbf{res}_z \{ \widetilde{A}(l(\overline{h})) l'(\overline{h}) \psi(\overline{h}) \varphi^{-1}(\overline{h}) z^{-m-1} \} \\ &= \frac{1}{\beta_m} \mathbf{res}_z \left\{ l'(\overline{h}) \psi(\overline{h}) \varphi^{-1}(\overline{h}) z^{-m-1} \left(\sum_{k=0}^{\infty} \alpha_k a_k (l(\overline{h}))^k \right) \right\} \\ &= \sum_{k=0}^{m} a_k \frac{\alpha_k}{\beta_m} \mathbf{res}_z \{ l'(\overline{h}) \psi(\overline{h}) \varphi^{-1}(\overline{h}) (l(\overline{h}))^k z^{-m-1} \} \\ &\quad \text{(using the change of variable } z = h(w)) \\ &= \sum_{k=0}^{m} a_k \frac{\alpha_k}{\beta_m} \mathbf{res}_w \{ \varphi^{-1}(w) l'(w) h'(w) \psi^{-k+1}(w) f^{-m-1}(w) w^{-m+k-1} \} \\ &= \sum_{k=0}^{m} a_k d_{mk}. \qquad \square \end{aligned}$$

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* For $k > m$ each added term of sum is equal to 0 according to the definition of **res** operator.

THEOREM 3 (on classification [5–7]). *The pairs of inverse relations of the simplest type, of Gould type, of Tchebycheff type, of Legendre type, of Legendre–Tchebycheff type, of Abel type, of ordinary and exponential types* ([23], *Table* 2.1–2.5, 3.1–3.3) *and of Lagrange type* ([23], *Ch.* 4, §5) *are all of the type* $R^q = R^q(\alpha_m, \beta_k; \varphi, f, \psi)$.

THEOREM 4 (on combinatorial characterization). *Matrices of binomial coefficients, Stirling numbers* (*usual and generalized*) *of the first and second kind and many others numbers belong to the type* $R^q = R^q(\alpha_m, \beta_k; \varphi, f, \psi)$.

The proofs of those theorems are by comparison of integral representation for combinatorial numbers (see [27], [4], pp. 68–92, [5], pp. 269–274 and others) with general term of the matrix of type R. For example, the formula (9) for binomial coefficients implies that $\binom{n}{k} = \binom{n}{n-k}$ is of type $R(1, 1; 1, 1, 1 + w)$; the known formula for the Stirling numbers of second kind

$$s_2(m, k) = \frac{m!}{k!}\mathbf{res}_z(-1 + \exp z)^k z^{-m-1}$$

implies that $s_2(m, k)$ is of type $R(1/m!, 1/k!; 1, (\mathrm{e}^z - 1)/z, 1)$; the formula

$$\alpha^{m-k}/(m-k)! = \mathbf{res}_z(\mathrm{e}^{\alpha z} z^{-m+k-1}), \quad \alpha = \text{const},$$

implies that exponential coefficients $\alpha^{m-k}/(m-k)!$ is of the type $R(1, 1; \mathrm{e}^{\alpha z}, 1, 1)$.

THEOREM 5 (on product). *Let the sequence* $\{\alpha_m\}$, $\alpha_m \neq 0$, *be fixed. Then the product of two matrices* $A = (a_{mk})$ *and* $B = (b_{mk})$ *of the type* $R(\alpha_m, \alpha_k; \varphi, f, \psi)$ *is a matrix the same type.*

Proof. In correspondence with Lemma 1, represent matrix A by the formula of type $\overline{R}$ and matrix B by the formula of type $\underline{R}$, i.e.

$$a_{mk} = \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\, \varphi_1(w)\psi^m(w)w^{-m+k-1}$$

and

$$b_{mk} = \frac{\alpha_k}{\alpha_m}\mathbf{res}_z\, \varphi_2(z)f^k(z)z^{-m+k-1},$$

where series $\varphi_1, \varphi_2, \psi, f \in L_0$. Then

$$\begin{aligned} d_{mk} &= \sum_{s=k}^{m} a_{ms}b_{sk} \\ &= \sum_{s=k}^{m} \frac{\alpha_s}{\alpha_m}\mathbf{res}_w\, \varphi_1(w)\psi^m(w)w^{-m+s-1}\frac{\alpha_k}{\alpha_s}\mathbf{res}_z\, \varphi_2(z)f^k(z)z^{-s+k-1} = \sum_{s=0}^{\infty}\dots \\ &= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\left\{\varphi_1(w)\psi^m(w)w^{-m-1}\left[\sum_{s=0}^{\infty} w^s\mathbf{res}_z(\varphi_2(z)f^k(z)z^k)z^{-s-1}\right]\right\} \end{aligned}$$

$$\begin{aligned}
&\text{(by the substitution rule, } z = w)\\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\{\varphi_1(w)\psi^m(w)w^{-m-1}[\varphi_2(z)f^k(z)z^k)]_{z=w}\}\\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\{\varphi_1(w)\varphi_2(w)\psi^m(w)f^k(w)w^{-m+k-1}\},
\end{aligned}$$

where $\varphi_1, \varphi_2, \psi, f \in L_0$. □

THEOREM 6 (on decomposition). *A matrix* (d_{mk}) *of the type* $R(\alpha_m, \beta_k; \varphi, f, \psi)$ *splits into the product of three matrices* (a_{mk}), (b_{mk}) *and* (c_{mk}) *of the types*

$$R(\alpha_m, \gamma_k; \varphi_1, 1, \psi),\ R(\gamma_m, \xi_k; \varphi_2, 1, 1) \quad \textit{and} \quad R(\xi_m, \beta_k; \varphi_3, f, 1), \tag{11}$$

where $\varphi = \varphi_1\varphi_2\varphi_3$, $\varphi_1(0)\varphi_2(0)\varphi_3(0) \neq 0$ *and sequences of nonzero numbers* $\{\gamma_m\}$, $\{\xi_k\}$ *are arbitrary.*

Proof. We have

$$\begin{aligned}
d_{mk} &= \sum_{r=0}^{m}\sum_{s=0}^{r} a_{mr}b_{rs}c_{sk}\\
&= \sum_{r=0}^{m}\sum_{s=0}^{r} \frac{\gamma_r}{\alpha_m}\mathbf{res}_w\, \varphi_1(w)\psi(w)^m w^{-m+r-1}\times\\
&\quad\times \frac{\xi_s}{\gamma_r}\mathbf{res}_v\, \varphi_2(v)v^{-r+s-1} \times \frac{\beta_k}{\xi_s}\mathbf{res}_u\, \varphi_3(u)f^k(u)u^{-s+k-1}\\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\ldots^{\star}\\
&= \frac{\beta_k}{\alpha_m}\mathbf{res}_w\left\{\varphi_1(w)\psi(w)^m w^{-m-1}\left[\sum_{r=0}^{\infty} w^r\mathbf{res}_v\, \varphi_2(v)v^{-r-1}\times\right.\right.\\
&\quad\left.\left.\times\left(\sum_{s=0}^{\infty} v^s\mathbf{res}_u\, \varphi_3(u)f^k(u)u^{-s+k-1}\right)\right]\right\}.
\end{aligned}$$

To finish the proof, it suffices to sum over s and r by the substitution rule for the **res** operator in variables u and v with changes $u = v$ and $v = w$ respectively. □

Remark 1. In [5], this theorem was proved with $\gamma_m = \xi_k = 1$ and $\varphi_2 = \varphi_3 = 1$ (see also [7, 8]). The above-considered theorem is stronger than the analogous result in [5], while the scheme of the proof remains almost the same. The presence of new weighting coefficients γ_m and ξ_k, $m, k = 0, 1, 2, \ldots$, and represntation $\varphi = \varphi_1\varphi_2\varphi_3$ allows us to formulate new results on algebraic characterization of pairs of Riordan type, generates new identities of Riordan type and introduces new objects (methods) as, e.g., the Lagrange summation matrix below.

* For $r > m$, or $s > r$ each added term of the sum is equal 0 according to definition of the **res** operator.

THEOREM 7 (on algebraic characterization). *Let the sequence* $\overline{\alpha} = \{\alpha_m\}, \alpha_m \neq 0$*, is fixed.*

(a) *The set of all matrices of the type* $R(\alpha_m, \alpha_k; \varphi, f, \psi)$ *forms a subgroup of the group* $TN(C)$ *of all triangular matrices. Denote this group as* $R(\overline{\alpha}) = R(\overline{\alpha}; \varphi, f, \psi)$*. If* $\varphi(0) = f(0) = \psi(0) = 1$*, then group* $R(\overline{\alpha})$ *is unitriangular.*

(b) *Groups* $R(\overline{\alpha}; \varphi, 1, \psi)$ *and* $R(\overline{\alpha}; \varphi, f, 1)$ *are isomorphic.*

(c) *Every set of matrices of the types* $R(\alpha_m, \alpha_k; \varphi, 1, 1)$*,* $R(\alpha_m, \alpha_k; 1, f, 1)$ *and* $R(\alpha_m, \alpha_k; 1, 1, \psi)$ *forms a subgroup of the group* $R(\overline{\alpha})$*. Denote them as* $R(\overline{\alpha}; \varphi, 1, 1)$*,* $R(\overline{\alpha}; 1, f, 1)$ *and* $R(\overline{\alpha}; 1, 1, \psi)$ *respectively.*

(d) *The subgroups* $R(\overline{\alpha}; \varphi, 1, 1)$*,* $R(\overline{\alpha}; 1, f, 1)$ *and* $R(\overline{\alpha}; 1, 1, \psi)$ *have the only pairwise common element* $I = (\delta_{mk})_{m,k=0,1,\ldots}$.

(e) *The group* $R(\overline{\alpha}; \varphi, f, \psi)$ *decomposes into the product of its three proper subgroups* $R(\overline{\alpha}; 1, 1, \psi)$*,* $R(\overline{\alpha}; \varphi, 1, 1)$ *and* $R(\overline{\alpha}; 1, f, 1)$.

Proof. Part (a) of the statement follows from the theorems on product and inversion immediately if we show that the identity element $I = (\delta_{mk})_{m,k=0,1,\ldots}$ belongs to the set matrices $R(\overline{\alpha}; \varphi, f, \psi)$. Indeed, letting $\varphi(w) = f(w) = \psi(w) = 1$, we have $c_{mk} = \frac{\alpha_k}{\alpha_m} \mathbf{res}_w\, w^{-m+k-1} = \frac{\alpha_k}{\alpha_m} \delta_{mk}$.

Part (b) of the statement follows from the result of Lemma 1 which states one-to-one correspondence between the elements of sets $R(\overline{\alpha}; \varphi, f, 1)$ and $R(\overline{\alpha}; \varphi, 1, \psi)$.

(c) This statement in extended form is as follows:

(1) The set of matrices of the type $R(\overline{\alpha}; \varphi, 1, 1)$ is a group under matrix multiplication with following properties: $R(\overline{\alpha}; \varphi_1, 1, 1) * R(\overline{\alpha}; \varphi_2, 1, 1) = R(\overline{\alpha}; \varphi_1\varphi_2, 1, 1)$; $R^{(-1)}(\overline{\alpha}; \varphi, 1, 1) = R(\{\frac{\alpha_m}{\alpha_k}\}; 1/\varphi, 1, 1)$; $I = (\delta_{mk})_{m,k=0,1,\ldots}$ is the the identity element.

(2) The set of matrices of the type $R(\overline{\alpha}; 1, f, 1)$ is a group under matrix multiplication with following properties:

$$R(\overline{\alpha}; 1, f_1, 1) * R(\overline{\alpha}; 1, f_2, 1) = R(\overline{\alpha}; 1, h_2(h_1(w))/w), 1),$$

where $h_1 = h_1(w) = wf_1(w) \in L_1, h_2 = h_2(w) = wf_2(w) \in L_1$; $R^{(-1)}(\overline{\alpha}; 1, f, 1) = R(\{\frac{\alpha_m}{\alpha_k}\}; 1, \bar{h}/w, 1)$; $I = (\delta_{mk})_{m,k=0,1,\ldots}$ is the identity element.

(3) The set of matrices of the type $R(\overline{\alpha}; 1, 1, \psi)$ is a group under matrix multiplication with following properties:

$$R(\overline{\alpha}; 1, 1, \psi_1) * R(\overline{\alpha}; 1, 1, \psi_2) = R(\overline{\alpha}; 1, 1, (l_1(l_2(z))/z)^{-1}),$$

where $l_1 = l_1(w) = w/\psi_1(w) \in L_1$, $l_2 = l_2(w) = w/\psi_2(w) \in L_1$; $R^{(-1)}(\overline{\alpha}; 1, 1, \psi) = R(\{\frac{\alpha_m}{\alpha_k}\}; 1, 1, z/\bar{l}(z))$; $I = (\delta_{mk})_{m,k=0,1,\ldots}$ is the the identity element.

These three claims are easily proved with the help of straightforward computations and inversion theorem:

(1) Let $\varphi_1(w), \varphi_2(w) \in L_0$, $A = (a_{mk})$, $B = (b_{mk})$, where

$$a_{mk} = \frac{\alpha_k}{\alpha_m} \mathbf{res}_w\, \varphi_1(w) w^{-m+k-1}$$

and

$$b_{mk} = \frac{\alpha_k}{\alpha_m}\mathbf{res}_z\, \varphi_2(z)z^{-m+k-1}, \quad m, k = 0, 1, 2, \ldots.$$

Then

$$\begin{aligned}
d_{m,k} &= \sum_{s=k}^{m} a_{ms}b_{sk} \\
&= \sum_{s=k}^{m} \frac{\alpha_s}{\alpha_m}\mathbf{res}_w\, \varphi_1(w)w^{-m+s-1}\frac{\alpha_k}{\alpha_s}\mathbf{res}_z\, \varphi_2(z)z^{-s+k-1} = \sum_{s=0}^{\infty}\cdots \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\left\{\varphi_1(w)w^{-m-1}\left[\sum_{s=0}^{\infty} w^s\mathbf{res}_z(z^k\varphi_2(z)z^{-s-1})\right]\right\} \\
&\quad \text{(by substitution rule, the change } z = w\text{)} \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\{\varphi_1(w)\varphi_2(w)w^{-m-1}\}.
\end{aligned}$$

(2) Let $f_1(w)$, $f_2(w) \in L_0$, $A = (a_{mk})$, $B = (b_{mk})$, where

$$a_{mk} = \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\, f_1^k(w)w^{-m+k-1}$$

and

$$b_{mk} = \frac{\alpha_k}{\alpha_m}\mathbf{res}_z\, f_2^k(z)z^{-m+k-1}, \quad m, k = 0, 1, 2, \ldots.$$

Then

$$\begin{aligned}
d_{m,k} &= \sum_{s=k}^{m} a_{ms}b_{sk} \\
&= \sum_{s=k}^{m} \frac{\alpha_s}{\alpha_m}\mathbf{res}_w\, f_1^s(w)w^{-m+s-1}\frac{\alpha_k}{\alpha_s}\mathbf{res}_z\, f_2^k(z)z^{-s+k-1} = \sum_{s=0}^{\infty}\cdots \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\left\{w^{-m-1}\left[\sum_{s=0}^{\infty}(f_1(w)w)^s\mathbf{res}_z(z^k f_2^k(z)z^{-s-1})\right]\right\} \\
&\quad \text{(by inversion rule, the change } z = wf_1(w) = h_1(w) \in L_1\text{)} \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\{(h_2(h_1(w)))^k w^{-m-1}\} \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\{(h_2(h_1(w)/w))^k w^{-m+k-1}\}.
\end{aligned}$$

(3) Let $\psi_1(w)$, $\psi_2(w) \in L_0$, $A = (a_{mk})$, $B = (b_{mk})$, where

$$a_{mk} = \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\, \psi_1^m(w)w^{-m+k-1}$$

and

$$b_{mk} = \frac{\alpha_k}{\alpha_m}\mathbf{res}_z\, \psi_2^m(z)z^{-m+k-1}, \quad m, k = 0, 1, 2, \dots.$$

Then

$$\begin{aligned}
d_{m,k} &= \sum_{s=k}^{m} a_{ms}b_{sk} \\
&= \sum_{s=k}^{m} \frac{\alpha_s}{\alpha_m}\mathbf{res}_w\, \psi_1^m(w)w^{-m+s-1}\frac{\alpha_k}{\alpha_s}\mathbf{res}_z\, \psi_2^s(z)z^{-s+k-1} = \sum_{s=0}^{\infty}\dots \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\left\{\psi_1^m(w)w^{-m-1}\left[\sum_{s=0}^{\infty} w^s\mathbf{res}_z(z^k\psi_2^s(z)z^{-s-1})\right]\right\} \\
&\quad \text{(by inversion rule, the change } z = \bar{l}_2(w) \in L_1) \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\{\psi_1^m(w)w^{-m-1}\bar{l}_2^k(w)/\psi_2(\bar{l}_2(w))l_2'(\bar{l}_2(w))\} \\
&\quad \text{(by change } w = l_2(z) \in L_1) \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_z\{(l_1(l_2(z)))^{-m}l_2^{-1}(z)z^k l_2'(z)/\psi_2(z)l_2'(z)\} \\
&= \frac{\alpha_k}{\alpha_m}\mathbf{res}_z\{(l_1(l_2(z)/z)^{-m}\ z^{-m+k-1}\}.
\end{aligned}$$

The second part of each claim (1)–(3) follows from the first part.

In order to prove statement (d), we have to check the conditions under which the matrices of the considered types are equal, i.e. under which conditions for $m, k = 0, 1, 2, \dots$, the following equalities hold:

$$\frac{\alpha_k}{\alpha_m}\mathbf{res}_w\, \varphi(w)w^{-m+k-1} \tag{12}$$

$$= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\, f^k(w)w^{-m+k-1} \tag{13}$$

$$= \frac{\alpha_k}{\alpha_m}\mathbf{res}_w\, \psi^m(w)w^{-m+k-1}. \tag{14}$$

By setting $k = m$ in (12)–(14), we have $\mathbf{res}_w\, \varphi(w)w^{-1} = \mathbf{res}_w\, f^m(w)w^{-1} = \mathbf{res}_w\, \psi^m(w)w^{-1} \Leftrightarrow \varphi(0) = f^m(0) = \psi^m(0), m = 0, 1, 2, \dots \Rightarrow \varphi(0) = f(0) = \psi(0) = 1$. By setting $k = 0$ in (12)–(14), we have $b_m = \mathbf{res}_w\, \varphi(w)w^{-m-1} = \mathbf{res}_w\, w^{-m-1} = \mathbf{res}_w\, \psi^m(w)w^{-m-1}, m = 0, 1, 2, \dots \Leftrightarrow b_m = \delta_{m0} = \mathbf{res}_w\, \psi^m(w)w^{-m-1}, m = 0, 1, 2, \dots$. From this, the statement about $R(\overline{\alpha}; 1, f, 1) \cap R(\overline{\alpha}; \varphi, 1, 1)$ and $R(\overline{\alpha}; 1, f, 1) \cap R(\overline{\alpha}; 1, 1, \psi)$ follows. In order to prove it for $R(\overline{\alpha}; \varphi, 1, 1) \cap R(\overline{\alpha}; 1, 1, \psi)$, observe that $b_m = \mathbf{res}_w\, \psi^m(w)w^{-m-1}, m = 0, 1, 2, \dots \Rightarrow \varphi(z) = \sum_{m=0}^{\infty} b_m z^m = \sum_{m=0}^{\infty} z^m \mathbf{res}_w\, \psi^m(w)w^{-m-1}$ (by inversion rule) $\Rightarrow \varphi(z) = 1/\psi(\bar{l}(z))l'(\bar{l}(z))$.

Setting $k = 1$ in (12) and (14) we have $\mathbf{res}_w\, \varphi(w)w^{-m} = \mathbf{res}_w\, \psi^m(w)w^{-m}, m = 1, 2, \dots \Leftrightarrow \mathbf{res}_w\, \varphi(w)w^{-m-1} = \mathbf{res}_w\, \psi^{m+1}(w)w^{-m-1}, m = 0, 1, 2, \dots \Rightarrow \varphi(z) =$

$\sum_{m=0}^{\infty} z^m \mathbf{res}_w\, \psi^{m+1}(w)w^{-m-1}$ = (by inversion rule) $= 1/l'(\bar{l}(z)) \Rightarrow \varphi(z) = 1/l'(\bar{l}(z)) = 1/\psi(\bar{l}(z))l'(\bar{l}(z)) \Rightarrow \psi(\bar{l}(z)) = 1 \Rightarrow \psi(w) = \varphi(w) = 1$.

The statement (e) follows from the theorem on decomposition. □

DEFINITION. We say that the matrix (8) is a summation method of type $R = R(\alpha_m, \beta_k; \varphi, f, \psi)$, if all c_{mk} are nonnegative and $\lim_{m\to\infty} c_{mk} = 0$ for any k. We say that the summation methods of types $R(\alpha_m, c_k; 1, 1, \psi)$, $R(c_m, d_k; \varphi, 1, 1)$ and $R(d_m, \beta_k; 1, f, 1)$ are summation methods of Lagrange, of Voronoy, and analytic, respectively.

THEOREM 8. *Let $A(w)$, $B(w)$, $C(w)$ and $D(w)$ be generating functions for sequences α_m, β_k, c_k and d_k respectively. For a summation matrix of type R to be regular* ([11])*, it is necessary and sufficient that $A(l(w))l'(w)\psi(w) = \varphi(w)B(h(w))$.*

Similarly, summation matrices of Lagrange, Voronoy, and analytic will be regular, if an only if, respectively, $\psi(w)l'(w)A(l(w)) = C(w)$, $C(w) = \varphi D(w)$ and $D(w) = B(wf(w))$.

The proof follows immediately from the Toepliz–Shur theorem [11].

THEOREM 9 (on functional-theoretical characterization, [5]). *The well-known summation matrices of divergent series due to Vallée-Poussin, Obreshkov, Cezàro, Euler, $P(q, r, s)$, general methods of Lagrange, Voronoy, Gronwall, etc., are particular cases of the regular summation method of type R. A regular summation method of type R splits into the product of summation methods of Lagrange, of Voronoy, and analytic.*

Remark 2. Summation method of Lagrange is introduced here for the first time. Classic Gronwall, Voronoy, and analytic summation methods of divergent series are the methods of type $R(1/b_m, 1; g(w)(1 - wf(w)), f(w), 1)$, $R(1/b_m, 1; g(w)(1 - w), 1, 1)$ and $R(1, 1; (1 - h(w)), f(w), 1)$ respectively, where $g(w)$, $f(w) \in L_0$ and $g(w) = 1 + \sum_{m=1}^{\infty} b_m w^m$. The second part of the last theorem is an extension of the known result in the divergent summing theory that the Gronwall matrix splits into the product of matrices of summing divergent series of Voronoy and analytic.

4. Riordan Arrays and Riordan Group

The concepts of a Riordan group and Riordan Array has been introduced in 1991 by Shapiro *et al.* ([26]). The group is quite easily developed but unifies many themes in enumeration, including of the generalized concept of Renewal Array defined by Rogers in 1978 ([28] and [1, 19–21], etc.). Their basic idea was to define a class of infinite lower triangular arrays with properties analogous to those those of the Pascal triangle whose elements.

A Riordan Array is an infinite lower triangular array $D = \{d_{nk}\}_{n,k\in N}$, defined by a pair of formal power series $(d(t), h(t))$, $d(t), h(t) \in L_0$, such that the generic element is the nth coefficient in the series $d(t)(th(t))^k$:

$$d_{nk} = [t^n]d(t)(th(t))^k, \quad n, k = 0, 1, 2, \ldots, d_{nk} = 0 \text{ for } k > n. \tag{15}$$

Here it is always assumed that $d(0) \neq 0$; if we also have $h(0) \neq 0$, then the Riordan Array is said to be *proper*; in the proper-case the diagonal elements d_{nn} are different from zero for all $n \in N$. Proper Riordan Arrays are characterized by the following basic property ([28]): a matrix $\{d_{nk}\}_{n,k\in N}$ is a proper Riordan Array iff there exists a sequence $A = \{a_i\}_{i\in N}$ with $a_0 \neq 0$ s.t. every element $d_{n+1,k+1}$ can be expressed as a linear combination, with coefficient in A of the elements in the preceding row, starting from the preceding column: $d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots$.

The Riordan group is a set of infinite lower triangular matrices of type (15). Shapiro ([26], etc.) often denotes a Riordan matrix D by $D = (g(x), f(x))$, $g(x) \in L_0$, $f(x) \in L_1$. We denote the set of Riordan matrices by R^*. R^* is a group under matrix multiplication with the following properties: $(g(x), f(x)) * (u(x), v(x)) = (g(x)u(f(x)), v(f(x)))$, $I = (1, x)$ is the identity element. The inverse of D is given by $D^{(-1)} = (1/g(\overline{f}(x)), \overline{f}(x))$.

It was shown in [21] that the members of the Riordan group of the form $(xf'(x)/f(x), f(x))$ belong to a subgroup denoted by PW; note the following:

(i) The identity $(1, x) = x(x)'/x, x) \in PW$.

(ii) The product

$$\begin{aligned}
&(xf'(x)/f(x), f(x)) * (xh'(x)/h(x), h(x))\\
&\quad = \left(\frac{xf'(x)}{f(x)}\frac{f(x)h'(f(x))}{h(f(x))}, h(f(x))\right)\\
&\quad = \left(\frac{x(h(f(x)))'}{h(f(x))}, h(f(x))\right) \in PW.
\end{aligned}$$

(iii) The inverse of $(xf'(x)/f(x), f(x))$ is

$$\left(\frac{1}{\overline{f}(x)\frac{f'(\overline{f}(x))}{f(\overline{f}(x))}}, \overline{f}(x)\right) = \left(x\frac{(\overline{f})'(x)}{\overline{f}(x)}, \overline{f}(x)\right) \in PW.$$

Another similar early examples involving the *Bell subgroup* $(g(x), xg(x))$ are by Jabotinsky ([13, 14], [26], p. 238).

Let $f(x), g(x) \in L_0$ as usually. Let us now (following [1]) define the operation which we call *Lagrange product*: $f(x) \otimes g(x) = f(x)g(xf(x))$. This product is associative, distributive, and it has an identity: $f(x) \otimes 1 = f(x) = 1 \otimes f(x)$. Let $y = xf(x) \in L_1$. The inverse element of a series $f(x) \in L_0$ is denoted by $\overline{f}(y) = 1/f(\bar{l}(y))$, where $x = l(y) = yf(y)$. The group $(L_0, \otimes)$ is called the *Lagrange group* (see [1]).

THEOREM 10 (on inclusion). *Let* $\varphi(0) = f(0) = \psi(0) = 1$. *Then*

(a) *The Riordan group coincides with the group* $R(\{1\}; \varphi, f, 1)$*, the Bell group coincides with the group* $R(\{1\}; f, f, 1)$*, the* PW *group coincides with the group* $R(\{1\}; (wf(w))'/f(w), f, 1)$.
(b) *The PW group is isomorphic to the group* $R(\{1\}; 1, 1, \psi)$ (*see Lemma* 1)*; the Bell group is isomorphic to the Lagrange group.*
(c) *The group* $R(\{1\}; \varphi, f, \psi)$ *decomposes into a product of three own subgroups – PW, Voronoy and Bell.*
(d) *The Riordan group decomposes into a product of two own subgroups – Voronoy and PW, or into a product of two own subgroups – Voronoy and Bell.*

Proof. (a) This statement is easily proven by comparisons of corresponding matrices. In the first case

$$c_{mk} = \mathbf{res}_z(\varphi(z) f^k(z) z^{-m+k-1}), \quad m, k = 0, 1, 2, \ldots .$$

In the second case

$$c_{mk} = \mathbf{res}_z(f^{k+1}(z) z^{-m+k-1}), \quad m, k = 0, 1, 2, \ldots .$$

In the third case

$$c_{mk} = \mathbf{res}_z((zf(z))' f^{k-1}(z) z^{-m+k-1}), \quad m, k = 0, 1, 2, \ldots .$$

(b) After change of variable $w = zf(z) = h(z) \in L_1$, $z = \overline{h}(w) \in L_1$, we have

$$c_{mk} = \mathbf{res}_z((zf(z))' f^{k-1}(z) z^{-m+k-1}) = \mathbf{res}_z((w/\overline{h}(w))^m w^{-m+k-1}),$$
$$m, k = 0, 1, 2, \ldots .$$

To prove this part of statement (b), observe the following: the result of the operation $\otimes$ for Lagrange group $(L_0, \otimes) : g(x) = f(x)g(xf(x))$, $f(x), g(x) \in L_0$, and the result of matrix operation $a_m = \sum_{k=0}^{\infty} c_{mk} g_k$, $m = 0, 1, 2, \ldots$ do coincide. Indeed, functional relation in this case is

$$\begin{aligned} A(x) &= \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} x^m \left\{ \sum_{k=0}^{\infty} c_{mk} g_k \right\} \\ &= \sum_{m=0}^{\infty} x^m \left\{ \sum_{k=0}^{\infty} g_k \mathbf{res}_z(f^{k+1}(z) z^{-m+k-1}) \right\} \\ &= \sum_{m=0}^{\infty} x^m \mathbf{res}_z(f(z) g(wf(z) z^{-m-1}) = f(x) g(wf(x)). \end{aligned}$$

Statements (c) and (d) follow directly from the splitting theorem and the statement (b). $\square$

EXAMPLE 2. Inverse identities of Legendre–Tshebyshev type ([23], Table 2.6, relation 5). Let $p > 0$, $r > 0$, and in (1)

$$C = (c_{mk}) = \left(\binom{rm+p}{m-k} - (r-1)\binom{rm+p}{m-k-1}\right),\quad m, k = 0, 1, 2, \ldots. \tag{16}$$

Then:

(a) matrix (16) defines the relation of the type

$$R^1(1, 1; (1+z)^p(1-(r-1)z), 1, (1+z)^r);$$

(b) inverse identities defined by matrix (16) are equivalent to the following functional identities

$$A(z(1+z)^{-r}) = (1+z)^{p+1}B(z), \qquad B(z) = (1+z)^{-p-1}A(z(1+z)^{-r}).$$

(c) matrix $D = (d_{mk})$ (inverse of the matrix (16)) is defined as

$$D = (d_{mk}) = \left((-1)^{m-k}\binom{m+p+rk-k}{m-k}\right).$$

(d) matrices C and D can be expanded as $C = ABI$, $D = IB^{-1}A^{-1}$, where I is the identity matrix,

$$A = (a_{mk}) = \left(\binom{mr}{m-k}\right),$$

$$B = (b_{mk}) = \left(\binom{p}{m-k} - (r-1)\binom{p}{m-k-1}\right),$$

$$A^{-1} = (a^{(-1)}_{mk}) = \left((-1)^{m-k}\binom{m+rk-k-1}{m-k}\right),$$

$$B^{-1} = (b^{(-1)}_{mk}) = \left((-1)^{m-k}\binom{p+m-k}{m-k}\right).$$

(e) matrix relations

$$CD = I, \qquad C = ABI, \qquad D = IB^{-1}A^{-1},$$

$$ABB^{-1}A^{-1} = I, \qquad BB^{-1}A^{-1} = A^{-1}, \qquad ABB^{-1} = A$$

generate the following combinatorial identities

$$\sum_{s=k}^{m}(-1)^{s-k}\left\{\binom{rm+p}{m-s} - (r-1)\binom{rm+p}{m-s-1}\right\} \times$$
$$\times \binom{s+p+rk-k}{s-k} = \delta(m,k), \quad m, k = 0, 1, 2, \ldots,$$

$$\sum_{s=k}^{m}\binom{mr}{m-s}\left\{\binom{p}{s-k}-(r-1)\binom{p}{s-k-1}\right\}$$
$$=\left(\binom{rm+p}{m-k}-(r-1)\binom{rm+p}{m-k-1}\right),\quad m,k=0,1,2,\ldots,$$

$$\sum_{s=k}^{m}\binom{p+m-s}{m-s}\binom{s+rk-k-1}{s-k}$$
$$=\binom{m+p+rk-k}{m-k},\quad m,k=0,1,2,\ldots,$$

$$\sum_{n=k}^{m}\sum_{t=n}^{m}\sum_{s=t}^{m}(-1)^{t-k}\binom{mr}{m-s}\left(\binom{p}{s-t}-(r-1)\binom{p}{s-t-1}\right)\times$$
$$\times\binom{p+t-n}{t-n}\binom{n+rk-k-1}{n-k}=\delta(m,k),\quad m,k=0,1,2,\ldots,$$

$$\sum_{t=k}^{m}\sum_{s=t}^{m}\left(\binom{p}{m-s}-(r-1)\binom{p}{m-s-1}\right)\times$$
$$\times(-1)^{s-k}\binom{p+s-t}{s-t}\cdot\binom{t+rk-k-1}{t-k}$$
$$=(-1)^{m-k}\binom{m+rk-k-1}{m-k},\quad m,k=0,1,2,\ldots,$$

$$\sum_{t=k}^{m}\sum_{s=t}^{m}\binom{mr}{m-s}\left(\binom{p}{s-t}-(r-1)\binom{p}{s-t-1}\right)\cdot(-1)^{t-k}\binom{p+t-k}{t-k}$$
$$=\binom{mr}{m-k},\quad m,k=0,1,2,\ldots.$$

Proof. From the definition of the general term of matrix C and from the integral representation of binomial coefficient (9) and taking into account (11), it follows that

$$c_{mk}=\mathbf{res}_z(1+z)^{rm+p}z^{-m+k-1}-(r-1)\mathbf{res}_z(1+z)^{rm+p}z^{-m+k}$$
$$=\mathbf{res}_z\big((1+z)^{rm+p}(1-(r-1)z)z^{-m+k-1}\big).$$

Comparison of this expression for c_{mk} with (8) proves the claim (a) of this example, if we let

$$\alpha_m=\beta_k=1,\qquad \varphi(z)=(1+z)^p(1-(r-1)z),$$

$$f(z) = 1, \qquad \psi(z) = (1+z)^r.$$

Other claims follow from properties of the operator **res** and of the relations of type R^1. □

5. Conclusion

We hope that the results of Theorems 5, 7, 8 are new in the theory of Riordan Arrays and in the theory of Riordan groups, and that it is easy to find a combinatorial interpretation and applications for them. Results of Theorems 2, 3, 8 and 9 are natural and not surprising for several reasons. An integral representation of the type R typically appears in the evaluation of combinatorial sums of different kinds (see [5], main theorem). This allows one to give a combinatorial interpretation to summation formulae, related to matrices of the type R. Weighting coefficients α_m and β_k could be interpreted for example as a number of terms or a value of the sum under investigation (see [2], etc., Example 3). Operations of multiplication, substitution and inversion in Cauchy algebra of series, hidden in the construction of matrix of the R type, also have a combinatorial interpretation (see [10, 12] and many others), including combinatorial interpretation and various proofs for one and multinomial inversion Lagrange formulas ([9]), explaining in every particular case the algebraic structure of the enumeration object under investigation. The result of Theorem 5 plays a similar role (compare to the result of Theorem 8). Example 2 of the generation of inverse identities of Legendre–Tshebyshev type can be viewed as an extension of the Riordan approach. The representation of combinatorial numbers a_n, $n = 0, 1, 2, \ldots$, as well as their generating functions $A(w) = \sum_{n=0}^{\infty} a_n w^n$ by an infinite triangular (*semicirculant*) matrices is usual procedure in combinatorial analysis (see, to example, [12], §1,3 and remark in [25], p. 43). Wide class of combinatorial schemes including Riordan Arrays and Riordan groups are used in tree enumerations.

Note that construction (8) and the results of Theorems 2–8 can be easily extended in several variants to the multidimensional case with the help of the main theorem in [5]. Also results of this paper can be extended to a wide class of difference and q-difference relations by following the nice results of Krattenchaler ([15–18]). These results and their combinatorial interpretation is the direction of our future investigations.

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