# Orthogonal Laurent-polynomials and continued fractions associated with log-normal distributions 

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Abstract: This paper describes properties and computational procedures related to orthogonal Laurent-polynomials, continued fractions, two-point Padé approximants, strong moment problems and L-Gaussian quadrature associated with $\log$-normal distribution functions $\phi(t)$ defined by $\phi^{\prime}(t)=\left(q^{1 / 2} / 2 \kappa \sqrt{\pi}\right) \mathrm{e}^{-(\ln t / 2 \kappa)^{2}}, 0<q<1, q=\mathrm{e}^{-2 \kappa^{2}}$. Log-normal distributions have recently been found to be applicable in weather research related to hurricanes. They are also of particular interest since one can obtain many explicit expressions for associated functions, formulae and other constructions.

Keywords: Orthogonal Laurent-polynomials, continued fractions, two-point Padé approximants, log-normal distribution.

## 1. Introduction

A distribution function $\phi(t)$ on $(0, \infty)$ (i.e., a bounded nondecreasing function with infinitely many points of increase), whose moments

$$
\begin{equation*}
c_{k}:=\int_{0}^{\infty}(-t)^{k} \mathrm{~d} \phi(t), \quad k=0, \pm 1, \pm 2, \ldots \tag{1.1}
\end{equation*}
$$

all exist, induces an inner product

$$
\begin{equation*}
(R, S):=\int_{0}^{\infty} R(t) S(t) \mathrm{d} \phi(t), \quad R, S \in \Lambda \tag{1.2}
\end{equation*}
$$

where $\Lambda$ denotes the space of Laurent polynomials (L-polynomials)

$$
\begin{equation*}
\Lambda:=\left[\sum_{k=m}^{n} a_{k} z^{k}: a_{k} \in \mathbb{R},-\infty<m \leqslant k \leqslant n<+\infty\right] . \tag{1.3}
\end{equation*}
$$

[^0]For $m=0, \pm 1, \pm 2, \ldots$,

$$
\begin{equation*}
H_{0}^{(m)}:=1 \quad \text { and } \quad H_{k}^{(m)}:=\operatorname{det}\left(c_{m+i+j}\right)_{i, j=0}^{k-1}, \quad k=1,2,3, \ldots, \tag{1.4}
\end{equation*}
$$

denote the Hankel determinants associated with $\left\{c_{k}\right\}$. It is well known [8] that

$$
\begin{equation*}
H_{2 n+1}^{(-2 n)}>0, \quad H_{2 n}^{(-2 n+1)}>0, \quad H_{2 n}^{(-2 n)}>0, \quad H_{2 n-1}^{(-2 n+1)}<0, \quad n=1,2,3, \ldots \tag{1.5}
\end{equation*}
$$

We are interested here in sequences $\left\{Q_{n}(z)\right\}$ of L-polynomials defined by $Q_{0}(z):=1$ and

$$
\begin{align*}
& Q_{2 n}(z):=\frac{(-1)^{n}}{H_{2 n}^{(-2 n+1)}}\left|\begin{array}{cccc}
c_{-2 n} & \cdots & c_{-1} & (-z)^{-n} \\
\vdots & & \vdots & \vdots \\
c_{-1} & \cdots & c_{2 n-2} & (-z)^{n-1} \\
c_{0} & \cdots & c_{2 n-1} & (-z)^{n}
\end{array}\right|, \quad n=1,2,3, \ldots,  \tag{1.6a}\\
& Q_{2 n+1}(z):=\frac{(-1)^{n}}{H_{2 n+1}^{(-2 n)}}\left|\begin{array}{cccc}
c_{-2 n-1} & \cdots & c_{-1} & (-z)^{-n-1} \\
\vdots & & \vdots & \vdots \\
c_{-1} & \cdots & c_{2 n-1} & (-z)^{n-1} \\
c_{0} & \cdots & c_{2 n} & (-z)^{n}
\end{array}\right|, \quad n=0,1,2, \ldots \tag{1.6~b}
\end{align*}
$$

It follows from (1.6) that the $Q_{n}(z)$ have the form

$$
\begin{equation*}
Q_{2 n}(z)=\sum_{j=-n}^{n} q_{2 n, j} z^{j}, \quad Q_{2 n+1}(z)=\sum_{j=-n-1}^{n} q_{2 n+1, j} z^{j}, \quad n=0,1,2, \ldots \tag{1.7a}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{2 n,-n}=q_{2 n+1,-n-1}=1, \quad q_{2 n, n}=\frac{H_{2 n}^{(-2 n)}}{H_{2 n}^{(-2 n+1)}}, \quad q_{2 n+1, n}=\frac{H_{2 n+1}^{(-2 n-1)}}{H_{2 n+1}^{(-2 n)}} \tag{1.7b}
\end{equation*}
$$

and satisfy the orthogonality conditions

$$
\begin{align*}
& \left(Q_{2 n}, z^{m}\right)=\left\{\begin{array}{ll}
0, & -n \leqslant m \leqslant n-1, \\
\frac{H_{2 n+1}^{(-2 n)}}{H_{2 n}^{(-2 n+1)},} & m=n,
\end{array} \quad n=1,2,3, \ldots,\right.
\end{align*}\left(\begin{array}{ll}
0, & -n \leqslant m \leqslant n,  \tag{1.8a}\\
\left(Q_{2 n+1}, z^{m}\right)= \begin{cases}H_{2 n+2}^{(-2 n-2)} \\
H_{2 n+1}^{(-2 n)} & \end{cases} & m=-n-1, \tag{1.8b}
\end{array},\right.
$$

$[6,7,9,10]$ (see also [5] for further examples of orthogonal L-polynomials). Setting $\|Q\|^{2}:=(Q, Q)$ one obtains for $n, m=0,1,2, \ldots,\left(Q_{n}, Q_{m}\right)=0$ for $m \neq n$, and

$$
\begin{align*}
& \left\|Q_{2 n}\right\|^{2}=q_{2 n, n}\left(Q_{2 n}, z^{n}\right)=\frac{H_{2 n}^{(-2 n)} H_{2 n+1}^{(-2 n)}}{\left(H_{2 n}^{(-2 n+1)}\right)^{2}}  \tag{1.9a}\\
& \left\|Q_{2 n+1}\right\|^{2}=\left(Q_{2 n+1}, z^{-n-1}\right)=\frac{H_{2 n+2}^{(-2 n-2)}}{H_{2 n+1}^{(-2 n)}} \tag{1.9b}
\end{align*}
$$

Our purpose in this paper is to investigate log-normal distribution functions $\phi(t)$ defined by

$$
\begin{equation*}
\phi^{\prime}(t):=w(t):=\frac{q^{1 / 2}}{2 \kappa \sqrt{\pi}} \mathrm{e}^{-(\ln t / 2 \kappa)^{2}}, \quad 0<q<1, \quad q=\mathrm{e}^{-2 \kappa^{2}} \tag{1.10}
\end{equation*}
$$

and the associated orthogonal L-polynomials $Q_{n}(z)$, L-Gaussian quadrature formulae, strong Stieltjes moment problems, two-point Padé approximants and positive T-fractions

$$
\begin{equation*}
\mathrm{K}_{n=1}^{\infty}\left(\frac{F_{n} z}{1+G_{n} z}\right)=\frac{F_{1} z \mid}{\mid 1+G_{1} z}+\frac{F_{2} z \mid}{\mid 1+G_{2} z}+\frac{F_{3} z \mid}{\mid 1+G_{3} z}+\cdots, \quad F_{n}>0, G_{n}>0, \tag{1.11}
\end{equation*}
$$

which correspond to the pair ( $L_{0}, L_{\infty}$ ) of formal power series (fps)

$$
\begin{equation*}
L_{0}:=\sum_{k-1}^{\infty}-c_{-k} z^{k}, \quad L_{\infty}:=\sum_{k=0}^{\infty} c_{k} z^{-k} . \tag{1.12}
\end{equation*}
$$

One reason for interest in the log-normal distribution functions is that their moments (1.1) have the tractable form

$$
\begin{equation*}
c_{k}:=\int_{0}^{\infty}(-t)^{k} w(t) \mathrm{d} t=(-1)^{k} q^{-k^{2} / 2-k}, \quad k=0, \pm 1, \pm 2, \ldots \tag{1.13}
\end{equation*}
$$

which lead to relatively simple, explicit expressions for the Hankel determinants (1.4) and many related formulae and constructions. Another motivation for this study is that log-normal distribution functions have been found to be applicable to research on hurricanes (see, for example, [13] and references given therein).

Orthogonal polynomials with respect to the weight functions (1.10) were introduced by Stieltjes [16] and Wigert [18] (see also [17, p. 33] and [4]). Orthogonal L-polynomials with respect to (1.10), first studied by Pastro [14], have the form (1.7a), but are normalized so that $q_{2 n, n}=q_{2 n+1, n}=1$. His starting point is to write down closed-form expressions for the $Q_{n}(z)$ in terms of $q$-shifted factorials

$$
(a ; q)_{n}:= \begin{cases}1, & n=0  \tag{1.14}\\ (1-a)(1-a q) \cdots\left(1-a q^{n-1}\right), & n=1,2,3, \ldots\end{cases}
$$

(See [2] and [15] for more information on these and related topics.) Pastro proves orthogonality and biorthogonality of his $Q_{n}(z)$ using the $q$-binomial theorem, computes the norms $\left\|Q_{n}\right\|$, and states the three-term recurrence relations. Our approach to the subject is to begin by deriving the positive T-fraction (1.11) corresponding to (1.12). This is done by two different procedures: first using quotient-difference relations; second by computing Hankel determinants directly. The latter are used to give explicit expressions for formulae (1.7b), (1.8) and (1.9). Known properties of positive T-fractions are then applied to give the determinant formulae (1.6) for the $Q_{n}(z)$, to obtain recurrence relations (3.2), and to prove that the related strong Stieltjes moment problem for $\left\{c_{n}\right\}$ is indeterminate (Theorem 2). The latter result is also established directly by giving other distribution functions that generate the same sequence of moments (1.13). Explicit expressions for the $Q_{n}(z)$ are obtained in Theorem 3 by using the recurrence relations (3.2). We conclude by stating in (3.8) the L-Gaussian quadrature formulae associated with log-normal distribution functions.

## 2. Positive T-fractions

Throughout the remainder of this paper we assume that $0<q<1$ is given and that the related log-normal moment sequence $\left\{c_{n}\right\}$ is defined by (1.13).

Theorem 1. The positive T-fraction (1.11) corresponding to the pair ( $L_{0}, L_{\infty}$ ) of fps

$$
\begin{equation*}
L_{0}:=\sum_{k=1}^{\infty}(-1)^{k-1} q^{-k^{2} / 2+k} z^{k}, \quad L_{\infty}:=\sum_{k=0}^{\infty}(-1)^{k} q^{-k^{2} / 2-k} z^{-k} \tag{2.1}
\end{equation*}
$$

has the coefficients given by

$$
\begin{align*}
& F_{1}:=q^{1 / 2}, \quad F_{n}:=q^{-n+3 / 2}\left(1-q^{n-1}\right), \quad n=2,3,4, \ldots,  \tag{2.2a}\\
& G_{n}:=q^{1 / 2}, \quad n=1,2,3, \ldots . \tag{2.2b}
\end{align*}
$$

Proof. A straightforward induction argument applied to the FG-relations of McCabe and Murphy [12]

$$
\begin{array}{ll}
F_{1}^{(m)}=0, \quad G_{1}^{(m)}=\frac{-c_{-m-1}}{c_{-m}}, & m=0, \pm 1, \pm 2, \ldots, \\
F_{n+1}^{(m)}=F_{n}^{(m+1)}+G_{n}^{(m+1)}-G_{n}^{(m)}, & n=1,2,3, \ldots, m=0, \pm 1, \pm 2, \ldots, \\
G_{n+1}^{(m)}=\frac{F_{n+1}^{(m)}}{F_{n+1}^{(m-1)}} G_{n}^{(m-1)}, & n=1,2,3, \ldots, m=0, \pm 1, \pm 2, \ldots, \tag{2.3c}
\end{array}
$$

yields the array of coefficients, for $m=0, \pm 1, \pm 2, \ldots$,

$$
\begin{equation*}
F_{n}^{(m)}=q^{-m-n+3 / 2}\left(1-q^{n-1}\right), \quad n \geqslant 2, \quad G_{n}^{(m)}=q^{1 / 2}, \quad n \geqslant 1 . \tag{2.4}
\end{equation*}
$$

Formulas (2.2) then follow from the relations

$$
\begin{equation*}
F_{1}=-c_{-1}, \quad F_{n}=F_{n}^{(0)}, \quad n \geqslant 2, \quad G_{n}=G_{n}^{(0)}, \quad n \geqslant 1 . \tag{2.5}
\end{equation*}
$$

The assertion of correspondence is a consequence of [8, Theorem 7.19 and Algorithm 7.3.1].
It is noteworthy that formulae (2.2) can also be obtained directly from

$$
\begin{align*}
F_{1} & =-H_{1}^{(-1)}, \quad F_{n}=-\frac{H_{n-2}^{(-n+3)} H_{n}^{(-n)}}{H_{n-1}^{(-n+2)} H_{n-1}^{(-n+1)}}, \quad n \geqslant 2,  \tag{2.6}\\
G_{n} & =-\frac{H_{n-1}^{(-n+2)} H_{n}^{(-n)}}{H_{n}^{(-n+1)} H_{n-1}^{(-n+1)}}, \quad n \geqslant 1,
\end{align*}
$$

(see, for example, [11]) and the expressions

$$
\begin{equation*}
H_{k}^{(m)}=(-1)^{-k m}\left[q^{-(m+k-1)^{2} / 2-(m+k-1)}\right]^{k} q^{-k\left(k^{2}-1\right) / 6} \sum_{j=1}^{k-1}\left(1-q^{j}\right)^{k-j} \tag{2.7}
\end{equation*}
$$

Formulae (2.7) can be derived by showing that the Hankel determinants (1.4) are essentially of the Vandermonde type.

If $f_{n}(z), A_{n}(z)$ and $B_{n}(z)$ denote the $n$th approximant, numerator and denominator, respectively, of the positive T-fraction (1.11), then it is known [11] that $A_{n}(z)$ and $B_{n}(z)$ are polynomials of degree at most $n, f_{n}(z)=A_{n}(z) / B_{n}(z)$ is a rational function holomorphic at $z=0$ and $\infty$, and

$$
\begin{array}{ll}
\Lambda_{0}\left(f_{n}(z)\right)=\sum_{k=1}^{n}-c_{-k} z^{k}+z^{n+1} \alpha_{n}(z), & n=1,2,3, \ldots, \\
\Lambda_{\infty}\left(f_{n}(z)\right)=\sum_{k=0}^{n-1} c_{k} z^{-k}+z^{n} \beta_{n}(z), & n=1,2,3, \ldots, \tag{2.8b}
\end{array}
$$

where $\alpha_{n}$ and $\beta_{n}$ are rational functions holomorphic at $z=0$ and $\infty$. It follows that $f_{n}(z)$ is the ( $n, n$ ) two-point Padé approximant of order $(n+1, n)$ of ( $L_{0}, L_{\infty}$ ).

It is readily seen that the following continued fraction is equivalent to the positive T -fraction (1.11), (2.2) [8, Section 2.3]:

$$
\begin{equation*}
\frac{z \mid}{\mid e_{1}+d_{1} z}+\frac{z \mid}{\mid e_{2}+d_{2} z}+\frac{z \mid}{\mid e_{3}+d_{3} z}+\cdots, \tag{2.9a}
\end{equation*}
$$

where

$$
\begin{align*}
& e_{2 n+1}:=q^{n-1 / 2} \frac{\left(q ; q^{2}\right)_{n}}{\left(q^{2} ; q^{2}\right)_{n}}, \quad n \geqslant 0, \quad e_{2 n}:=q^{n} \frac{\left(q^{2} ; q^{2}\right)_{n-1}}{\left(q ; q^{2}\right)_{n}}, \quad n \geqslant 1,  \tag{2.9b}\\
& d_{n}:=q^{1 / 2} e_{n}, \quad n=1,2,3, \ldots \tag{2.9c}
\end{align*}
$$

Since $\left\{\left(q ; q^{2}\right)_{n}\right\}$ and $\left\{\left(q^{2} ; q^{2}\right)_{n}\right\}$ both converge to finite limits, it follows from (2.2) that $\sum_{1}^{\infty} e_{n}=q^{1 / 2} \sum_{1}^{\infty} d_{n}<\infty$. Hence by [11, Theorems 3.3 and 6.3] we obtain the next theorem.

Theorem 2. (A) The continued fractions (1.12) and (2.9), which have the same approximants $f_{n}(z)$, are divergent for all $0 \neq z \in \mathbb{C}$.
(B) The subsequences $\left\{f_{2 n}(z)\right\}$ and $\left\{f_{2 n+1}(z)\right\}$ converge uniformly on compact subsets of $S_{\pi}:=\{z \in \mathbb{C}:|\arg z|<\pi\}$ to different holomorphic functions $F_{0}(z)$ and $F_{1}(z)$, respectively.
(C) There exist distinct distribution functions $\phi_{0}(t)$ and $\phi_{1}(t)$ that generate the sequence $\left\{c_{n}\right\}$ of moments (1.13) and satisfy

$$
\begin{equation*}
F_{\nu}(z):=\lim _{n \rightarrow \infty} f_{2 n+\nu}(z)=z \int_{0}^{\infty} \frac{\mathrm{d} \phi_{\nu}(t)}{z+t}, \quad z \in S_{\pi}, \quad \nu=0,1 \tag{2.10}
\end{equation*}
$$

Hence the strong Stieltjes moment problem for $\left\{c_{n}\right\}$ is indeterminate and has infinitely many solutions.

The indeterminacy of the strong Stieltjes moment problem (Theorem 2) has been mentioned in [1]. By use of a beta type integral introduced by Ramanujan (see, for example, [3]) one can show that the sequence $\left\{c_{n}\right\}$ of moments (1.13) is also generated by a distribution function $\psi(t)$ defined by

$$
\begin{equation*}
\psi^{\prime}(t)=\frac{t^{1 / 2}}{a \prod_{j=0}^{\infty}\left(1+t q^{j}\right)\left(1+t^{-1} q^{j+1}\right)}, \quad a:=-\prod_{j=0}^{\infty} \frac{\left(1-q^{j+3 / 2}\right)\left(1-q^{j-1 / 2}\right)}{\left(1+q^{j+1}\right)}>0 \tag{2.11}
\end{equation*}
$$

It is as yet not known whether $\phi_{1}$ and/or $\phi_{2}$ in (2.10) overlap with the distribution functions defined by (1.10) and (2.11), respectively. A further distribution function generating (1.13) is given in [14]; others, for $q=\mathrm{e}^{-2}$, can be obtained from [4, p.73].

## 3. Orthogonal L-polynomials

Using well-known determinant expressions for the $n$th denominator $B_{n}(z)$ of the positive T-fraction (1.12), (2.2), one can show that

$$
\begin{equation*}
Q_{2 n}(z)=B_{2 n}(-z) / z^{n}, \quad Q_{2 n+1}(z)=B_{2 n+1}(-z) / z^{n+1}, \quad n=0,1,2, \ldots \tag{3.1}
\end{equation*}
$$

where the $Q_{n}(z)$ are defined by (1.6). The difference equations [8, (2.1.6)] satisfied by the $B_{n}(z)$ then yield three-term recurrence relations

$$
\begin{array}{ll}
Q_{0}(z)=1, \quad Q_{1}(z)=z^{-1}, & \\
Q_{2 n}(z)=\left(1-G_{2 n} z\right) Q_{2 n-1}(z)-F_{2 n} Q_{2 n-2}(z), & n=1,2,3, \ldots \\
Q_{2 n+1}(z)=\left(z^{-1}-G_{2 n+1}\right) Q_{2 n}(z)-F_{2 n+1} Q_{2 n-1}(z), & n=1,2,3, \ldots \tag{3.2c}
\end{array}
$$

By use of (3.2) one can prove the following theorem.
Theorem 3. (A) The orthogonal L-polynomials (1.6) can be expressed by

$$
\begin{equation*}
Q_{2 n}(z)=\sum_{j=-n}^{n} \frac{\left(q^{-2 n} ; q\right)_{n-j}}{(q ; q)_{n-j}} q^{(n-j)^{2} / 2+n z} j, \quad n=0,1, \ldots \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{2 n-1}(z)=-\sum_{j=-n}^{n-1} \frac{\left(q^{-2 n+1} ; q\right)_{n-j-1}}{(q ; q)_{n-j-1}} q^{(n-j-1)^{2} / 2+(2 n-1) / 2} z^{j}, \quad n=1,2, \ldots \tag{B}
\end{equation*}
$$

Proof. (A) It is easy to check that (3.3) and (3.4) hold for $n=0$ and $n=1$, respectively. Assume that (3.3) and (3.4) hold for all $n \geqslant k$. First we will show that (3.4) holds for $n=k+1$. From (3.2) we have

$$
\begin{aligned}
Q_{2 k+1}(z)= & \left(z^{-1}-G_{2 k+1}\right) Q_{2 k}(z)-F_{2 k+1} Q_{2 k-1}(z) \\
= & \left(z^{-1}-q^{1 / 2}\right) \sum_{j=-k}^{k} \frac{\left(q^{2 k} ; q\right)_{k-j}}{(q ; q)_{k-j}} q^{(k-j)^{2} / 2+k} z^{j} \\
& -q^{-(2 k+1)+3 / 2}\left(1-q^{2 k}\right) \sum_{j=-k}^{k-1}-\frac{\left(q^{-2 k+1} ; q\right)_{k-j-1}}{(q ; q)_{k-j-1}} q^{(k-j-1)^{2} / 2+(2 k-1) / 2} z^{j}
\end{aligned}
$$

$$
\begin{aligned}
& =-\frac{\left(q^{-2 k-1} ; q\right)_{2 k+1}}{(q ; q)_{2 k+1}} q^{2 k^{2}+3 k+1} z^{-k-1} \\
& +\sum_{j=-k}^{k-1}\left[\left(\frac{\left(q^{-2 k} ; q\right)_{k-j-1}}{(q ; q)_{k-j}}\right)\right. \\
& \times\left(q^{\left(k^{2}-2 k j+j^{2}+2 j+1\right) / 2}-q^{\left(k^{2}-2 k j+j^{2}+2 k+1\right) / 2}\right. \\
& -q^{\left(k^{2}-2 k j+j^{2}+2 k+1\right) / 2}+q^{\left(k^{2}-2 k j+j^{2}-2 j-1\right) / 2} \\
& -q^{\left(k^{2}-2 k j+j^{2}+2 j+1\right) / 2}+q^{\left(k^{2}-2 k j+j^{2}-2 k-1\right) / 2} \\
& \left.\left.+q^{\left(k^{2}-2 k j+j^{2}+2 k+1\right) / 2}-q^{\left(k^{2}-2 k j+j^{2}-2 j-1\right) / 2}\right)\right] z^{j} \\
& -q^{(2 k+1) / 2} z^{k} \\
& =-\sum_{j=-(k+1)}^{(k+1)-1}-\frac{\left(q^{-2(k+1)+1} ; q\right)_{(k+1)-j-1}}{(q ; q)_{(k+1)-j-1}} q^{((k+1)-j-1)^{2} / 2+(2(k+1)-1) / 2} z^{j} .
\end{aligned}
$$

A similar argument establishes (3.3) for $k+1$.
(B) follows by applying (2.7) to (1.7b), (1.8) and (1.9). (B) can also be proved directly using (3.3) and (3.4).

We conclude with remarks on L-Gaussian quadrature with respect to log-normal distribution functions (1.10). It is known [6, Theorem 2.2] that, for each $n \geqslant 1, Q_{n}(z)$ has $n$ zeros $t_{1}^{(n)}, t_{2}^{(n)}, \ldots, t_{n}^{(n)}$, all of which are distinct and lie on the infinite interval $0<t<\infty$. If

$$
\begin{equation*}
w_{j}^{(n)}:=\frac{1}{Q_{n}^{\prime}\left(t_{j}^{(n)}\right)} \int_{0}^{\infty} \frac{Q_{n}(t)}{t-t_{j}^{(n)}} w(t) \mathrm{d} t, \quad j=1,2, \ldots, n \tag{3.8a}
\end{equation*}
$$

then

$$
\begin{equation*}
\int_{0}^{\infty} F(t) w(t) \mathrm{d} t=\sum_{j=1}^{n} w_{j}^{(n)} F\left(t_{j}^{(n)}\right) \tag{3.8b}
\end{equation*}
$$

is valid for all $F(t) \in \Lambda_{-n, n-1}$, where

$$
\begin{equation*}
\Lambda_{m, n}:=\left[\sum_{k=m}^{n} a_{k} z^{k}: a_{k} \in \mathbb{R},-\infty<m \leqslant k \leqslant n<+\infty\right] \tag{3.9}
\end{equation*}
$$

(see, e.g., [6, Theorem 2.3]).

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