

Some Classes of Generating Functions for the Laguerre and Hermite Polynomials

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Abstract. In the first half of the article, we present two theorems which give, as special cases, a number of new classes of generating functions for the Laguerre polynomial. These formulae extend the recent results of Carlitz [2] and others. The latter part of our work deals with two theorems involving new generating functions for the Hermite and generalized Hermite polynomials, thus generalizing some well-known expansions. The method of proof adopted in this paper differs from that of previous workers.

1. Introduction. Extending the work done by Chatterjea [3] and Brown [1], Carlitz [2] gave the generating function

$$(1.1) \quad \sum_{n=0}^{\infty} t^n L_n^{a+bn}(x) = \frac{(1+v)^{a+1}}{1-bv} \exp(-xv),$$

where v satisfies $v = t(1+v)^{b+1}$, $v(0) = 0$, and a, b are arbitrary complex numbers. The Laguerre polynomial is defined in the usual way [6, Volume II, p. 188].

In our presentation, we derive two theorems which give expansions for the Laguerre polynomial. A special case of Theorem 1 is seen to be

$$(1.2) \quad \sum_{n=0}^{\infty} t^n L_n^{v+bn}[x(1+an)] \\ = (1-z)^{1-v} [1 - z(b+2-ax) + z^2(b+1)]^{-1} \exp[xz/(z-1)],$$

where $t = z(1-z)^b \exp[axz/(1-z)]$, v, a, b are arbitrary complex numbers and $|t| < 1$.

1. Letting $a = 0$ in (1.2) gives essentially (1.1).

Theorem 2 yields a number of new expressions. One class of generating functions is

$$(1.3) \quad \sum_{n=0}^{\infty} \frac{t^n}{(v+bn+n)} L_n^{v+bn}[x(1+an)] \\ = (v)^{-1} (1-z)^{-v} \exp[xz/(z-1)] {}_1F_1 \left[\begin{matrix} 1; \\ \frac{v+1+b}{1+b}; \end{matrix} \frac{xz(1+b-av)}{(1-z)(1+b)} \right],$$

where $t = z(1-z)^b \exp[axz/(1-z)]$ and $|t| < 1$.

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The right-hand side of (1.3) is the incomplete gamma function. See Chapter IV of Luke [9] for extensive treatment and bibliography of this function. A reduction of interest involves the case of $1 + b = av$, and we obtain

$$(1.4) \quad \sum_{n=0}^{\infty} \frac{t^n}{(1+an)} L_n^{v+avn-n} [x(1+an)] = (1-z)^{-v} \exp[xz/(z-1)],$$

where $t = z(1-z)^{av-1} \exp[axz/(1-z)]$ and $|t| < 1$. For $a = 0$ in (1.3), we deduce the known result and its extension given by Eq. (4.13) of [4].

A second class of generating functions, a consequence of Theorem 2, may be expressed as

$$(1.5) \quad \sum_{n=0}^{\infty} \frac{t^n}{(1+an)} L_n^{v+bn} [x(1+an)] = (1-z)^{-v} \exp[xz/(z-1)] {}_2F_1 \left[\begin{matrix} \frac{1+b-av}{a}, & 1; \\ & z \end{matrix} \right]$$

where $t = z(1-z)^b \exp[axz/(1-z)]$ and $|t| < 1$. The right-hand side of (1.5) is the incomplete beta function [9, p. 299]. Putting $b = a + av$, we have a simplification of interest:

$$(1.6) \quad \sum_{n=0}^{\infty} \frac{t^n}{(1+an)} L_n^{v+an+avn} [x(1+an)] = (1-z)^{-1-v} \exp[xz/(z-1)],$$

where $t = z(1-z)^{a+av} \exp[axz/(1-z)]$ and $|t| < 1$. Letting $a \rightarrow 0$ in (1.5) gives the known Carlitz expansion (1.1).

A third class of generating functions may be deduced from Theorem 2, which assumes the form

$$(1.7) \quad \sum_{n=0}^{\infty} \frac{t^n}{(n+1)} L_n^{v+bn} [x(1+an)] = \frac{\exp[(-xz+x-ax)/(1-z)]}{(1-z)^v z(v-b)} \left\{ {}_1F_1 \left[\begin{matrix} v-b; & \frac{x(a-1)}{1-z} \\ v-b+1; & \end{matrix} \right] - (1-z)^{v-b} {}_1F_1 \left[\begin{matrix} v-b; & x(a-1) \\ v-b+1; & \end{matrix} \right] \right\},$$

where $t = z(1-z)^b \exp[axz/(1-z)]$ and $|t| < 1$.

The three classes of generating functions given above are in fact particular examples of the more general expansion

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{t^n L_n^{v+bn} [x(1+an)]}{(l-v-bn-n)((l-v+s-bn-n)/(s))_l}, \\
 (1.8) \quad & = \frac{(1-z)^{l-v} \exp[xyz/(z-1)]}{l'!} \sum_{r=0}^{l'} \sum_{n=0}^{l+sr} \frac{(-z)^n (1-z)^{rs-n} (-l')_r (-l-rs)_n}{n! r! (l-v+rs-bn-n)} \\
 & \cdot {}_1F_1 \left[\begin{matrix} 1; \\ (v-l+b+1+bn+n-rs)/(b+1); \end{matrix} \frac{xz(b+1+al-av+ars)}{(1-z)(b+1)} \right],
 \end{aligned}$$

where $t = z(1-z)^b \exp[axz/(1-z)]$, v, a, b are arbitrary complex numbers, l, l', s nonnegative integers, and $|t| < 1$. Note that we have assumed l, l' , and s to be nonnegative integers to terminate the series on the right-hand side of (1.8). This condition is not necessary. The result is valid provided both sides of (1.8) exist for arbitrary values of the parameters involved.

Letting $a = 0$ and $l' = 0$ in (1.8) gives essentially Eq. (3.4) of Zeitlin [10].

As the Bessel polynomial is expressed as [6, Volume I, p. 195] and the Poisson-Charlier polynomial as [6, Volume I, p. 268], then the formulae given for the Laguerre polynomial may be converted to generate the above-mentioned polynomials.

Theorems 3 and 4 involve the Hermite and generalized Hermite polynomials. A special case is the generating function

$$\begin{aligned}
 (1.9) \quad & \sum_{n=0}^{\infty} \frac{(1+bn)^{1/2n}}{n!} t^n H_n [x(1+an)/(1+bn)^{1/2}] \\
 & = \exp[-z^2 - 2xz] [1 + 2bz^2 + 2axz]^{-1}
 \end{aligned}$$

where $t = (-z) \exp[bz^2 + 2axz]$ and $|2axz \exp[bz^2 + 2axz + 1]| < 1$. The Hermite polynomial is defined in [6, Volume II, p. 193]. The special case $a = b = 0$ in (1.9) gives the old and well-known result [6, Volume II, p. 194, Eq. (19)]. If $b = 0$ and $a = 0$ in (1.9), then the equations (2.14) and (2.16), respectively, of Cohen [4], present themselves.

From Theorem 4, one class of new generating functions that may be deduced is

$$\begin{aligned}
 (1.10) \quad & \sum_{n=0}^{\infty} \frac{t^n (1+bn)^{1/2(n-2)}}{n!} H_n [x(1+an)/(1+bn)^{1/2}] \\
 & = \exp[-z^2 - 2xz] {}_1F_1 \left[\begin{matrix} 1; \\ \frac{b+1}{b}; \end{matrix} 2xz(b-a)/b \right],
 \end{aligned}$$

where $t = (-z) \exp[bz^2 + 2axz]$ and $|2axz \exp[bz^2 + 2axz + 1]| < 1$. If $b \rightarrow 0$ and $a \rightarrow 0$ in (1.10), then Eqs. (2.14) and (2.17), respectively, of [4] are obtained. A special case of interest may be derived from (1.10) for $a = b$ to give

$$(1.11) \quad \sum_{n=0}^{\infty} \frac{t^n (1 + an)^{1/2(n-2)}}{n!} H_n [x(1 + an)^{1/2}] = \exp[-z^2 - 2xz],$$

where $t = (-z)\exp(az^2 + 2axz)$ and $|2axz \exp[az^2 + 2axz + 1]| < 1$.

A second class of generating functions is a special case of Theorem 4, and may be expressed as

$$(1.12) \quad \sum_{n=0}^{\infty} \frac{t^n (1 + bn)^{1/2n}}{n!(1 + an)} H_n [x(1 + an)/(1 + bn)^{1/2}] \\ = \exp[-z^2 - 2xz] {}_1F_1 \left[\begin{matrix} 1; \\ \frac{1 + 2a}{2a}; \end{matrix} \quad z^2(a - b)/a \right],$$

where $t = (-z)\exp[bz^2 + 2axz]$ and $|2axz \exp[bz^2 + 2axz + 1]| < 1$. Letting $b \rightarrow 0$ and $a \rightarrow 0$ in (1.12) give the known Eqs. (2.15) and (2.16), respectively, of [4].

2. Generating Functions. The method of proof in the four theorems is a modification and extension of the one employed in obtaining new generating functions for the Jacobi polynomial in [5]. As in [5], no appeal is made to the Lagrange Theorem. It differs from that of previous workers, including the approach in [4].

THEOREM 1. For α, β, s arbitrary complex numbers and r a positive integer

$$(2.1) \quad \sum_{k=0}^{\infty} \frac{t^k (\alpha + 1 + sk)_k}{k!} {}_rF_r \left[\begin{matrix} \Delta(-k, r); \\ \Delta(-\alpha - sk - k, r); \end{matrix} \quad x(\beta + sk) \right] \\ = (1 - z)^{\alpha+1} (1 + sz + rsy)^{-1} \exp[-\beta y],$$

where $t = (-z)(1 - z)^{-s-1} \exp(sy)$, $x = (-y)[(z - 1)/z]^r$, $(a)_k = \Gamma(a + k)/\Gamma(a)$, quotient of gamma functions $= a(a + 1)(a + 2) \dots (a + k - 1)$ for k a positive integer, and $\Delta(-k, r) = -k/r, (-k + 1)/r, \dots, (-k + r - 1)/r$. $|t| < 1$, and ${}_rF_r$ is the generalized hypergeometric polynomial [9, p. 155].

Proof. Let us begin with the expression

$$(2.2) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n y^m}{n!m!} D^n [x^{\alpha-\beta+n} \delta^m \{x^\beta (1 - x^s)^n (1 - x^{s'})^m\}],$$

where $D \equiv d/dx$ and $\delta \equiv x d/dx$. Now

$$(2.3) \quad D^n [x^{\alpha-\beta+n} \delta^m \{x^\beta (1 - x^s)^n (1 - x^{s'})^m\}]$$

$$(2.4) \quad = \sum_{k=0}^n \sum_{p=0}^m \frac{(-n)_k (-m)_p (\beta + sk + s'p)^m (\alpha + 1 + sk + s'p)_n x^{\alpha+sk+s'p}}{k!p!}.$$

Taking (2.3) and (2.4) for $x = 1$, and putting the results in (2.2), one has

$$(2.5) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^m \sum_{k=0}^n \frac{z^n y^m (-n)_k (-m)_p (\beta + sk + s'p)^m (\alpha + 1 + sk + s'p)_n}{k!p!n!m!}$$

$$(2.6) \quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-sz)^n (-s'y)^m (m+n)!}{n!m!}$$

$$(2.7) \quad = [1 + sz + s'y]^{-1}.$$

We go from (2.6) to (2.7) using the binomial expansions. Now applying the series transformation

$$(2.8) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^m \sum_{k=0}^n f(n, m, k, p) = \sum_{n, m, k, p=0}^{\infty} f(n+k, m+p, k, p)$$

to (2.5), and simplifying, gives

$$(2.9) \quad \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-z)^k (-y)^p \exp[y(\beta + sk + s'p)] (\beta + sk + s'p)^p (\alpha + 1 + sk + s'p)_k}{k!p!(1-z)^{\alpha+1+sk+k+s'p}}.$$

The summation of the series over n and m is achieved with the aid of the binomial and exponential expansions. Equating (2.7) and (2.9), and employing

$$(2.10) \quad \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} f(p, k) = \sum_{k=0}^{\infty} \sum_{p=0}^{\lfloor r/k \rfloor} f(p, k-rp)$$

with $r = s'/s$, and some algebra, gives the required theorem.

THEOREM 2. For α, β, s, s' arbitrary complex numbers, l, l', s'' nonnegative integers and r a positive integer

$$(2.11) \quad \sum_{k=0}^{\infty} \frac{(\alpha + 1 + sk)_k t^k}{(\alpha + l + 1 + sk)k!((\alpha + l + 1 + s'' + sk)/s'')_l} \\ \cdot {}_rF_r \left[\begin{matrix} \Delta(-k, r); \\ \Delta(-\alpha - sk - k, r); \end{matrix} \quad x(\beta + sk) \right] \\ = \frac{(1-z)^{\alpha+l+1} \exp(-\beta y)}{l'!} \cdot \sum_{q=0}^{l'} \sum_{k=0}^{l+s''q} \frac{(-z)^k (1-z)^{s''q-k} (-l')_q (-l-s''q)_k}{k!q!(\alpha + l + 1 + s''q + sk)} \\ \cdot {}_1F_1 \left[\begin{matrix} 1; \\ (\alpha + l + 1 + s' + s''r + sk)/s'; \end{matrix} \quad y(\beta - \alpha - l - 1 - s''q) \right],$$

where $t = (-z)(1-z)^{-s-1} \exp(sy)$, $x = (-y)[(z-1)/z]^r$, and $|t| < 1$. Other symbols are defined in Theorem 1.

Proof. We first evaluate the integral

$$(2.12) \quad \int_0^1 x^l (1 - x^{s''})^{l'} D^n [x^{\alpha-\beta+n} \delta^m \{x^\beta (1 - x^s)^n (1 - x^{s'})^m\}] dx.$$

Expanding the operators as in Theorem 1, we have

$$(2.13) \quad \sum_{p=0}^m \sum_{k=0}^n \frac{(-n)_k (-m)_p (\beta + sk + s'p)^m (\alpha + 1 + sk + s'p)_n}{k!p!} \cdot \int_0^1 x^{sk+s'p+l} (1 - x^{s''})^{l'} dx.$$

The integral in (2.13) may be evaluated to give

$$(2.14) \quad \frac{l'!}{(\alpha + l + 1 + sk + s'p)(\alpha + l + 1 + s'' + sk + s'p)/s'')_{l'}}.$$

Going back to (2.12), and expanding $(1 - x^{s''})^{l'}$ gives

$$(2.15) \quad \sum_{q=0}^{l'} \frac{(-l')_q}{q!} \int_0^1 x^{l+s''q} D^n [x^{\alpha-\beta+n} \delta^m \{x^\beta (1 - x^s)^n (1 - x^{s'})^m\}] dx.$$

Now integrating by parts n times, then expanding $(1 - x^s)^n$ and $(1 - x^{s'})^m$ and integrating again, (2.15) reduces to

$$(2.16) \quad \sum_{q=0}^{l'} \sum_{p=0}^m \sum_{k=0}^n \frac{(-l')_q (-l - s''q)_n (-n)_k (-m)_p (\beta + sk + s'p)^m}{q!p!k!(\alpha + l + 1 + s''q + sk + s'p)}.$$

Equating (2.13) and (2.14) to (2.16), multiplying both sides of the resulting equation by

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n y^m}{n!m!}$$

and proceeding as in Theorem 1, we have, after simplification

$$(2.17) \quad \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-z)^k (-y)^p \exp[y(\beta + sk + s'p)] (\beta + sk + s'p)^p (\alpha + 1 + sk + s'p)_k}{k!p!(1-z)^{\alpha+1+sk+k+s'p} (\alpha + 1 + l + sk + s'p)(\alpha + l + s'' + 1 + sk + s'p)/s'')_{l'}} \\ = \frac{1}{l'!} \sum_{q=0}^{l'} \sum_{k=0}^{l+s''q} \frac{(-z)^k (1-z)^{l+s''q-k} (-l')_q (-l - s''q)_k}{k!q!(\alpha + l + 1 + s''q + sk)} \\ \cdot {}_1F_1 \left[\begin{matrix} 1; \\ (\alpha + l + s' + 1 + s''q + sk)/s'; \end{matrix} y(\beta - \alpha - l - 1 - s''q) \right].$$

Applying the series transformation (2.10) and simplifying, gives Theorem 2.

THEOREM 3. For α, β, s' arbitrary complex numbers and r a positive integer

$$(2.18) \quad \sum_{p=0}^{\infty} \frac{t^p (\beta + s'p)^p}{p!} {}_rF_0 \left[\Delta(-p, r); \text{---}; \frac{x^r (\alpha + s'p)}{(\beta + s'p)^r} \right] \\ = \exp(-\beta y - \alpha z) (1 + s'y + rs'z)^{-1},$$

where $t = (-y)\exp(s'y + s'z)$, $x = (-z)(y)^{-r}$, and $|s'y \exp(s'y + s'z + 1)| < 1$. See Theorem 1 for other symbols.

Proof. Consider

$$(2.19) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n y^m}{n! m!} \delta^n [x^{\alpha-\beta} \delta^m \{x^\beta (1-x^s)^n (1-x^{s'})^m\}].$$

At $x = 1$ we have, after expansion

$$(2.20) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n y^m}{n! m!} \sum_{p=0}^m \sum_{k=0}^n \frac{(-n)_k (-m)_p (\alpha + sk + s'p)^n (\beta + sk + s'p)^m}{k! p!}.$$

But at $x = 1$, (2.19) also reduces to

$$(2.21) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-sz)^n (-s'y)^m (n+m)!}{n! m!}.$$

Equating (2.20) to (2.21), and proceeding as in Theorem 1, with applying series transformations and simplifying, gives

$$(2.22) \quad \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-z)^k (-y)^p \exp[y(\beta + sk + s'p) + z(\alpha + sk + s'p)] (\alpha + sk + s'p)^k (\beta + sk + s'p)^p}{k! p!} \\ = (1 + zs + ys')^{-1}.$$

Using the series transformation (2.10) and some manipulation results in Theorem 3. Note that for $s' = 0$ in the above theorem, the ${}_rF_0$ hypergeometric polynomial is essentially the generalized Hermite polynomial which occurs in probability problems. It has been studied by Gould and Hopper [7], Gupta and Jain [8], Cohen [4], and others.

THEOREM 4. For α, β, s, s' arbitrary complex numbers and r, r' positive integers (a)

$$(2.23) \quad \sum_{p=0}^{\infty} \frac{(\beta + s'p)^p t^p}{p! (\alpha + s'p)} {}_rF_0 \left[\Delta(-p, r); \text{---}; \frac{xr^r (\alpha + s'p)}{(\beta + s'p)^r} \right] \\ = \frac{\exp(-\alpha z - \beta y)}{(\alpha)} {}_1F_1 \left[\begin{matrix} 1; \\ (\alpha + s')/s'; \end{matrix} y(\beta - \alpha) \right],$$

where $t = (-y)\exp(s'y + s'z)$, $x = (-z)(y)^{-r}$, and $|s'y \exp(s'y + s'z + 1)| < 1$. (b)

$$(2.24) \quad \sum_{k=0}^{\infty} \frac{(\alpha + sk)^{k-1} t'^k}{k!} {}_rF_0 \left[\Delta(-k, r'); \text{---}; \frac{r'r'x'(\beta + sk)}{(\alpha + sk)^{r'}} \right] \\ = \frac{\exp(-\alpha z - \beta y)}{(\alpha)} {}_1F_1 \left[\begin{matrix} 1; \\ (\alpha + r's)/r's; \end{matrix} y(\beta - \alpha) \right]$$

where $t' = (-z)\exp(sy + sz)$, $x' = (-y)(z)^{-r'}$, and $|sz \exp(sy + sz + 1)| < 1$. Other symbols are defined in Theorem 1.

Proof. Let us evaluate the expression

$$(2.25) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n y^m}{n!m!} \int_0^1 x^{-1} \delta^n [x^{\alpha-\beta} \delta^m \{x^\beta (1-x^s)^n (1-x^{s'})^m\}] dx.$$

Expanding and operating as in the previous theorems, we have

$$(2.26) \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n y^m}{n!m!} \sum_{p=0}^m \sum_{k=0}^n \frac{(-n)_k (-m)_p (\beta + sk + s'p)^m (\alpha + sk + s'p)^{n-1}}{k!p!}.$$

Going back to (2.25) and integrating, we find the only contribution comes from $n = 0$ and hence (2.25) may be evaluated to give

$$(2.27) \quad \sum_{m=0}^{\infty} \frac{[y(\beta - \alpha)]^m m!}{m! (\alpha)(\alpha + s')/s'_m}.$$

Equating (2.26) and (2.27), using the series transformation, and reducing the summation over n and m , one obtains the expression

$$(2.28) \quad \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-z)^k (-y)^p (\beta + sk + s'p)^p (\alpha + sk + s'p)^{k-1} \exp[z(\alpha + sk + s'p) + y(\beta + sk + s'p)]}{k!p!} \\ = (1/\alpha)_1 F_1 \left[\begin{matrix} 1; \\ (\alpha + s')/s'; \end{matrix} y(\beta - \alpha) \right].$$

Now applying (2.10) to (2.28) twice, with p and k interchanged, and some algebra, gives the two parts of the theorem. Note that $r = s/s'$, and $r' = s'/s$.

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