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Inversion techniques and combinatorial identities. - A unified treatment for the ${}_7F_6$–series identities

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Abstract

As an extension of a useful inverse pair due to Gould–Hsu (1973), a general pair of reciprocal relations is established. The inversion technique for proving combinatorial identities, originated by Riordan (1968) and Greene & Knuth (1981), is developed systematically to explore the dual relations of Pfaff–Saalschutz and Dougall–Dixon formulae. Most of the known strange hypergeometric evaluations, covered in Bailey (1935), Slater (1966), Gessel & Staton (1982), Gasper (1989) and Gasper & Rahman (1990), and several new mysterious looking formulas are demonstrated almost mechanically.

1. Introduction

Recently, combinatorial computation has aroused new interest for its wide applications in mathematics, physics and computer science. As its main part, thousands of binomial identities have been realized, by combinatorists during the past few years, to be equivalent to the relatively few hypergeometric evaluations. Since then, attention has been turned to the hypergeometric relations. Through the mathematical world, the hypergeometric "windstorm" swept across, and the "disease" of its q–analogue has expanded rapidly, partly because of the Indian mathematical genius

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Ramanujan's romantic career. The vast literature along this direction leaves us an impression that (basic) hypergeometric series is full of magic power and treasure to be explored.

However, during the past years, one important discovery due to Gould and Hsu [13], in 1973, has been neglected completely by combinatorists although its special cases were rediscovered, rephrased in terms of Lagrange inversion, and used, by Gessel and Stanton [12], to investigate hypergeometric evaluations. To be precise, we restate Gould and Hsu's theorem as follows: Let $\{a_i\}$ and $\{b_i\}$ be any two complex sequences such that the polynomials defined by

$$\psi(x;n) = \prod_{k=0}^{n-1} \big(a_k + xb_k\big) \tag{1.1a}$$

differ from zero for non–negative integers x and n with $\psi(x;0) = 1$. Then there hold the inverse relations

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(k;n)\, g(k)\,, \tag{1.1b}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k + kb_k}{\psi(n;k+1)}\, f(k)\,. \tag{1.1c}$$

Originated from Riordan (1968), now it begins to be accepted by combinatorists, e.g. Andrews [1], Greene & Knuth [14], and Chu & Hsu [6–7], that inverse series relations are partially responsible for the proliferation of combinatorial identities. One implication of (1.1) is that for every relation of one form in this pair, there is a companion of the dual form. To prove one is to prove both. On one side, if one member of an inverse pair is a known relation, then the other member often provides a new formula. On another side, if the truth for one combinatorial summation formula in one form of the inverse pair needs to be confirmed (proved or disproved), then it would be transformed into checking its dual formulation which is often attributed to a routine combinatorial fact. Based on this observation, numerous combinatorial identities, e.g. the convolution formulas due to Abel and Hagen–Rothe, and the evaluations discovered by Dougall–Dixon–Kummer and Watson–Whipple, and conjectured by Gasper (cf. Gessel and Stanton [12]), have been revisited through (1.1) and in fact transformed into Euler's binomial theorem, Chu–Vandermonde–Gauss and Pfaff–Saalschutz formulae. For details the interested reader can consult the recent work [6, 7].

The main objective of the present paper is to establish an extended version of Gould–Hsu's theorem. By means of the inversion technique described above, most of the known strange hypergeometric evaluations, covered in Bailey [4], Slater [18], Gessel & Stanton [12], Gasper [9] and Gasper & Rahman [10–11], and several new mysterious–looking formulas are demonstrated, almost mechanically, to be the dual relations of Pfaff–Saalschutz and Dougall–Dixon formulas. Among them, the most striking example is the re–confirmation for one ${}_7F_6$–identity, conjectured by Gasper (1977) and proved recently by Zeilberger [8] and Gasper & Rahman [10–11].

Except for special indications, the usual notation of shifted factorial, binomial coefficient, and hypergeometric series from the monographs by Bailey [4], Slater [18] and Gasper & Rahman [11] will be adopted throughout the paper. To save space in writing, we will use

$$\left[\begin{matrix} a, b, \dots, c \\ \\ u, v, \dots, w \end{matrix}\right]_n = \frac{(a)_n\,(b)_n \cdots (c)_n}{(u)_n\,(v)_n \cdots (w)_n}$$

instead of factorial–fraction.

2. Inverse series relations

Theorem

With the ψ–polynomials defined by (1.1a), the inverse series relations

$$\text{(2.1a)} \quad f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(\lambda+k;n)\; \psi(-k;n)\; \frac{\lambda+2k}{(\lambda+n)_{k+1}}\; g(k)$$

$$\text{(2.1b)} \quad g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k+(\lambda+k)b_k}{\psi(\lambda+n;k+1)}\; \frac{a_k-kb_k}{\psi(-n;k+1)}\; (\lambda+k)_n\; f(k)$$

are valid provided that the sequence–transforms involved are non–singular, i.e. $\psi(\lambda+n;m+1)$, $\psi(-n;m+1)$ and $(\lambda+n)_{m+1}$ do not vanish for non–negative integers $m \le n$.

It is obvious that this pair of inversions is an extension of Gould and Hsu's because it reduces to (1.1) when λ tends to infinity. Similar to the role of Gould and Hsu's inversions, (2.1) may be used, systematically, to deal with the strange hypergeometrics, based on the so–called embedding machinery described in the introduction. The exhibition demonstrated in the next three sections will show that this approach can not only certify the know hypergeometric formulas, but also create several new strange evaluations.

Proof of the theorem. As the linear transformation defined by (2.1b) with fixed ψ is non–singular, it suffices to show that (2.1a) implies (2.1b). In fact, substituting (2.1a) into (2.1b) gives

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a_k+(\lambda+k)b_k}{\psi(\lambda+n;k+1)}\frac{a_k-kb_k}{\psi(-n;k+1)}(\lambda+k)_n$$
$$\times\sum_{m=0}^{k}(-1)^m\binom{k}{m}\psi(\lambda+m;k)\,\psi(-m;k)\,\frac{\lambda+2m}{(\lambda+k)_{m+1}}\,g(m)\ .$$

Interchanging the summation indices and noting that

$$\binom{n}{k}\binom{k}{m}=\binom{n}{m}\binom{n-m}{k-m},\qquad \frac{(\lambda+k)_n}{(\lambda+k)_{m+1}}=\frac{(\lambda)_n}{(\lambda)_m}\frac{(\lambda+n)_k}{(\lambda+m)_{k+1}}$$

we can reformulate this expression as

$$\sum_{m=0}^{n}\binom{n}{m}\frac{(\lambda)_n}{(\lambda)_m}\,g(m)$$
$$\times\sum_{k=m}^{n}(-1)^{m+k}\binom{n-m}{k-m}\frac{\psi(\lambda+m;k)\,\psi(-m;k)}{\psi(\lambda+n;k+1)\,\psi(-n;k+1)}\frac{(\lambda+n)_k}{(\lambda+m)_{k+1}}$$
$$\times(\lambda+2m)\big(a_k+(\lambda+k)b_k\big)(a_k-kb_k)\ .$$

It will reduce to $g(n)$ if we can verify that the inner sum equals $\delta_{m,n}$, the Kronecker delta, or equivalently, the orthogonal relation

(2.2a)
$$(\lambda+2m)\sum_{k=m}^{n}(-1)^{m+k}\binom{n-m}{k-m}\frac{\psi(\lambda+m;k)\,\psi(-m;k)}{\psi(\lambda+n;k+1)\,\psi(-n;k+1)}$$
$$\times\frac{(\lambda+n)_k}{(\lambda+m)_{k+1}}\big(a_k+(\lambda+k)b_k\big)(a_k-kb_k)=\delta_{m,n}$$

which simply follows from the summand–splitting

$$(\lambda+m+n)\binom{n-m}{k-m}\frac{\psi(\lambda+m;k)\,\psi(-m;k)}{\psi(\lambda+n;k+1)\,\psi(-n;k+1)}\frac{(\lambda+n)_k}{(\lambda+m)_{k+1}}$$
$$\times\big(a_k+(\lambda+k)b_k\big)(a_k-kb_k)$$

$$= \binom{n-m-1}{k-m} \frac{\psi(\lambda+m;k+1)\ \psi(-m;k+1)}{\psi(\lambda+n;k+1)\ \psi(-n;k+1)} \frac{(\lambda+n)_{k+1}}{(\lambda+m)_{k+1}}$$

$$+ \binom{n-m-1}{k-m-1} \frac{\psi(\lambda+m;k)\ \psi(-m;k)}{\psi(\lambda+n;k)\ \psi(-n;k)} \frac{(\lambda+n)_k}{(\lambda+m)_k}$$

and the diagonal–canceling. This completes the proof of the theorem. □

If we substitute (2.1b) into (2.1a), then the resulting orthogonality

$$\text{(2.2b)}\big(a_n(\lambda+n)b_n\big)\,(a_n-nb_n)\sum_{k=m}^{n}(-1)^{m+k}$$
$$\times \binom{n-m}{k-m} \frac{\psi(\lambda+k;n)\ \psi(-k;n)}{\psi(\lambda+k;m+1)\ \psi(-k;m+1)} \frac{(\lambda+k)_m}{(\lambda+k)_{n+1}}(\lambda+2k) = \delta_{m,n}$$

is much more tedious to confirm. In fact the author has not found a direct derivation.

In view of the relation between reciprocal transforms and matrix inverse pairs, (2.1a) and (2.1b) admits the following rotated forms.

Corollary

Assume the condition of the theorem. The system of equations

$$\text{(2.3a)}\quad F(n) = \sum_{k=n}^{N}(-1)^k\binom{k}{n}\ \psi(\lambda+n;k)\ \psi(-n;k)\ \frac{\lambda+2n}{(\lambda+k)_{n+1}}\ G(k)\,,\ \ (0\le n\le N)$$

is equivalent to the system

(2.3b)
$$G(n) = \sum_{k=n}^{N}(-1)^k\binom{k}{n} \frac{a_n+(\lambda+n)b_n}{\psi(\lambda+k;n+1)} \frac{a_n-nb_n}{\psi(-k;n+1)}(\lambda+n)_k\,F(k)\,,\ \ (0\le n\le N)$$

where N is an arbitrary non–negative integer or infinity.

One special pair of inverse relations implied in the works of Bressould [5] and Gasper [9] could also be derived from (2.2) and (2.3) by defining ψ–polynomials to be shifted–factorials.

Proposition

Let a, b, and μ be complex numbers such that $a+x+\mu y$, $b+x-\mu y$ and $x+\mu^{-1}(a-b)$ differ from zero for non–negative integers x and y. Then there hold the inverse series relations

$$f(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,(a+\mu k)_n\,(b-\mu k)_n\,\frac{\mu^{-1}(a-b)+2k}{\big(\mu^{-1}(a-b)+n\big)_{k+1}}\,g(k)\,, \tag{2.4a}$$

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,\frac{a+\mu k+k}{(a+\mu n)_{k+1}}\,\frac{b-\mu k+k}{(b-\mu n)_{k+1}}\,\big(\mu^{-1}(a-b)+k\big)_n\,f(k)\,; \tag{2.4b}$$

and their rotated forms

$$F(n)=\sum_{k\geq n}(-1)^k\binom{k}{n}\,(a+\lambda n)_k\,(b-\lambda n)_k\,\frac{\lambda^{-1}(a-b)+2n}{\big(\lambda^{-1}(a-b)+k\big)_{n+1}}\,G(k)\,, \tag{2.5a}$$

$$G(n)=\sum_{k\geq n}(-1)^k\binom{k}{n}\,\frac{a+\lambda n+n}{(a+\lambda k)_{n+1}}\,\frac{b-\lambda n+n}{(b-\lambda k)_{n+1}}\,\big(\lambda^{-1}(a-b)+n\big)_n\,F(n). \tag{2.5b}$$

3. Embedding technique on k–balanced hypergeometrics

Based on the transform

$$(a+k)_n=(a+n)_k(a)_n/(a)_k \tag{3.0a}$$

$$(c-k)_n=(c)_n(1-c)_k/(1-c-n)_k \tag{3.0b}$$

the Saalschutzian (i.e. one–balanced) and 2–balanced formulas will be rewritten in the form of (2.4b), and produce a family of curious hypergeometric evaluations through the dual relation (2.4a).

EXAMPLE 3.1: Saalschutz theorem $\Longleftrightarrow$ The first very well–poised ${}_7F_6$–summation.

The Saalschutz summation (cf. e.g. [4, p. 9])

(3.1a)

$${}_3F_2\left[\begin{matrix}1+2u-2v,\quad u+n,\ -n\\ 1+u-v-n,\ 1+2u-v+n\end{matrix}\right]=\left[\begin{matrix}1+u-v,\ 1+2u-v,\ v/2,\ (1+v)/2\\ v,\ v-u,\ u-(v-1)/2,\ u-(v-2)/2\end{matrix}\right]_n$$

can be restated in the form

$$(3.1\text{b})\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2u-v+2k}{(2u-v+n)_{k+1}}\,\frac{u-v}{(u-v-n)_{k+1}}\times(u+k)_n\,\frac{2u-v}{2u-v+2k}(u)_k(1+2u-2v)_k$$

$$=\left[\begin{matrix}u, & 1+u-v, & 2u-v, & v/2, & (1+v)/2\\ v, & 1-u+v, & 1+u-v/2, & u+(1-v)/2\end{matrix}\right]_n$$

and yield the dual relation

$$(3.1\text{c})\qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}(2u-v+k)_n\,(u-v-k)_n\,\frac{u+2k}{(u+n)_{k+1}}\times\left[\begin{matrix}u, & 1+u-v, & 2u-v, & v/2, & (1+v)/2\\ v, & 1-u+v, & 1+u-v/2, & u+(1-v)/2\end{matrix}\right]_k$$

$$=\frac{2u-v}{2u-v+2n}\,(u)_n\,(1+2u-2v)_n$$

which may be reformulated as the first interesting hypergeometric evaluation

(3.1d)

$${}_7F_6\left[\begin{matrix}u, & 1+u/2, & v/2, & (v+1)/2, & 1+u-v, & 2u-v+n, & -n\\ & u/2, & u+1-v/2, & u+(1-v)/2, & v, & 1-u+v-n, & 1+u+n\end{matrix}\right]$$

$$=\frac{2u-v}{2u-v+2n}\left[\begin{matrix}1+u, & 1+2u-2v\\ u-v, & 2u-v\end{matrix}\right]_n.$$

Example 3.2: 2–balanced series $\Longleftrightarrow$ The second very well–poised ${}_7F_6$–summation.

The nearly–poised (also a 2–balanced series, cf. e.g. [4, p. 30])

$$(3.2\text{a})\qquad {}_3F_2\left[\begin{matrix}2u-2v, & u+n, & -n\\ 1+u-v-n, & 1+2u-v+n\end{matrix}\right]$$

$$=\left[\begin{matrix}u-v, & 1+2u-v, & v/2, & (1+v)/2\\ 1+v, & v-u, & u-(v-1)/2, & u-(v-2)/2\end{matrix}\right]_n$$

can be restated in the form

(3.2b)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,\frac{2u-v+2k}{(2u-v+n)_{k+1}}\,\frac{u-v}{(u-v-n)_{k+1}}\,(u+k)_n\,\frac{2u-v}{2u-v+2k}\,(u)_k\,(2u-2v)_k$$

$$=\begin{bmatrix} u, & u-v, & 2u-v, & v/2, & (1+v)/2 \\ 1+v, & 1-u+v, & 1+u-v/2, & u+(1-v)/2 \end{bmatrix}_n$$

and yield the dual relation

(3.2c)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,(2u-v+k)_n\,(u-v-k)_n\,\frac{u+2k}{(u+n)_{k+1}}$$

$$\times\begin{bmatrix} u, & u-v, & 2u-v, & v/2, & (1+v)/2 \\ 1+v, & 1-u+v, & 1+u-v/2, & u+(1-v)/2 \end{bmatrix}_k$$

$$=\frac{2u-v}{2u-v+2n}\,(u)_n\,(2u-2v)_n$$

which may be reformulated as the second interesting hypergeometric evaluation

(3.2d)

$${}_7F_6\begin{bmatrix} u, & 1+u/2, & v/2, & (v+1)/2, & u-v, & 2u-v+n, & -n \\ & u/2, & u+1-v/2, & u+(1-v)/2, & 1+v, & 1-u+v-n, & 1+u+n \end{bmatrix}$$

$$=\frac{2u-v}{2u-v+2n}\begin{bmatrix} 1+u, & 2u-2v \\ u-v, & 2u-v \end{bmatrix}_n .$$

EXAMPLE 3.3: Saalschutz series $\Longleftrightarrow$ The third very well–poised ${}_7F_6$–summation.

The Saalschutzian series (also a nearly–poised series, cf. e.g. [4, p. 30])

(3.3a)

$${}_4F_3\begin{bmatrix} 2+2u-2v, & 1+(1+2u-v)/2, & u+n, & -n \\ & (1+2u-v)/2, & 2+u-v-n, & 2+2u-v+n \end{bmatrix}$$

$$=\begin{bmatrix} 1+u-v, & 2+2u-v, & v/2, & (1+v)/2 \\ v, & v-u-1, & u-(v-1)/2, & u-(v-2)/2 \end{bmatrix}_n$$

can be restated in the form

(3.3b)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+2u-v+2k}{(1+2u-v+n)_{k+1}}\;\frac{1+u-v}{(1+u-v-n)_{k+1}}\;(u+k)_n\,(u)_k\,(2+2u-2v)_k$$

$$=\left[\begin{matrix}u, & 1+u-v, & 1+2u-v, & v/2, & (1+v)/2\\ v, & v-u, & 1+u-v/2, & u+(1-v)/2 & \end{matrix}\right]_n$$

and yield the dual relation

(3.3c)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+2u-v+k)_n\,(1+u-v-k)_n\,\frac{u+2k}{(u+n)_{k+1}}$$

$$\times\left[\begin{matrix}u, & 1+u-v, & 1+2u-v, & v/2, & (1+v)/2\\ v, & v-u, & 1+u-v/2, & u+(1-v)/2 & \end{matrix}\right]_k$$

$$=\;(u)_n\;(2+2u-2v)_n$$

which may be reformulated as the third interesting hypergeometric evaluation

(3.3d)
$${}_7F_6\left[\begin{matrix}u, & 1+u/2, & v/2, & (v+1)/2, & 1+u-v, & 1+2u-v+n, & -n\\ & u/2, & u+1-v/2, & u+(1-v)/2, & v, & v-u-n, & 1+u+n\end{matrix}\right]$$

$$=\left[\begin{matrix}1+u, & 2+2u-2v\\ 1+u-v, & 1+2u-v\end{matrix}\right]_n.$$

Example 3.4: 2–balanced series $\Longleftrightarrow$ The fourth very well–poised ${}_7F_6$–summation.

The nearly–poised series (also a 2–balanced series, cf. e.g. [4, p. 30])

(3.4a)

$${}_4F_3\left[\begin{matrix}1+2u-2v, & 1+(1+2u-v)/2, & u+n, & -n\\ & (1+2u-v)/2, & 2+u-v-n, & 2+2u-v+n\end{matrix}\right]$$

$$=\left[\begin{matrix}v-u, & 2+2u-v, & v/2, & (1+v)/2\\ 1+v, & v-u-1, & u-(v-1)/2, & u-(v-2)/2\end{matrix}\right]_n$$

can be restated in the form

(3.4b)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+2u-v+2k}{(1+2u-v+n)_{k+1}}\,\frac{1+u-v}{(1+u-v-n)_{k+1}}\,(u+k)_n\,(u)_k\,(1+2u-2v)_k$$
$$=\left[\begin{matrix}u, & u-v, & 1+2u-v, & v/2, & (1+v)/2\\ 1+v, & v-u, & 1+u-v/2, & u+(1-v)/2\end{matrix}\right]_n$$

and yield the dual relation

(3.4c)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+2u-v+k)_n\,(1+u-v-k)_n\,\frac{u+2k}{(u+n)_{k+1}}$$
$$\times\left[\begin{matrix}u, & u-v, & 1+2u-v, & v/2, & (1+v)/2\\ 1+v, & v-u, & 1+u-v/2, & u+(1-v)/2\end{matrix}\right]_k$$
$$=\ (u)_n\ (1+2u-2v)_n$$

which may be reformulated as the fourth interesting hypergeometric evaluation

(3.4d)

$${}_7F_6\left[\begin{matrix}u, & 1+u/2, & v/2, & (v+1)/2, & u-v, & 1+2u-v+n, & -n\\ & u/2, & u+1-v/2, & u+(1-v)/2, & 1+v, & v-u-n, & 1+u+n\end{matrix}\right]$$
$$=\left[\begin{matrix}1+u, & 1+2u-2v\\ 1+u-v, & 1+2u-v\end{matrix}\right]_n\ .$$

All the formulas demonstrated above but the last one (cf. Bailey [3], or [4, p. 98]) have not appeared in the literature explicitly.

4. Strange hypergeometric evaluations

For terminating hypergeometrics, the most general evaluation is the Dougall–Dixon theorem

$$
{}_7F_6\left[\begin{matrix} a, & 1+a/2, & b, & c, & d, & e, & f \\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\right]
$$
$$
=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-b-e)\Gamma(1+a-c-d)}
$$
$$
\times\frac{\Gamma(1+a-b-c-e)\Gamma(1+a-b-d-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a-c-e)\Gamma(1+a-d-e)\Gamma(1+a-b-c-d-e)}
$$

provided that one of b, c, d, e and f is a non–positive integer and $1+2a=b+c+d+e+f$. In fact, this formula has been the starting point for many combinatorial computations.

By means of the transforms (3.0) and relations

$$
\text{(4.0a)} \qquad (u)_{mk}=m^{mk}(u)_k\,(u+1/m)_k\ldots\big(u+(m-1)/m\big)_k
$$

$$
\text{(4.0b)} \qquad (v+n)_{mk}=(v)_{mk}\,(v+mk)_n/(v)_n
$$

$$
\text{(4.0c)} \qquad (w-n)_{mk}=(w)_{mk}(1-w)_n/(1-w-mk)_n
$$

the Dougall–Dixon theorem will be specified to telescope in (2.1a) and generate, through (2.1b), the dual formulas. From this process, several strange hypergeometric evaluations will be found, unexpectedly.

Example 4.1: Dougall–Dixon theorem $\Longleftrightarrow$ Gessel and Stanton's strange evaluation.

Rewrite the special Dougall–Dixon formula

(4.1a)

$$
{}_7F_6\left[\begin{matrix} a+d+1/2, & 1+(a+d+1/2)/2, & d+b+1/2, & d-b+1, & a+n/2, & a+(1+n)/2, & -n \\ & (a+d+1/2)/2, & 1+a-b, & a+b+1/2, & d-(n-3)/2, & d-(n-2)/2, & a+d+n+3/2 \end{matrix}\right]
$$
$$
=\left[\begin{matrix} 2b, & 1-2b, & a-d, & a+d+3/2 \\ 2+2d, & -1-2d, & 1+a-b, & a+b+1/2 \end{matrix}\right]_n
$$

in the form

(4.1b)
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,(2a+2k)_n\,(-1-2d-2k)_n\,\frac{2k+a+d+1/2}{(n+a+d+1/2)_{k+1}}$$
$$\times\begin{bmatrix}a, a+1/2,\ d+b+1/2,\ 1+d-b,\ a+d+1/2\\ \\ 1+d,\ d+3/2,\ a+b+1/2,\ 1+a-b\end{bmatrix}_k$$
$$=\begin{bmatrix}2a,\ 2b,\ 1-2b,\ a-d,\ a+d+1/2\\ \\ 1+a-b,\ a+b+1/2,\ 2+2d\end{bmatrix}_n.$$

Its dual relation

(4.1c)
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\,\frac{2a+3k}{(2a+2n)_{k+1}}\,\frac{-1-2d-k}{(-1-2d-2n)_{k+1}}\,(k+a+d+1/2)_n$$
$$\times\begin{bmatrix}2a,\ 2b,\ 1-2b,\ a-d,\ a+d+1/2\\ \\ 1+a-b,\ a+b+1/2,\ 2+2d\end{bmatrix}_k$$
$$=\begin{bmatrix}a, a+1/2,\ d+b+1/2,\ 1+d-b,\ a+d+1/2\\ \\ 1+d,\ d+3/2,\ a+b+1/2,\ 1+a-b\end{bmatrix}_n$$

can be reformulated as one strange evaluation of Gessel and Stanton [12, Eq. (1.7)]

(4.1d)
$${}_7F_6\begin{bmatrix}2a, & 2b, & 1-2b, & 1+2a/3, & a-d, & a+d+n+1/2, & -n\\ \\ 1+a-b, & a+b+1/2, & 2a/3, & 1+2d, & -2d-2n, & 1+2a+2n\end{bmatrix}$$
$$=\begin{bmatrix}1+2a\\ \\ 1+2d\end{bmatrix}_{2n}\begin{bmatrix}1+d-b,\ d+b+1/2\\ \\ 1+a-b,\ a+b+1/2\end{bmatrix}_n.$$

EXAMPLE 4.2: Dougall–Dixon theorem $\Longleftrightarrow$ Gessel and Stanton's strange evaluation.

Rewrite the special Dougall–Dixon formula

(4.2a)
$$ {}_7F_6\left[\begin{matrix} a+d, & 1+(a+d)/2, & b+d, & a-d+b+1/2, & a+n, & -n/2, & (1-n)/2 \\ & (a+d)/2, & 1+a-b, & b+1/2, & 1+d-n, & 1+a+d+n/2, & a+d+(n+1)/2 \end{matrix}\right] $$
$$ = \left[\begin{matrix} a-b+1/2,\ b,\ -2d,\ 2a+2d+1 \\ 2a-2b+1,\ 2b,\ -d,\ a+d+1/2 \end{matrix}\right]_n $$

in the form

$$ \sum_{k\ge 0} \binom{n}{2k} (a+k)_n\,(-d-k)_n\,\frac{4k+2a+2d}{(n+2a+2d)_{2k+1}} \tag{4.2b} $$
$$ \times \left[\begin{matrix} 1/2,\ a,\ a+d,\ b+d,\ a-b+1/2 \\ b+1/2,\ d+1,\ 1+a-b \end{matrix}\right]_k 4^k $$
$$ = \left[\begin{matrix} a,\ b,\ a-b+1/2,\ 2a+2d,\ -2d \\ 2b,\ 1+2a-2b,\ a+d+1/2 \end{matrix}\right]_n . $$

Its dual relation

(4.2c)
$$ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a+3k/2}{(a+n/2)_{k+1}}\,\frac{-d+k/2}{(-d-n/2)_{k+1}}\,(k+2a+2d)_n $$
$$ \times \left[\begin{matrix} a,\ b,\ a-b+1/2,\ 2a+2d,\ 2d \\ 2b,\ 1+2a-2b,\ a+d+1/2 \end{matrix}\right]_k $$
$$ = \begin{cases} 0, & \text{(n-odd)} \\ 4^m \left[\begin{matrix} 1/2,\ a,\ a+d,\ b+d,\ a-b+d+1/2 \\ b+1/2,\ d+1,\ 1+a-b \end{matrix}\right]_m, & \text{(n=2m)} \end{cases} $$

can be reformulated as another evaluation of Gessel and Stanton [12, Eq. (1.8)]

(4.2d)

$$
{}_7F_6\left[\begin{matrix} a, & b, & a-b+1/2, & 1+2a/3, & 1-2d, & 2a+2d+n, & -n \\ & 2b, & 2a-2b+1, & 2a/3, & a+d+1/2, & 1+a+n/2, & 1-d-n/2 \end{matrix}\right]
$$

$$
= \begin{cases} 0, & \text{(n-odd)} \\ \left[\begin{matrix} 1/2,\ a,\ b+d,\ a-b+d+1/2 \\ b+1/2,\ d+1,\ 1+a-b,\ a+d+1/2 \end{matrix}\right]_m, & \text{(n=2m)}\ . \end{cases}
$$

Both (4.1d) and (4.2d) are due to Gessel and Stanton [12], where their discoveries were motivated by a special pair of Gosper's conjectures and accomplished by series–rearrangement.

EXAMPLE 4.3: Dougall–Dixon theorem $\Longleftrightarrow$ Whipple formula.

Rewrite the special Dougall–Dixon formula

(4.3a)

$$
{}_7F_6\left[\begin{matrix} a, & 1+a/2, & 2a-2b, & b+n/2, & b+(1+n)/2, & -n/2, & (1-n)/2 \\ & a/2, & 1-a+2b, & a-b+1-n/2, & a-b+(1-n)/2, & 1+a+n/2, & a+(n+1)/2 \end{matrix}\right]
$$

$$
= (-1)^n \left[\begin{matrix} 2a+1, & 2a-2b, & 2b-a+1/2 \\ a+1/2, & 2b-2a, & 4b-2a+1 \end{matrix}\right]_n
$$

in the form

(4.3b)

$$
\sum_{k\geq 0}\binom{n}{2k}\frac{2a+4k}{(2a+n)_{2k+1}}\frac{2a-2b}{(2a-2b-n)_{2k+1}}(2b+2k)_n\left[\begin{matrix} a, & 2a-2b \\ 1, & 1-a+2b \end{matrix}\right]_k (1)_{2k}(2b)_{2k}
$$

$$
= (-1)^n \left[\begin{matrix} 2a,\ 2b,\ 2a-2b,\ 2b-a+1/2 \\ a+1/2,\ 1-2a+2b,\ 1-2a+4b \end{matrix}\right]_n .
$$

Its dual relation

(4.3c)

$$\sum_{k=0}^{n}\binom{n}{k}(2a+k)_n\,(2a-2b-k)_n\,\frac{2b+2k}{(2b+n)_{k+1}}$$
$$\times\left[\begin{matrix}2a,\ 2b,\ 2a-2b,\ -a+2b+1/2\\ a+1/2,\ 1-2a+2b,\ 1-2a+4b\end{matrix}\right]_k$$
$$=\begin{cases}0, & \text{(n-odd)}\\ \left[\begin{matrix}a,\ 2a-2b\\ 1,\ 1-a+2b\end{matrix}\right]_m (1)_n\,(2b)_n\ , & \text{(n=2m)}\end{cases}$$

can be reformulated as a terminating Whipple [19, Eq. (15.73)] formula

(4.3d)

$${}_6F_5\left[\begin{matrix}2b,\ 1+b,\ 2b-a+1/2,\ 2a-2b, & 2a+n, & -n\\ b, & a+1/2,\ 1-2a+4b,\ 1-2a+2b-n,\ 1+2b+n\end{matrix};-1\right]$$
$$=\begin{cases}0, & \text{(n-odd)}\\ \left[\begin{matrix}1/2,\ \ b+1/2,\ 1+b,\ 2a-2b\\ a+1/2,\ a-b,\ a-b+1/2,\ 1-a+2b\end{matrix}\right]_m\ , & \text{(n=2m)}\ .\end{cases}$$

EXAMPLE 4.4: Dougall–Dixon theorem $\Longleftrightarrow$ Gasper's strange evaluation.

Rewrite the special Dougall–Dixon formula

(4.4a)

$${}_7F_6\left[\begin{matrix}2a+b, & 1+a+b/2, & a+2b, & a+n/3, & a+(1+n)/3, & a+(2+n)/3, & -n\\ & a+b/2, & 1+a-b, & 1+a+b-n/3, & a+b+(2-n)/3, & a+b+(1-n)/3, & 1+2a+b+n\end{matrix}\right]$$
$$=\left[\begin{matrix}1+2a+b, & 3b\\ 1+a-b, & -3a-3b\end{matrix}\right]_n\left[\begin{matrix}3a\\ 1+3a+3b\end{matrix}\right]_{2n}$$

in the form

(4.4b)

$$\sum_{k=0}(-1)^k\binom{n}{k}(3a+3k)_n\,(-3a-3b-3k)_n\,\frac{2a+b+2k}{(2a+b+n)_{k+1}}$$
$$\times\begin{bmatrix} a+2b,\ 2a+b \\ 1+a-b \end{bmatrix}_k \begin{bmatrix} 3a \\ 1+3a+3b \end{bmatrix}_{3k}$$
$$=\begin{bmatrix} 3a,\ 2a+b,\ 3b \\ 1+a-b \end{bmatrix}_n \begin{bmatrix} 1-3b \\ 1+3a+3b \end{bmatrix}_{2n}.$$

Its dual relation

(4.4c)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{3a+4k}{(3a+3n)_{k+1}}\,\frac{-3a-3b-2k}{(-3a-3b-3n)_{k+1}}\,(2a+b+k)_n$$
$$\times\begin{bmatrix} 3a,\ 2a+b,\ 3b \\ 1+a-b \end{bmatrix}_k \begin{bmatrix} 1-3b \\ 1+3a+3b \end{bmatrix}_{2k}$$
$$=\begin{bmatrix} a+2b,\ 2a+b \\ 1+a-b \end{bmatrix}_n \begin{bmatrix} 3a \\ 1+3a+3b \end{bmatrix}_{3n}$$

can be reformulated as a terminating Gasper's formula (cf. [9, Eq. (5.23)] and[10, Eq. (1.3)])

(4.4d)

$${}_7F_6\begin{bmatrix} 3a,\ 3b, & (1-3b)/2, & (2-3b)/2, & 1+3a/4, & 2a+b+n, & -n \\ 1+a-b, & (1+3a+3b)/2, & (3a+3b)/2, & 3a/4, & 1-3a-3b-3n, & 1+3a+3n \end{bmatrix}$$
$$=\begin{bmatrix} a+2b \\ 1+a-b \end{bmatrix}_n \begin{bmatrix} 3a+1 \\ 3a+3b \end{bmatrix}_{3n}.$$

EXAMPLE 4.5: Dougall–Dixon theorem $\Longleftrightarrow$ The first new strange evaluation.

Rewrite the special Dougall–Dixon formula

(4.5a)

$$
{}_7F_6\left[\begin{matrix} a,\ 1+a/2, & a-b,\ a+b+n, & (1-n)/3, & (2-n)/3, & -n/3 \\ a/2, & 1+b,\ 1-b-n, & a+(2+n)/3, & a+(1+n)/3, & 1+a+n/3 \end{matrix}\right]
$$

$$
= \left[\begin{matrix} a,\ 1+3a \\ b, 1+3b \end{matrix}\right]_n \left[\begin{matrix} 3b \\ 3a \end{matrix}\right]_{2n}
$$

in the form

(4.5b)

$$
\sum_{k\geq 0} (-1)^k \binom{n}{3k} (a+b+k)_n\, (b-k)_n\, \frac{3a+6k}{(3a+n)_{3k+1}} \left[\begin{matrix} a,\ a+b,\ a-b \\ 1+b,\ 1-b \end{matrix}\right]_k (3k)!/k!
$$

$$
= \left[\begin{matrix} a,\ a+b,\ 3a \\ 1+3b \end{matrix}\right]_n \left[\begin{matrix} 3b \\ 3a \end{matrix}\right]_{2n} .
$$

Its dual relation

(4.5c)

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a+b+4k/3}{(a+b+n/3)_{k+1}} \frac{b+2k/3}{(b-n/3)_{k+1}} (3a+k)_n \left[\begin{matrix} a,\ a+b,\ 3a \\ 1+3b \end{matrix}\right]_k \left[\begin{matrix} 3b \\ 3a \end{matrix}\right]_{2k}
$$

$$
= \begin{cases} \left[\begin{matrix} a,\ a+b,\ a-b \\ 1+b,\ 1-b \end{matrix}\right]_m n!/m!\ , & \text{(n=3m)} \\ 0, & \text{(otherwise)} \end{cases}
$$

can be reformulated as the first new strange evaluation in this section

(4.5d)

$$
{}_7F_6\left[\begin{matrix} a,\ a+b,\ 1+3(a+b)/4, & (1+3b)/2, & (2+3b)/2, & 3a+n, & -n \\ 1+3b, & 3(a+b)/4, & 3a/2, & (1+3a)/2, & 1+b-n/3, & 1+a+b+n/3 \end{matrix}\right]
$$

$$
= \begin{cases} \left[\begin{matrix} a,\ 1+a+b,\ a-b \\ 1,\ 1+b,\ -b \end{matrix}\right]_m n!/(3a)_n\ , & \text{(n=3m)} \\ 0, & \text{(otherwise)} . \end{cases}
$$

EXAMPLE 4.6: Dougall–Dixon theorem $\Longleftrightarrow$ The second new strange evaluation.

Rewrite the special Dougall–Dixon formula

(4.6a)

$$ {}_7F_6\left[\begin{matrix} a, & 1+a/2, & 2a+n-1/2, & -n/4, & (1-n)/4, & (2-n)/4, & (3-n)/4 \\ & a/2, & -a-n+3/2, & a+1+n/4, & a+(3+n)/4, & a+(2+n)/4, & a+(1+n)/4 \end{matrix}\right] $$

$$ = 2^n \, \frac{[a, -1+2a,\ 1+4a]_n}{[-1+2a, -2+4a, 4a]_{2n}} \, (-2+4a)_{3n} $$

in the form

(4.6b)

$$ \sum_{k\geq 0} \binom{n}{4k} (k+2a-1/2)_n \, (-k+a-1/2)_n \, \frac{4a+8k}{(4a+n)_{4k+1}} $$

$$ \times \left[\begin{matrix} a,\ 2a-1/2 \\ -a+3/2 \end{matrix}\right]_k (4k)!/k! = 2^{-3n} \, \frac{(4a)_n \, (-2+4a)_{3n}}{(4a)_{2n}} \ . $$

Its dual relation

(4.6c)

$$ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{-(1/2)+2a+5k/4}{\big(-(1/2)+2a+n/4\big)_{k+1}} \, \frac{-(1/2)+a+3k/4}{\big(-(1/2)+a-n/4\big)_{k+1}} \, (4a+k)_n $$

$$ \times \frac{(4a)_k \, (-2+4a)_{3k}}{(4a)_{2k}} \, 2^{-3k} $$

$$ = \begin{cases} \left[\begin{matrix} a,\ 2a-1/2 \\ -a+3/2 \end{matrix}\right]_m n!/m! \ , & \text{(n=4m)} \\ 0, & \text{(otherwise)} \end{cases} $$

can be reformulated as the second new strange evaluation in this section

(4.6d)

$$ {}_6F_5\left[\begin{matrix} (-1+4a)/3,\ 4a/3,\ (1+4a)/3,\ (3+8a)/5, & 4a+n, & -n \\ 2a, \quad 2a+1/2,\ (-2+8a)/5,\ a+(2-n)/4,\ 2a+(2+n)/4 & & \end{matrix} ;27/32\right] $$

$$ = \begin{cases} \left[\begin{matrix} a,\ 2a+1/2 \\ 1,\ -a+1/2 \end{matrix}\right]_m n!/(4a)_n \ , & \text{(n=4m)} \\ 0, & \text{(otherwise)}\ . \end{cases} $$

EXAMPLE 4.7: Dougall–Dixon theorem $\Longleftrightarrow$ The third new strange evaluation.

Rewrite the special Dougall–Dixon formula

(4.7a)

$$
{}_7F_6\left[\begin{matrix} \frac{(1+6x)}{4}, & \frac{(9+6x)}{8}, & x+\frac{n}{3}, & x+\frac{1+n}{3} & x+\frac{2+n}{3}, & \frac{-n}{2}, & \frac{(1-n)}{2} \\ & \frac{(1+6x)}{8}, & \frac{(15+6x-4n)}{12}, & \frac{(11+6x-4n)}{12}, & \frac{(7+6x-4n)}{12}, & \frac{(5+6x+2n)}{4}, & \frac{(3+6x+2n)}{4} \end{matrix}\right]
$$

$$
= (-1)^n\,\frac{3x+3/2}{3x+1/2}\,\frac{3x+n+1/2}{3x+n+3/2}\,\frac{\big((1+6x)/4\big)_{2n}}{\big((3+6x)/4\big)_n\big(-(3+6x)/4\big)_n}
$$

in the form

(4.7b)

$$
\sum_{k\ge 0}\binom{n}{2k}(3k+3x)_n\big(-3k-(3+6x)/4\big)_n\,\frac{4k+3x+1/2}{(n+3x+1/2)_{2k+1}}
$$

$$
\times\left[\begin{matrix} 3x \\ (7+6x)/4 \end{matrix}\right]_{3k}\big((1+6x)/4\big)_k\,(2k)!/k!
$$

$$
= (-1)^n\left[\begin{matrix} 3x,\ 3x+3/2 \\ 3x+5/2,\ (3+6x)/4 \end{matrix}\right]_n\big((1+6x)/4\big)_{2n}.
$$

Its dual relation

(4.7c)

$$
\sum_{k=0}^{n}\binom{n}{k}\frac{3x+5k/2}{(3x+3n/2)_{k+1}}\,\frac{-(3+6x+2k)/4}{\big(-(3+6x+6n)/4\big)_{k+1}}\,(3x+k+1/2)_n
$$

$$
\times\left[\begin{matrix} 3x,\ 3x+3/2 \\ 3x+5/2,\ (3+6x)/4 \end{matrix}\right]_k\big((1+6x)/4\big)_{2k}
$$

$$
= \begin{cases} 0, & \text{(n-odd)} \\ \left[\begin{matrix} 3x \\ (7+6x)/4 \end{matrix}\right]_{3m}\big((1+6x)/4\big)_m\,n!/m!\,, & \text{(n=2m)} \end{cases}
$$

can be reformulated as the third new strange evaluation in this section

(4.7d)

$$
{}_6F_5\left[\begin{matrix} 3x,\ 1+6x/5,\ (1+6x)/8,\ (5+6x)/8,\ \ 3x+n+1/2, & -n \\ 6x/5,\ \ 3x+1/2,\ \ (3+6x)/4,\ 1+3x+3n/2,\ (1-6x-6n)/4 \end{matrix};-4\right]
$$

$$
= \begin{cases} 0, & \text{(n-odd)} \\ \left[\begin{matrix} 1/2 \\ (3+6x)/4 \end{matrix}\right]_m \left[\begin{matrix} 1+3x \\ (3+6x)/4 \end{matrix}\right]_{3m}, & \text{(n=2m)} \end{cases}.
$$

EXAMPLE 4.8: Dougall–Dixon theorem $\Longleftrightarrow$ The fourth new strange evaluation.

Rewrite the special Dougall–Dixon formula

(4.8a)

$$
{}_7F_6\left[\begin{matrix} y+\frac{1}{4}, & \frac{9+4y}{8}, & y+\frac{n}{2}, & y+\frac{1+n}{2}, & \frac{-n}{3}, & \frac{1-n}{3}, & \frac{2-n}{3} \\ & \frac{1+4y}{8}, & \frac{5-2n}{4}, & \frac{3-2n}{4}, & y+\frac{15+4n}{12}, & y+\frac{11+4n}{12}, & y+\frac{7+4n}{12} \end{matrix}\right]
$$

$$
= (-1)^n\, 3\, \frac{n-1/2}{n-3/2}\, \frac{(y+1/4)_n\,(3y+7/4)_n}{(3y+3/4)_{2n}}
$$

in the form

(4.8b)

$$
\sum_{k\geq 0}(-1)^k\binom{n}{3k}(2y+2k)_n\,(-2k-1/2)_n\,\frac{3y+6k+3/4}{(3y+n+3/4)_{3k+1}}
$$

$$
\times\left[\begin{matrix} 2y \\ 3/2 \end{matrix}\right]_{2k}(y+1/4)_k\,(3k)!/k!
$$

$$
= (-1)^n\, 3\frac{n-1/2}{n-3/2}\,\frac{[-1/2,2y,y+1/4,3y+3/4]_n}{(3y+3/4)_{2n}}\ .
$$

Its dual relation

(4.8c)

$$
\sum_{k=0}^{n}\binom{n}{k}\frac{2y+5k/3}{(2y+2n/3)_{k+1}}\,\frac{-(3-2k)/6}{\big(-(3+4n)/6\big)_{k+1}}\,(3y+k+3/4)_n
$$

$$\times \frac{k-1/2}{-(3-2k)/6} \frac{[-1/2, 2y, y+1/4, 3y+3/4]_k}{(3y+3/4)_{2k}}$$

$$= \begin{cases} \begin{bmatrix} 2y \\ 3/2 \end{bmatrix}_{2m} (y+1/4)_m \ n!/m! \ , & \text{(n=3m)} \\ 0, & \text{(otherwise)} \end{cases}$$

can be reformulated as the fourth new strange evaluation in this section

(4.8d)

$$ {}_6F_5 \left[\begin{matrix} 1/2, & 1+6y/5, & 2y, & y+1/4, & 3y+n+3/4, & -n \\ & 6y/5, & (3+12y)/8, & (7+12y)/8, & (3-4n)/6, & 1+2y+2n/3 \end{matrix} ; -1/4 \right]$$

$$= \begin{cases} \begin{bmatrix} 1/3, & 2/3, & y+1/2, & y+1 \\ 1/4, & 3/4, & y+7/12, & y+11/12 \end{bmatrix}_m , & \text{(n=3m)} \\ 0, & \text{(otherwise)} \ . \end{cases}$$

Remark. Similarly, one can show that the original Dougall formula

(4.9)

$$ {}_7F_6 \left[\begin{matrix} a, 1+a/2, & b, & c, & d, & 1+2a-b-c-d+n, & -n \\ a/2, & 1+a-b, & 1+a-c, & 1+a-d, & b+c+d-a-n, & 1+a+n \end{matrix} \right]$$

$$= \begin{bmatrix} 1+a, & 1+a-b-c, & 1+a-b-d, & 1+a-c-d \\ 1+a-b-c-d, & 1+a-b, & 1+a-c, & 1+a-d \end{bmatrix}_n$$

with 5-free parameters, is self reciprocal, i.e., the dual relation is the same as the original one under parameter replacement. An exceptional case of the Dougall–Dixon theorem

(4.10)

$$ {}_7F_6 \left[\begin{matrix} 2a+\frac{1}{4}, & a+\frac{9}{8}, & a+\frac{n}{4}, & a+\frac{1+n}{4}, & a+\frac{2+n}{4}, & a+\frac{3+n}{4}, & -n \\ & a+\frac{1}{8}, & a+\frac{5-n}{4}, & a+\frac{3-n}{4}, & a+\frac{3-n}{4}, & a+\frac{2-n}{4}, & 2a+n+\frac{5}{4} \end{matrix} \right]$$

$$= \delta_{0,n}$$

with 2 free parameters, leads to trivial reciprocal relations under the embedding process.

5. Reversal embeddings on the Dougall–Dixon theorem

Sometimes a terminating hypergeometric summation $\sum_{k=0}^{n} T(n,k) = S(n)$ can not be expressed as one of the relations in (2.1), but its reversal $\sum_{k=0}^{n} T(n, n-k) = S(n)$ can be expressed in this form. In this case, the dual relation will create some "mysterious–looking" formulas. By means of (3.0), (4.0) and transforms

$$(5.0a) \quad (x - mn + n)_{mn-mk} = (-1)^{mn+mk+n}\ (x - mk)_n\ (1 - x)_{mn-n}/(1 - x)_{mk}$$

$$(5.0b) \quad (z - mn - n)_{mn-mk} = (-1)^{mn+mk}\ (1 - z + mk)_n^{-1}\ (1 - z)_{mn+n}/(1 - z)_{mk}$$

some striking examples are demonstrated as follows.

EXAMPLE 5.1: Dougall–Dixon formula $\Longleftrightarrow$ One "very–strange" evaluation.

For $\varepsilon = 0$ and 1, the reversal of Dougall–Dixon formula

(5.1a)

$$ {}_7F_6\left[\begin{matrix} A-2n, & 1+\frac{A-2n}{2}, & -\frac{\varepsilon+n}{2}, & -\frac{\varepsilon+n-1}{2}, & A-u-n, & A-v-n, & \varepsilon+u+v-n+\frac{1}{2} \\ & \frac{A-2n}{2}, & 1+A+\frac{\varepsilon-3n}{2}, & A+\frac{1+\varepsilon-3n}{2}, & 1+u-n, & 1+v-n, & -\varepsilon+A-u-v-n+\frac{1}{2} \end{matrix}\right]$$
$$= \frac{(-A)_{2n}}{\big(-\varepsilon-2A\big)_{3n}}\left[\begin{matrix} -\varepsilon-A+1/2, & -\varepsilon-2u, & -\varepsilon-2v, & \varepsilon-2A+2u+2v+1 \\ & -u, & -v, & \varepsilon-A+u+v+1/2 \end{matrix}\right]_n$$

may be expressed as, after adding some extra zero–terms

(5.1b)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\big(-\varepsilon-2A+2k\big)_n\big(-\varepsilon-2k\big)_n\frac{-A+2k}{(-A+n)_{k+1}}$$
$$\times\left[\begin{matrix} 1, & -u, & -v, & \varepsilon-A+u+v+1/2 \\ 1-A+u, & 1-A+v, & -\varepsilon-u-v+1/2 \end{matrix}\right]_k\frac{(-\varepsilon-2A)_{2k}}{(1+\varepsilon)_{2k}}$$
$$= \left[\begin{matrix} 1, & -A, & -\varepsilon-A+1/2, & -\varepsilon-2u, & -\varepsilon-2v, & 1+\varepsilon-2A+2u+2v \\ & 1+\varepsilon, & 1-A+u, & 1-A+u, & -\varepsilon-u-v+1/2 & \end{matrix}\right]_n$$

whose dual relation

(5.1c)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{-\varepsilon-2A+3k}{(-\varepsilon-2A+2n)_{k+1}}\,\frac{-\varepsilon-k}{(-\varepsilon-2n)_{k+1}}\,(-A+k)_n$$
$$\times\left[\begin{matrix}1,\ -A,\ -\varepsilon-A+1/2,\ -\varepsilon-2u,\ -\varepsilon-2v,\ 1+\varepsilon-2A+2u+2v\\ 1+\varepsilon,\ 1-A+u,\ 1-A+v,\ -\varepsilon-u-v+1/2\end{matrix}\right]_k$$
$$=\frac{(-\varepsilon-2A)_{2n}}{(1+\varepsilon)_{2n}}\left[\begin{matrix}1,\ -u,\ -v,\ \varepsilon-A+u+v+1/2\\ 1-A+u,\ 1-A+v,\ -\varepsilon-u-v+1/2\end{matrix}\right]_n$$

can be reformulated in hypergeometrics

(5.1d)

$${}_8F_7\left[\begin{matrix}-\varepsilon-A+\frac{1}{2}, & 1, & 1-\frac{\varepsilon+2A}{3}, & -\varepsilon-2u, & -\varepsilon-2v, & 1+\varepsilon-2A+2u+2v, & -A+n, & -n\\ & \varepsilon, & -\frac{\varepsilon+2A}{3}, & 1-A+u, & 1-A+v, & -\varepsilon-u-v+\frac{1}{2}, & 1-\varepsilon-2n, & 1-\varepsilon-2A+2n\end{matrix}\right]$$
$$=\frac{(1-\varepsilon-2A)_{2n}}{(\varepsilon)_{2n}}\left[\begin{matrix}1,\ -u,\ -v,\ \varepsilon-A+u+v+1/2\\ -A,\ 1-A+u,\ 1-A+v,\ -\varepsilon-u-v+1/2\end{matrix}\right]_n.$$

For $\varepsilon=1$, this formula results in the strange evaluation:

(5.1e)

$${}_7F_6\left[\begin{matrix}-A-1/2,(2-2A)/3,-1-2u,-1-2v,2-2A+2u+2v,-A+n, & -n\\ -(1+2A)/3,\ 1-A+u,\ 1-A+v,\quad -u-v-1/2,\ -2n, & -2A+2n\end{matrix}\right]$$
$$=\left[\begin{matrix}-A+1/2,-u,-v,-A+u+v+3/2\\ 1/2,1-A+u,1-A+v,-u-v-1/2\end{matrix}\right]_n.$$

When ε tends to 0, the limiting version of (5.1d) multiplied by ε will return to (5.1e) after the parameter replacements $n\longrightarrow n+1,\ A\longrightarrow A+2,\ u\longrightarrow u+1,\ v\longrightarrow v+1$ have been performed.

EXAMPLE 5.2: Dougall–Dixon formula $\Longleftrightarrow$ Two "very–strange" evaluations.

For $\varepsilon = 0$, 1 and 2, the reversal of Dougall–Dixon formula

(5.2a)

$$
{}_7F_6\left[\begin{matrix} -2n-\frac{\varepsilon+2a}{3}, & 1-n-\frac{\varepsilon+2a}{6}, & -\frac{\varepsilon+2n}{3}, & -\frac{\varepsilon-1+2n}{3}, & -\frac{\varepsilon-2+2n}{3}, & -n-\frac{2a-b}{3}, & -n-\frac{2a+b-\varepsilon}{3} \\ & -n-\frac{\varepsilon+2a}{6}, & \frac{3-2a-4n}{3}, & \frac{2-2a-4n}{3}, & \frac{1-2a-4n}{3}, & 1-n-\frac{\varepsilon+b}{3}, & 1-n-\frac{2\varepsilon-b}{3} \end{matrix}\right]
$$

$$
= \frac{\big((\varepsilon+2a)/3\big)_{2n}\,(1+2a-\varepsilon)_{2n}}{(2a)_{4n}} \left[\begin{matrix} b,\ \varepsilon-b \\ (\varepsilon+b)/3, (2\varepsilon-b)/3 \end{matrix}\right]_n
$$

may be expressed as, after adding some extra zero–terms

(5.2b)

$$
\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2a+3k)_n\,(-\varepsilon-3k)_n\,\frac{2k+(\varepsilon+2a)/3}{\big(n+(\varepsilon+2a)/3\big)_{k+1}}
$$

$$
\times\left[\begin{matrix} 1, (\varepsilon+b)/3, (2\varepsilon-b)/3 \\ 1+(2a-b)/3,\ 1-(\varepsilon-2a-b)/3 \end{matrix}\right]_k \frac{(2a)_{3k}}{(1+\varepsilon)_{3k}}
$$

$$
= \frac{(1+2a-\varepsilon)_{2n}}{(1+\varepsilon)_{2n}} \left[\begin{matrix} 1,\ b,\ \varepsilon-b,\ (\varepsilon+2a)/3 \\ 1+(2a-b)/3,\ 1-(\varepsilon-2a-b)/3 \end{matrix}\right]_n
$$

whose dual relation

(5.2c)

$$
\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2a+4k}{(2a+3n)_{k+1}}\,\frac{-\varepsilon-2k}{(-\varepsilon-3n)_{k+1}}\big(k+(\varepsilon+2a)/3\big)_n
$$

$$
\times\left[\begin{matrix} 1,\ b,\ \varepsilon-b,\ (\varepsilon+2a)/3 \\ 1+(2a-b)/3,\ 1-(\varepsilon-2a-b)/3 \end{matrix}\right]_k \left[\begin{matrix} 1+2a-\varepsilon \\ 1+\varepsilon \end{matrix}\right]_{2k}
$$

$$
= \frac{(2a)_{3n}}{(1+\varepsilon)_{3n}} \left[\begin{matrix} 1,\ (\varepsilon+b)/3,\ (2\varepsilon-b)/3 \\ 1+(2a-b)/3,\ 1-(\varepsilon-2a-b)/3 \end{matrix}\right]_n
$$

can be reformulated in hypergeometrics

(5.2d)

$$
{}_8F_7\left[\begin{matrix} a+\frac{1-\varepsilon}{2}, & 1, & a+\frac{2-\varepsilon}{2}, & 1+\frac{a}{2}, & b, & \varepsilon-b, & n+\frac{\varepsilon+2a}{3}, & -n \\ & \frac{\varepsilon}{2}, & \frac{\varepsilon+1}{2}, & \frac{a}{2}, & 1+\frac{2a-b}{3}, & 1+\frac{2a+b-\varepsilon}{3}, & 1-\varepsilon-3n, & 1+2a+3n \end{matrix}\right]
$$

$$
= \frac{(1+2a)_{3n}}{(\varepsilon)_{3n}} \left[\begin{matrix} 1,\ (\varepsilon+b)/3,\ (2\varepsilon-b)/3 \\ (\varepsilon+2a)/3, 1+(2a-b)/3, 1+(2a+b-\varepsilon)/3 \end{matrix}\right]_n .
$$

This formula is the unified version of a pair of very strange hypergeometric evaluations. One of these for $\varepsilon = 1$

$$
{}_7F_6\left[\begin{matrix} a, & 1+\frac{a}{2}, & a+\frac{1}{2}, & b, & 1-b, & n+\frac{2a+1}{3}, & -n \\ & \frac{a}{2}, & \frac{1}{2}, & \frac{2a-b+3}{3}, & \frac{2a+b+2}{3}, & -3n, & 1+2a+3n \end{matrix}\right] \tag{5.2e}
$$

$$
= \left[\begin{matrix} (1+b)/3, (2-b)/3, (2a+2)/3, (2a+3)/3 \\ 1/3,\ 2/3,\ (3+2a-b)/3,\ (2+2a+b)/3 \end{matrix}\right]_n
$$

is the original Gosper Conjecture (1977) (see also Gessel & Stanton [12]), which was recently confirmed by Ekhad [8] by computer certification and Gasper & Rahman (cf. [10, Eq. (1.6)] and [11, Eq. (3.8.17)]) for its non–terminating version. Another for $\varepsilon = 2$

(5.2f)

$$
{}_7F_6\left[\begin{matrix} a, & 1+\frac{a}{2}, & a-\frac{1}{2}, & b, & 2-b, & n+\frac{2a+2}{3}, & -n \\ & \frac{a}{2}, & \frac{3}{2}, & \frac{2a-b+3}{3}, & \frac{2a+b+1}{3}, & -1-3n, & 1+2a+3n \end{matrix}\right]
$$

$$
= \left[\begin{matrix} (2+b)/3, (4-b)/3, (2a+1)/3, (2a+3)/3 \\ 2/3,\ 4/3,\ (3+2a-b)/3,\ (1+2a+b)/3 \end{matrix}\right]_n
$$

is due to Gasper and Rahman (cf. [10, Eq. (4.7)] and [11, p. 100]), where its non–terminating form was established either. When ε tends to 0, the limiting version of (5.2d) multiplied by ε will return to (5.2f) after the parameter replacements $n \longrightarrow n+1,\ a \longrightarrow a-2,\ b \longrightarrow b-1$ have been performed.

Example 5.3: Dougall–Dixon formula $\Longleftrightarrow$ Three "very–strange" evaluations.

For $\varepsilon = 0,\ 1,\ 2$ and 3, the reversal of Dougall–Dixon formula

(5.3a)

$$
{}_7F_6\left[\begin{matrix} x-2n, & 1-n+\frac{x}{2}, & -\frac{\varepsilon+3n}{4}, & -\frac{\varepsilon-1+3n}{4}, & -\frac{\varepsilon-2+2n}{4}, & -\frac{\varepsilon-3+2n}{4}, & \varepsilon+2x-n-\frac{1}{2} \\ & -n-\frac{x}{2}, & x+\frac{\varepsilon+4-5n}{4}, & x+\frac{\varepsilon+3-5n}{4}, & x+\frac{(\varepsilon+2-5n}{4}, & x+\frac{\varepsilon+1-5n}{4}, & -\varepsilon-x-n+\frac{3}{2} \end{matrix}\right]
$$

$$
= \ 2^{3n}\ \frac{(-x)_{2n}\,(1-2\varepsilon-4x)_{2n}}{\big(-\varepsilon-4x\big)_{5n}}\ \left[\begin{matrix} -2+3\varepsilon+4x, -\varepsilon-2x+3/2 \\ \varepsilon+x-1/2 \end{matrix}\right]_n
$$

may be expressed as, after adding some extra zero–terms

(5.3b)

$$
\sum_{k=0}^{n}(-1)^k\binom{n}{k}(-\varepsilon-4x+4k)_n\,(-\varepsilon-4k)_n\,\frac{-x+2k}{(-x+n)_{k+1}}
$$

$$
\times\left[\begin{matrix} 1,\varepsilon+x-1/2 \\ -\varepsilon-2x+3/2 \end{matrix}\right]_k\left[\begin{matrix} -\varepsilon-4x \\ 1+\varepsilon \end{matrix}\right]_{4k}
$$

$$
= 2^{3n}\,\frac{(1-2\varepsilon-4x)_{2n}}{(1+\varepsilon)_{3n}}\ [1,-x,-2+3\varepsilon+4x]_n
$$

whose dual relation

(5.3c)

$$
\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{-\varepsilon-4x+5k}{(-\varepsilon-4x+4n)_{k+1}}\,\frac{-\varepsilon-3k}{(-\varepsilon-4n)_{k+1}}\,(-x+k)_n
$$

$$
\times\,\frac{(1-2\varepsilon-4x)_{2k}}{(1+\varepsilon)_{3k}}\,[1,-x,-2+3\varepsilon+4x]_k\ 2^{3k}
$$

$$
= \left[\begin{matrix} 1,\varepsilon+x-1/2 \\ -\varepsilon-2x+3/2 \end{matrix}\right]_n\left[\begin{matrix} -\varepsilon-4x \\ 1+\varepsilon \end{matrix}\right]_{4n}
$$

can be reformulated in hypergeometrics

(5.3d)

$$
{}_7F_6\left[\begin{matrix} -2+3\varepsilon+4x, & 1, & -\varepsilon-2x+1, & -\varepsilon-2x+\frac{1}{2}, & 1-\frac{\varepsilon+4x}{5}, & -x+n, & -n & \\ & \frac{\varepsilon}{3,} & \frac{\varepsilon+1}{3,} & \frac{\varepsilon+2}{3,} & -\frac{\varepsilon+4x}{5}, & 1-\varepsilon-4n, & 1-\varepsilon4x+4n & ;\frac{32}{27} \end{matrix}\right]
$$

$$= \frac{(1-\varepsilon-4x)_{4n}}{(\varepsilon)_{4n}} \left[\begin{matrix} 1, & \varepsilon+x-1/2 \\ -x, & -\varepsilon-2x+3/2 \end{matrix}\right]_n .$$

It can be displayed, explicitly for $\varepsilon = 1,\ 2,$ and 3, as three very strange evaluations:

(5.3e)

$${}_6F_5\left[\begin{matrix} 1+4x, & -2x, & -2x-1/2, & (4-4x)/5, & -x+n, & -n \\ & 1/3, & 2/3, & -(1+4x)/5, & -4n, & -4x+4n \end{matrix} ; 32/27\right]$$

$$= \left[\begin{matrix} -x+1/4,\ -x+2/4,\ -x+3/4,\ x+1/2 \\ 1/4,\ 2/4,\ 3/4,\ -2x+1/2 \end{matrix}\right]_n$$

(5.3f)

$${}_6F_5\left[\begin{matrix} 4+4x, & -1-2x, & -2x-3/2, & (3-4x)/5, & -x+n, & -n \\ & 2/3, & 4/3, & -(2+4x)/5, & -1-4n, & -1-4x+4n \end{matrix} ; 32/27\right]$$

$$= \left[\begin{matrix} -x-1/4,\ -x+1/4,\ -x+2/4,\ x+3/2 \\ 2/4,\ 3/4,\ 5/4,\ -2x-1/2 \end{matrix}\right]_n$$

(5.3g)

$${}_6F_5\left[\begin{matrix} 7+4x, & -2-2x, & -2x-\frac{5}{2}, & \frac{2-4x}{5,} & -x+n, & -n \\ & \frac{4}{3,} & \frac{5}{3,} & -\frac{3+4x}{5}, & -2-4n, & -2-4x+4n \end{matrix} ; \frac{32}{27}\right]$$

$$= \left[\begin{matrix} -x-2/4,\ -x-1/4,\ -x+1/4,\ x+5/2 \\ 3/4,\ 5/4,\ 6/4,\ -2x-3/2 \end{matrix}\right]_n .$$

When ε tends to 0, the limiting version of (5.3d) multiplied by ε will return to (5.3g) after the parameter replacements $n \longrightarrow n+1,\ x \longrightarrow x+2$ have been performed.

Example 5.4: Dougall–Dixon formula$\Longleftrightarrow$ Two "very–strange" evaluations.

For $\varepsilon = 0,\ 1,$ and 2, the reversal of Dougall–Dixon formula

(5.4a)

$$ {}_6F_5\left[\begin{matrix} -\varepsilon-2n+\frac{1}{2}, & -n+\frac{5-2\varepsilon}{4}, & -\frac{2\varepsilon+4n}{5}, & -\frac{2\varepsilon-1+4n}{5}, & -\frac{2\varepsilon-2+4n}{5}, & -\frac{2\varepsilon-3+4n}{5}, & -\frac{2\varepsilon-4+4n}{5} \\ & -n+\frac{1-2\varepsilon}{4}, & \frac{15-6\varepsilon-12n}{10,} & \frac{13-6\varepsilon-12n}{10,} & \frac{11-6\varepsilon-12n}{10,} & \frac{9-6\varepsilon-12n}{10,} & \frac{7-6\varepsilon-12n}{10} \end{matrix}\right] $$

$$ = 5^{2n}[\varepsilon+1/2, \varepsilon-1/2, \varepsilon-3/2]_{2n}/(3\varepsilon-5/2)_{6n} $$

may be expressed as, after adding some extra zero–terms

(5.4b)

$$ \sum_{k=0}^{n}(-1)^k\binom{n}{k}(3\varepsilon+5k-5/2)_n\,(-2\varepsilon-5k)_n\,\frac{\varepsilon+2k-1/2}{(\varepsilon+n-1/2)_{k+1}} $$

$$ \times\begin{bmatrix} 3\varepsilon-5/2 \\ 2\varepsilon+1 \end{bmatrix}_{5k}(1)_k = (5/4)^{2n}\,\frac{(\varepsilon-3/2)_{2n}}{(1+\varepsilon)_{2n}}\,[1, \varepsilon-1/2]_n $$

whose dual relation

(5.4c)

$$ \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{3\varepsilon+6k-5/2}{(3\varepsilon+5n-5/2)_{k+1}}\;\frac{-2\varepsilon-4k}{(-2\varepsilon-5n)_{k+1}}\;(\varepsilon+k-1/2)_n $$

$$ \times\frac{(\varepsilon-3/2)_{2k}}{(1+\varepsilon)_{2k}}\,[1, \varepsilon-1/2]_k\,(5/4)^{2k} = \begin{bmatrix} 3\varepsilon-5/2 \\ 2\varepsilon+1 \end{bmatrix}_{5n}(1)_n $$

can be reformulated in hypergeometrics

(5.4d)

$$ {}_6F_5\left[\begin{matrix} \frac{2\varepsilon-1}{4}, & 1, & \frac{2\varepsilon-3}{4}, & \frac{7+6\varepsilon}{12}, & \varepsilon+n-\frac{1}{2}, & -n \\ & \frac{\varepsilon}{2}, & \frac{\varepsilon+1}{2}, & \frac{-5+6\varepsilon}{12}, & 1-2\varepsilon-5n, & 3\varepsilon+5n-\frac{3}{2} \end{matrix};\frac{25}{16}\right] $$

$$ = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}_n\begin{bmatrix} 3\varepsilon-3/2 \\ 2\varepsilon \end{bmatrix}_n. $$

We can exhibit it, for $\varepsilon = 1$ and 2, as two very strange evaluations:

(5.4e)
$$
{}_5F_4\left[\begin{matrix}1/4, & -1/4, 13/12, & n+1/2, & -n \\ & 1/2, \quad 1/12, & 5n+3/2, & -1-5n\end{matrix}\,;25/16\right]
= \left[\begin{matrix}1\\1/2\end{matrix}\right]_n \left[\begin{matrix}3/2\\2\end{matrix}\right]_{5n},
$$

(5.4f)
$$
{}_5F_4\left[\begin{matrix}1/4, & 3/4, 19/12, & n+3/2, & -n \\ & 3/2, \quad 7/12, & 5n+9/2, & -3-5n\end{matrix}\,;25/16\right]
= \left[\begin{matrix}1\\3/2\end{matrix}\right]_n \left[\begin{matrix}9/2\\4\end{matrix}\right]_{5n}.
$$

When ε tends to 0, the limiting version of (5.4d) multiplied by ε will return to (5.4f) after the parameter replacement $n \longrightarrow n+1$ has been performed.

Except for (5.2e) and (5.2f), all the formulas demonstrated in this section do not fit into the known hypergeometric relations. The author believes them to be new confidently.

Concluding remarks

The examples exhibited in this paper are sufficient to convince that the inversion technique described is indeed an efficient approach for verifying and discovering combinatorial identities. As the computations involved are quite trivial (only the transforms between hypergeometric series and inverse relations), it is very convenient to check the truth of any given combinatorial sum in the forms of (2.1a-2.1b) by inversion machinery. In fact, it can be shown that *almost all terminating hypergeometric identities are the dual relations of only three hypergeometric formulae named after Chu–Vandermonde–Gauss, Pfaff–Saalschutz and Dougall–Dixon–Kummer, as long as a general reciprocal pair (2.1) is accepted in advance.* Moreover, the applicable aspects of (2.1a) and (2.1b) have not been exhausted. The author believes that the potential this pair of reciprocal relations needs to be tapped further. For example, the dual relations of a very–well–poised and four–balanced evaluation (instead of Dougall–Dixon theorem) due to Lakin [16] (cf. Askey [2]) would result in the strange evaluations parallel to those displayed in Section 3 and 4. The author hopes that some new striking combinatorial relationships hidden behind (2.1a-b) will be found soon.

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