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SOME NOVEL GENERATING FUNCTIONS OF EXTENDED JACOBI POLYNOMIALS BY GROUP THEORETIC METHOD

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1. Introduction

The extended Jacobi polynomials defined by Patil and Thakare [1]

$$F_n(\alpha,\beta;x) = \frac{(-1)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}\Big(\frac{\lambda}{b-a}\Big)^n \times D^n[(x-a)^{n+\alpha}(b-x)^{n+\beta}], \tag{1.1}$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and λ is a number such that $\frac{\lambda}{b-a} > 0$, satisfy the ordinary differential equation [2]

$$[(x-a)(b-x)D^2 + \{(\alpha+1)(b-x)-(\beta+1)(x-a)\}D + n(1+\alpha+\beta+n)]y = 0. \tag{1.2}$$

Very recently, attempts have been made [2, 3] in connection with the derivation of generating functions of the extended Jacobi polynomials from the Lie-group viewpoint.

The aim of the present paper is to investigate some novel generating relations of the extended Jacobi polynomial $F_n(\alpha,\beta;x)$ by the application of L. Weisner's group-theoretic method [4] which is vividly presented in the monograph by E.B. McBride [5]. It may be of interest to remark that in course of constructing a Lie algebra we obtain a pair of linear partial differential operators which simultaneously raise (lower) and lower (raise) the parameters α and β of the polynomial under consideration. We would like to mention that our results differ from the traditional concept of a generating function for orthogonal polynomials.

Replacing $\frac{\mathrm{d}}{\mathrm{d}x}$ by $\frac{\partial}{\partial x}$, α by $y\frac{\partial}{\partial y}$, β by $z\frac{\partial}{\partial z}$ and y by $u(x,y,z)$ in (1.2), we get the partial differential equation

$$\text{(2.1)}\qquad \Big[(x-a)(b-x)\frac{\partial^2}{\partial x^2}+\Big\{\Big(y\frac{\partial}{\partial y}+1\Big)(b-x)-\Big(z\frac{\partial}{\partial z}+1\Big)(x-a)\Big\}\frac{\partial}{\partial x} + n\Big(1+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+n\Big)\Big]u(x,y,z)=0.$$

Thus we see that $u(x,y,z)=F_n(\alpha,\beta;x)y^\alpha z^\beta$ is a solution of (2.1), since $F_n(\alpha,\beta;x)$ is a solution of (1.2).

We now define linear partial differential operators

$$\text{(2.2)}\qquad \begin{aligned} A_1 &= y\frac{\partial}{\partial y},\\ A_2 &= z\frac{\partial}{\partial z},\\ A_3 &= (x-b)yz^{-1}\frac{\partial}{\partial x}+y\frac{\partial}{\partial z},\\ A_4 &= (x-a)y^{-1}z\frac{\partial}{\partial x}+z\frac{\partial}{\partial y} \end{aligned}$$

such that

$$\text{(2.3)}\qquad \begin{aligned} A_1[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= \alpha F_n(\alpha,\beta;x)y^\alpha z^\beta,\\ A_2[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= \beta F_n(\alpha,\beta;x)y^\alpha z^\beta,\\ A_3[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= (\beta+n)F^n(\alpha+1,\beta-1;x)y^{\alpha+1}z^{\beta-1},\\ A_4[F_n(\alpha,\beta;x)y^\alpha z^\beta] &= (n+\alpha)F_n(\alpha-1,\beta+1;x)y^{\alpha-1}z^{\beta+1}. \end{aligned}$$

The commutator relations satisfied by A_i (i = 1, 2, 3, 4) are

$$\text{(2.4)}\qquad \begin{aligned} [A_1,A_2]&=0, & [A_2,A_3]&=-A_3,\\ [A_1,A_3]&=A_3, & [A_2,A_4]&=A_4,\\ [A_1,A_4]&=-A_4, & [A_3,A_4]&=A_1-A_2 \end{aligned}$$

where $[A,B]u=(AB-BA)u$.

Thus we arrive at the following theorem:

Theorem. *The set of operators $\{1, A_i\ (i=1,2,3,4)\}$ where 1 stands for the identity operator, generates a Lie algebra $\mathcal{L}$.*

It can be shown that the partial differential operator L,

$$L = (x-a)(b-x)\frac{\partial^2}{\partial x^2} + \Big[\Big(y\frac{\partial}{\partial y}+1\Big)(b-x) - \Big(z\frac{\partial}{\partial z}+1\Big)(x-a)\Big]\frac{\partial}{\partial x} + n\Big(1 + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + n\Big) \tag{2.5}$$

can be related to A_i ($i = 1, 2, 3, 4$) as follows:

$$L = -A_3A_4 + A_1A_2 + A_1 + n(1 + A_1 + A_2 + n). \tag{2.6}$$

Now one can easily verify that L commutes with each A_i ($i = 1, 2, 3, 4$), i.e.,

$$[L, A_i] = 0. \tag{2.7}$$

The extended form of the groups generated by A_i ($i = 1, 2, 3, 4$) are

$$\begin{aligned} e^{a_1A_1}f(x,y,z) &= f(x, e^{a_1}y, z),\\ e^{a_2A_2}f(x,y,z) &= f(x, y, e^{a_2}z),\\ e^{a_3A_3}f(x,y,z) &= f\Big(x + a_3\frac{(x-b)y}{z}, y, z + a_3y\Big),\\ e^{a_4A_4}f(x,y,z) &= f\Big(x + a_4\frac{(x-a)y}{z}, y + a_4z, z\Big). \end{aligned} \tag{2.8}$$

From he above we get

$$\begin{aligned} &e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}f(x,y,z)\\ &\quad = f\Big(\Big(x + a_4\frac{(x-a)z}{y}\Big)\Big[1 + a_3\Big(a_4 + \frac{y}{z}\Big)\Big] - a_3b\Big(a_4 + \frac{y}{z}\Big),\\ &\qquad e^{a_1}y\Big(1 + \frac{a_4}{y}z\Big), e^{a_2}z\Big\{1 + a_3\Big(\frac{y}{z} + a_4\Big)\Big\}\Big). \end{aligned} \tag{2.9}$$

3. Generating functions

Now it follows from (2.1) that $u_1(x,y,z) = F_n(\alpha,\beta;x)y^\alpha z^\beta$ is a solution of the system

$$\begin{aligned} Lu &= 0,\\ (A_1 - \alpha)u &= 0;\\ Lu &= 0,\\ (A_2 - \beta)u &= 0;\\ Lu &= 0,\\ (A_1 + A_2 - \alpha - \beta)u &= 0. \end{aligned}$$

From (2.7) one can easily verify that

$$S\big(L\big(F_n(\alpha,\beta;x)y^\alpha z^\beta\big)\big) = L\big(S\big(F_n(\alpha,\beta;x)y^\alpha z^\beta\big)\big) = 0,$$

where

$$S = \mathrm{e}^{a_4A_4}\mathrm{e}^{a_3A_3}\mathrm{e}^{a_1A_2}\mathrm{e}^{a_1A_1}.$$

Therefore $S\big(F_n(\alpha,\beta;x)y^\alpha z^\beta\big)$ is annihilated by L.

Putting $a_1 = a_2 = 0$ and replacing $f(x,y,z)$ by $F_n(\alpha,\beta;x)y^\alpha z^\beta$ in (2.9), we get

$$\tag{3.1} \begin{aligned} &\mathrm{e}^{a_4A_4}\mathrm{e}^{e_3A_3}[F_n(\alpha,\beta;x)y^\alpha z^\beta] \\ &= F_n\Big(\alpha,\beta;\Big\{x + a_4\frac{(x-a)z}{y}\Big\}\Big\{1 + a_3\Big(a_4 + \frac{y}{z}\Big)\Big\} - a_3b\Big(a_4 + \frac{y}{z}\Big)\Big) \\ &\quad \times y^\alpha\Big(1 + \frac{a_4}{y}z\Big)^\alpha \times z^\beta\Big\{1 + a_3\Big(a_4 + \frac{y}{z}\Big)\Big\}^\beta. \end{aligned}$$

However,

$$\tag{3.2} \begin{aligned} &\mathrm{e}^{a_4A_4}\mathrm{e}^{e_3A_3}[F_n(\alpha,\beta;x)y^\alpha z^\beta] \\ &= \sum_{p=0}^{\infty}\frac{(-a_4)^p}{p!}(-n-\alpha-k)_p\sum_{k=0}^{\infty}\frac{(-a_3)^k}{k!}(-\beta-n)_k \\ &\quad \times F_n(\alpha+k-p,\beta-k+p;x)y^{\alpha+k-p}z^{\beta-k+p}. \end{aligned}$$

Equating (3.1) and (3.2), we get

$$\tag{3.3} \begin{aligned} &\Big(1 + a_4\frac{y}{z}\Big)^\alpha\Big\{1 + a_3\Big(a_4 + \frac{y}{z}\Big)\Big\}^\beta F_n\Big(\alpha,\beta;\Big\{x + a_4\frac{(x-a)z}{y}\Big\} \\ &\quad \times \Big\{1 + a_3\Big(a_4 + \frac{y}{z}\Big)\Big\} - a_3b\Big(a_4 + \frac{y}{z}\Big)\Big) \\ &= \sum_{p=0}^{\infty}\frac{(-a_4)^p}{p!}(-n-\alpha-k)_p\sum_{k=0}^{\infty}\frac{(-a_3)^k}{k!}(-\beta-n)_k \\ &\quad \times F_n(\alpha+k-p,\beta-k+p;x)y^{k-p}z^{-k+p}. \end{aligned}$$

We now consider the following cases:

Case 1. Putting $a_3 = 1$, $a_4 = 0$ and replacing $\frac{a_3y}{z}$ by $-t$, we get

$$\tag{3.4} (1-t)^\beta F_n\big(\alpha,\beta;x-(x-b)t\big) = \sum_{k=0}^{\infty}\frac{(-\beta-n)_k}{k!}F_n(\alpha+k,\beta-k;x)t^k.$$

Case 2. Putting $a_3 = 0$, $a_4 = 1$ and substituting $\frac{a_4 z}{y} = t$, we get

$$(3.5) \quad (1+t)^{\alpha} F_n\big(\alpha, \beta; x + (x-a)t\big) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} (-n-\alpha)_p F_n(\alpha - p, \beta + p; x) t^p.$$

Case 3. Putting $a_3 = \frac{1}{w}$, $a_4 = 1$ and $\frac{z}{y} = t$, we get

$$\begin{aligned}(3.6) \quad & (1+t)^{\alpha}\Big(1 + \frac{1}{w}\Big(1 + \frac{1}{t}\Big)\Big)^{\beta} \\ & \times F_n\Big(\alpha, \beta; \{x + (x-a)t\}\Big\{1 + \frac{1}{w}\Big(1+\frac{1}{t}\Big)\Big\} - \frac{b}{w}\Big(1 + \frac{1}{t}\Big)\Big) \\ & = \sum_{p=0}^{\infty} \frac{(-t)^p}{p!} (-n-\alpha-k)_p \sum_{k=0}^{\infty} \frac{(-\frac{1}{wt})^k}{k!} (-\beta - n)_k \\ & \times F_n(\alpha + k - p, \beta - k + p; x).\end{aligned}$$

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