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Research Article

A Note on Some Identities of Frobenius-Euler Numbers and Polynomials

J. Choi,[1] D. S. Kim,[2] T. Kim,[3] and Y. H. Kim[1]

[1] *Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[2] *Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*
[3] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*

Correspondence should be addressed to D. S. Kim, dskim@sogang.ac.kr
and T. Kim, tkkim@kw.ac.kr

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The purpose of this paper is to give some identities on the Frobenius-Euler numbers and polynomials by using the fermionic p-adic q-integral equation on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic absolute value on $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. Assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

Let f be a continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic q-integral on $\mathbb{Z}_p$ for f is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (\text{see [1]}). \tag{1.1}$$

From (1.1), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + (1+q) \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) q^l, \tag{1.2}$$

where $n \in \mathbb{N}$ and $f_n(x) = f(x+n)$ (see [1]). The ordinary Euler polynomials $E_n(x)$ are defined by

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.3}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1–10]). In the special case, $x = 0$, $E_n(0) = E_n$ is called the nth Euler number.

As the extension of (1.3), the Frobenius-Euler polynomials are defined by

$$\frac{1-q}{e^t-q}e^{xt} = \sum_{n=0}^{\infty} H_n(q,x)\frac{t^n}{n!}, \quad \text{(see [2])}. \tag{1.4}$$

In the special case, $x = 0$, $H_n(q,0) = H_n(q)$ is called the nth Frobenius-Euler number.

By (1.3) and (1.4), we easily get $H_n(-1,x) = E_n(x)$.

From (1.4), we note that

$$H_n(q,x) = \sum_{l=0}^{n} \binom{n}{l} H_l(q)x^{n-l} = (H(q)+x)^n, \quad \text{(see [2])}, \tag{1.5}$$

with the usual convention about replacing $H(q)^n$ by $H_n(q)$.

In this paper, we consider the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the Frobenius-Euler numbers and polynomials. From these p-adic q-integrals on $\mathbb{Z}_p$, we derive some new and interesting identities on the Frobenius-Euler numbers and polynomials.

2. Identities on the Frobenius-Euler Numbers

From (1.2) and (1.4), we can derive the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{1+q^{-1}}{e^t+q^{-1}}e^{xt} = \sum_{n=0}^{\infty} H_n\left(-q^{-1},x\right)\frac{t^n}{n!}. \tag{2.1}$$

Thus, by (2.1), we get Witt's formula for $H_n(-q^{-1},x)$ as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = H_n\left(-q^{-1},x\right), \quad n \in \mathbb{Z}_+. \tag{2.2}$$

By (1.5) and (2.1), we get

$$\left(H\left(-q^{-1}\right)+1\right)^n + q^{-1}H_n\left(-q^{-1}\right) = \begin{cases} 1+q^{-1}, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{2.3}$$

with the usual convention about replacing $H(-q^{-1})^n$ by $H_n(-q^{-1})$.

From (1.5) and (2.3), we note that

$$H_0\left(-q^{-1}\right)=1, \qquad H_n\left(-q^{-1},1\right)+q^{-1}H_n\left(-q^{-1}\right)=0, \quad \text{if } n>0. \tag{2.4}$$

By (2.1) and (2.2), we get

$$\int_{\mathbb{Z}_p}(y+1-x)^n d\mu_{-q}(y)=(-1)^n\int_{\mathbb{Z}_p}(y+x)^n d\mu_{-q^{-1}}(y). \tag{2.5}$$

Therefore, by (2.5), we obtain the following lemma.

Lemma 2.1. *For $n\in\mathbb{Z}_+$, one has*

$$H_n\left(-q^{-1},1-x\right)=(-1)^n H_n(-q,x). \tag{2.6}$$

From (2.3), we can derive the following:

$$\begin{aligned}
q^2H_n\left(-q^{-1},2\right)-q^2-q&=q^2\sum_{l=0}^{n}\binom{n}{l}\left(H\left(-q^{-1}\right)+1\right)^l-q^2-q\\
&=q\sum_{l=1}^{n}\binom{n}{l}q\left(H\left(-q^{-1}\right)+1\right)^l-q\\
&=-q\sum_{l=0}^{n}\binom{n}{l}H_l\left(-q^{-1}\right)\\
&=-(1+q)\delta_{0,n}+H_n\left(-q^{-1}\right),
\end{aligned} \tag{2.7}$$

where $\delta_{k,n}$ is the Kronecker symbol.

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For $n\in\mathbb{Z}_+$, one has*

$$H_n\left(-q^{-1},2\right)=1+q^{-1}-q^{-2}(1+q)\delta_{0,n}+q^{-2}H_n\left(-q^{-1}\right). \tag{2.8}$$

First we consider the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the nth Frobenius-Euler polynomials as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} H_n\left(-q^{-1}, x\right) d\mu_{-q}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} H_l\left(-q^{-1}\right) \int_{\mathbb{Z}_p} x^{n-l} d\mu_{-q}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} H_l\left(-q^{-1}\right) H_{n-l}\left(-q^{-1}\right), \quad \text{where } n \in \mathbb{Z}_+. \end{aligned} \tag{2.9}$$

On the other hand, by (2.5) and Lemma 2.1, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} H_n\left(-q^{-1}, x\right) d\mu_{-q}(x) = (-1)^n \int_{\mathbb{Z}_p} H_n(-q, 1-x) d\mu_{-q}(x) \\ &= (-1)^n \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(-q) \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-q}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) \int_{\mathbb{Z}_p} (x-1)^l d\mu_{-q}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) H_l\left(-q^{-1}, -1\right). \end{aligned} \tag{2.10}$$

From Lemma 2.1, Theorem 2.2, and (2.10), we note that

$$\begin{aligned} I_1 &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) H_l\left(-q^{-1}, -1\right) \\ &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n} H_{n-l}(-q) H_l(-q, 2) \\ &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n} H_{n-l}(-q) \left\{1 + q - q^2\left(1 + q^{-1}\right)\delta_{0,l} + q^2 H_l(-q)\right\} \\ &= (-1)^n (1+q)((1+q)\delta_{0,n} - qH_n(-q)) - H_n(-q)\left(q + q^2\right)(-1)^n \\ &\quad + (-1)^n q^2 \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(-q) H_l(-q). \end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}H_l\left(-q^{-1}\right)H_{n-l}\left(-q^{-1}\right) = (-1)^n(1+q)\left((1+q)\delta_{0,n} - 2qH_n(-q)\right) + (-1)^n q^2 \sum_{l=0}^{n}\binom{n}{l}H_{n-l}(-q)H_l(-q). \tag{2.12}$$

In particular, for $n \in \mathbb{N}$, one has

$$\sum_{l=0}^{n}\binom{n}{l}H_l\left(-q^{-1}\right)H_{n-l}\left(-q^{-1}\right) = 2(-1)^{n+1}q(1+q)H_n(-q) + (-1)^n q^2 \sum_{l=0}^{n}\binom{n}{l}H_{n-l}(-q)H_l(-q). \tag{2.13}$$

Let us consider the following fermionic p-adic q-integral on $\mathbb{Z}_p$ for the product of Bernoulli and Frobenius-Euler polynomials as follows:

$$\begin{aligned} I_2 &= \int_{\mathbb{Z}_p} B_m(x)H_n\left(-q^{-1},x\right)d\mu_{-q}(x) \\ &= \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}\left(-q^{-1}\right)\int_{\mathbb{Z}_p} x^{k+l}d\mu_{-q}(x) \\ &= \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}\left(-q^{-1}\right)H_{k+l}\left(-q^{-1}\right). \end{aligned} \tag{2.14}$$

It is known that $B_n(x) = (-1)^n B_n(1-x)$.

On the other hand, by Lemma 2.1, we get

$$\begin{aligned} I_2 &= (-1)^{m+n}\int_{\mathbb{Z}_p} B_m(1-x)H_n(-q,1-x)d\mu_{-q}(x) \\ &= (-1)^{m+n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}(-q)\int_{\mathbb{Z}_p}(1-x)^{k+l}d\mu_{-q}(x) \\ &= (-1)^{m+n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}(-q)\left\{(1+q) - q^2\left(1+q^{-1}\right)\delta_{0,k+l} + q^2H_{k+l}(-q)\right\} \\ &= (-1)^{m+n}(1+q)B_m(1)H_n(-q,1) - \left(q^2+q\right)(-1)^{m+n}B_mH_n(-q) \\ &\quad + q^2(-1)^{m+n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}(-q)H_{k+l}(-q). \end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $m, n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
&\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}\left(-q^{-1}\right)H_{k+l}\left(-q^{-1}\right) \\
&\quad= (-1)^{m+n}(1+q)B_m(1)H_n(-q,1) - \left(q^2+q\right)(-1)^{m+n}B_mH_n(-q) \\
&\qquad+ q^2(-1)^{m+n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}(-q)H_{k+l}(-q).
\end{aligned} \tag{2.16}$$

In particular, for $m \in \mathbb{N} - \{1\}$, $n \in \mathbb{N}$, one has

$$\begin{aligned}
&\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}\left(-q^{-1}\right)H_{k+l}\left(-q^{-1}\right) \\
&\quad= 2(-1)^{m+n+1}\left(q^2+q\right)B_mH_n(-q) \\
&\qquad+ q^2(-1)^{m+n}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}H_{n-l}(-q)H_{k+l}(-q).
\end{aligned} \tag{2.17}$$

It is known that $x^n = (1/(n+1))\sum_{l=0}^{n}\binom{n+1}{l}B_l(x)$. Let us consider the following fermionic p-adic q-integral on $\mathbb{Z}_p$:

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\int_{\mathbb{Z}_p} B_l(x)d\mu_{-q}(x). \\
&= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k}\int_{\mathbb{Z}_p} x^k d\mu_{-q}(x) \\
&= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k}H_k\left(-q^{-1}\right).
\end{aligned} \tag{2.18}$$

Therefore by (2.18), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, one has*

$$H_n\left(-q^{-1}\right) = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k}H_k\left(-q^{-1}\right). \tag{2.19}$$

From (1.3), we can derive the following:

$$x^n = E_n(x) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(x). \tag{2.20}$$

Let us take the fermionic p-adic q-integral on $\mathbb{Z}_p$ in (2.20) as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} E_n(x)d\mu_{-q}(x) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\int_{\mathbb{Z}_p} E_l(x)d\mu_{-q}(x)\\ &= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}H_l\left(-q^{-1}\right) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}H_k\left(-q^{-1}\right).\end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}$, one has*

$$H_n\left(-q^{-1}\right) = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}H_l\left(-q^{-1}\right) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}H_k\left(-q^{-1}\right). \tag{2.22}$$

By Theorems 2.5 and 2.6, we obtain the following corollary.

Corollary 2.7. *For $n \in \mathbb{N}$, one has*

$$\begin{aligned}&\frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k}H_k\left(-q^{-1}\right)\\ &\quad= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}H_l\left(-q^{-1}\right) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}H_k\left(-q^{-1}\right).\end{aligned} \tag{2.23}$$

By (1.3), we easily get $E_n(x) = (-1)^n E_n(1-x)$.

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) \\
&= (-1)^n \int_{\mathbb{Z}_p} E_n(1-x)d\mu_{-q}(x) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l \int_{\mathbb{Z}_p} E_l(1-x)d\mu_{-q}(x) \\
&= (-1)^n \sum_{l=0}^{n}\binom{n}{l}E_{n-l}\int_{\mathbb{Z}_p}(1-x)^l d\mu_{-q}(x) \\
&\quad + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}E_{l-k}\int_{\mathbb{Z}_p}(1-x)^k d\mu_{-q}(x) \\
&= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}(-1)^{n-l}H_l\left(-q^{-1},-1\right) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}(-1)^{l-k}H_k\left(-q^{-1},-1\right) \\
&= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}(-1)^{n}H_l(-q,2) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}(-1)^{l}H_k(-q,2).
\end{aligned} \tag{2.24}$$

Therefore, by (2.24), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$, one has*

$$\begin{aligned}
H_n\left(-q^{-1}\right) &= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}(-1)^{n}H_l(-q,2) \\
&\quad + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}(-1)^{l}H_k(-q,2).
\end{aligned} \tag{2.25}$$

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