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Research Article

A Note on Some Identities of Frobenius-Euler Numbers and Polynomials

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The purpose of this paper is to give some identities on the Frobenius-Euler numbers and polynomials by using the fermionic p-adic q-integral equation on \mathbb{Z}_p .

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of \mathbb{Q}_p , respectively. Let \mathbb{N} be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic absolute value on \mathbb{C}_p is normalized so that $|p|_p = 1/p$. Assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

Let f be a continuous function on \mathbb{Z}_p . Then the fermionic p-adic q-integral on \mathbb{Z}_p for f is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \quad (\text{see } [1]).$$
 (1.1)

From (1.1), we note that

$$q^{n}I_{-q}(f_{n}) = (-1)^{n}I_{-q}(f) + (1+q)\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l)q^{l},$$
(1.2)

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where $n \in \mathbb{N}$ and $f_n(x) = f(x+n)$ (see [1]). The ordinary Euler polynomials $E_n(x)$ are defined by

$$\frac{2}{e^t + 1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$
(1.3)

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1–10]). In the special case, x = 0, $E_n(0) = E_n$ is called the nth Euler number.

As the extension of (1.3), the Frobenius-Euler polynomials are defined by

$$\frac{1-q}{e^t-q}e^{xt} = \sum_{n=0}^{\infty} H_n(q,x)\frac{t^n}{n!}, \quad \text{(see [2])}.$$
 (1.4)

In the special case, x = 0, $H_n(q, 0) = H_n(q)$ is called the nth Frobenius-Euler number.

By (1.3) and (1.4), we easily get $H_n(-1, x) = E_n(x)$.

From (1.4), we note that

$$H_n(q,x) = \sum_{l=0}^{n} {n \choose l} H_l(q) x^{n-l} = (H(q) + x)^n, \text{ (see [2])},$$
 (1.5)

with the usual convention about replacing $H(q)^n$ by $H_n(q)$.

In this paper, we consider the fermionic p-adic q-integral on \mathbb{Z}_p for the Frobenius-Euler numbers and polynomials. From these p-adic q-integrals on \mathbb{Z}_p , we derive some new and interesting identities on the Frobenius-Euler numbers and polynomials.

2. Identities on the Frobenius-Euler Numbers

From (1.2) and (1.4), we can derive the following:

$$\int_{\mathbb{Z}_n} e^{(x+y)t} d\mu_{-q}(y) = \frac{1+q^{-1}}{e^t+q^{-1}} e^{xt} = \sum_{n=0}^{\infty} H_n(-q^{-1}, x) \frac{t^n}{n!}.$$
 (2.1)

Thus, by (2.1), we get Witt's formula for $H_n(-q^{-1}, x)$ as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = H_n\left(-q^{-1}, x\right), \quad n \in \mathbb{Z}_+.$$
(2.2)

By (1.5) and (2.1), we get

$$\left(H\left(-q^{-1}\right)+1\right)^{n}+q^{-1}H_{n}\left(-q^{-1}\right)=\begin{cases}1+q^{-1}, & \text{if } n=0,\\0, & \text{if } n>0,\end{cases}$$
(2.3)

with the usual convention about replacing $H(-q^{-1})^n$ by $H_n(-q^{-1})$.

From (1.5) and (2.3), we note that

$$H_0(-q^{-1}) = 1$$
, $H_n(-q^{-1}, 1) + q^{-1}H_n(-q^{-1}) = 0$, if $n > 0$. (2.4)

By (2.1) and (2.2), we get

$$\int_{\mathbb{Z}_{p}} (y+1-x)^{n} d\mu_{-q}(y) = (-1)^{n} \int_{\mathbb{Z}_{p}} (y+x)^{n} d\mu_{-q^{-1}}(y).$$
 (2.5)

Therefore, by (2.5), we obtain the following lemma.

Lemma 2.1. *For* $n \in \mathbb{Z}_+$ *, one has*

$$H_n(-q^{-1}, 1-x) = (-1)^n H_n(-q, x).$$
 (2.6)

From (2.3), we can derive the following:

$$q^{2}H_{n}(-q^{-1},2) - q^{2} - q = q^{2} \sum_{l=0}^{n} \binom{n}{l} \left(H(-q^{-1}) + 1\right)^{l} - q^{2} - q$$

$$= q \sum_{l=1}^{n} \binom{n}{l} q \left(H(-q^{-1}) + 1\right)^{l} - q$$

$$= -q \sum_{l=0}^{n} \binom{n}{l} H_{l}(-q^{-1})$$

$$= -(1+q)\delta_{0,n} + H_{n}(-q^{-1}),$$

$$(2.7)$$

where $\delta_{k,n}$ is the Kronecker symbol.

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For* $n \in \mathbb{Z}_+$ *, one has*

$$H_n(-q^{-1},2) = 1 + q^{-1} - q^{-2}(1+q)\delta_{0,n} + q^{-2}H_n(-q^{-1}).$$
(2.8)

First we consider the fermionic *p*-adic *q*-integral on \mathbb{Z}_p for the *n*th Frobenius-Euler polynomials as follows:

$$I_{1} = \int_{\mathbb{Z}_{p}} H_{n}(-q^{-1}, x) d\mu_{-q}(x)$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_{l}(-q^{-1}) \int_{\mathbb{Z}_{p}} x^{n-l} d\mu_{-q}(x)$$

$$= \sum_{l=0}^{n} \binom{n}{l} H_{l}(-q^{-1}) H_{n-l}(-q^{-1}), \text{ where } n \in \mathbb{Z}_{+}.$$
(2.9)

On the other hand, by (2.5) and Lemma 2.1, we get

$$I_{1} = \int_{\mathbb{Z}_{p}} H_{n}(-q^{-1}, x) d\mu_{-q}(x) = (-1)^{n} \int_{\mathbb{Z}_{p}} H_{n}(-q, 1 - x) d\mu_{-q}(x)$$

$$= (-1)^{n} \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(-q) \int_{\mathbb{Z}_{p}} (1 - x)^{l} d\mu_{-q}(x)$$

$$= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) \int_{\mathbb{Z}_{p}} (x - 1)^{l} d\mu_{-q}(x)$$

$$= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} H_{n-l}(-q) H_{l}(-q^{-1}, -1).$$
(2.10)

From Lemma 2.1, Theorem 2.2, and (2.10), we note that

$$I_{1} = \sum_{l=0}^{n} {n \choose l} (-1)^{n-l} H_{n-l}(-q) H_{l}(-q^{-1}, -1)$$

$$= \sum_{l=0}^{n} {n \choose l} (-1)^{n} H_{n-l}(-q) H_{l}(-q, 2)$$

$$= \sum_{l=0}^{n} {n \choose l} (-1)^{n} H_{n-l}(-q) \left\{ 1 + q - q^{2} \left(1 + q^{-1} \right) \delta_{0,l} + q^{2} H_{l}(-q) \right\}$$

$$= (-1)^{n} (1+q) ((1+q)\delta_{0,n} - qH_{n}(-q)) - H_{n}(-q) (q+q^{2}) (-1)^{n}$$

$$+ (-1)^{n} q^{2} \sum_{l=0}^{n} {n \choose l} H_{n-l}(-q) H_{l}(-q).$$

$$(2.11)$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.3. *For* $n \in \mathbb{Z}_+$ *, one has*

$$\sum_{l=0}^{n} {n \choose l} H_l(-q^{-1}) H_{n-l}(-q^{-1}) = (-1)^n (1+q) ((1+q)\delta_{0,n} - 2qH_n(-q))$$

$$+ (-1)^n q^2 \sum_{l=0}^{n} {n \choose l} H_{n-l}(-q) H_l(-q).$$
(2.12)

In particular, for $n \in \mathbb{N}$ *, one has*

$$\sum_{l=0}^{n} {n \choose l} H_l(-q^{-1}) H_{n-l}(-q^{-1}) = 2(-1)^{n+1} q (1+q) H_n(-q)$$

$$+ (-1)^n q^2 \sum_{l=0}^{n} {n \choose l} H_{n-l}(-q) H_l(-q).$$
(2.13)

Let us consider the following fermionic p-adic q-integral on \mathbb{Z}_p for the product of Bernoulli and Frobenius-Euler polynomials as follows:

$$I_{2} = \int_{\mathbb{Z}_{p}} B_{m}(x) H_{n}(-q^{-1}, x) d\mu_{-q}(x)$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} {m \choose k} {n \choose l} B_{m-k} H_{n-l}(-q^{-1}) \int_{\mathbb{Z}_{p}} x^{k+l} d\mu_{-q}(x)$$

$$= \sum_{k=0}^{m} \sum_{l=0}^{n} {m \choose k} {n \choose l} B_{m-k} H_{n-l}(-q^{-1}) H_{k+l}(-q^{-1}).$$
(2.14)

It is known that $B_n(x) = (-1)^n B_n(1 - x)$.

On the other hand, by Lemma 2.1, we get

$$I_{2} = (-1)^{m+n} \int_{\mathbb{Z}_{p}} B_{m}(1-x) H_{n}(-q, 1-x) d\mu_{-q}(x)$$

$$= (-1)^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) \int_{\mathbb{Z}_{p}} (1-x)^{k+l} d\mu_{-q}(x)$$

$$= (-1)^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) \left\{ (1+q) - q^{2} \left(1+q^{-1} \right) \delta_{0,k+l} + q^{2} H_{k+l}(-q) \right\}$$

$$= (-1)^{m+n} (1+q) B_{m}(1) H_{n}(-q, 1) - \left(q^{2} + q \right) (-1)^{m+n} B_{m} H_{n}(-q)$$

$$+ q^{2} (-1)^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l}(-q) H_{k+l}(-q).$$

$$(2.15)$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For* m, $n \in \mathbb{Z}_+$, *one has*

$$\sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l} (-q^{-1}) H_{k+l} (-q^{-1})$$

$$= (-1)^{m+n} (1+q) B_m (1) H_n (-q,1) - (q^2+q) (-1)^{m+n} B_m H_n (-q)$$

$$+ q^2 (-1)^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l} (-q) H_{k+l} (-q).$$
(2.16)

In particular, for $m \in \mathbb{N} - \{1\}$, $n \in \mathbb{N}$, one has

$$\sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l} (-q^{-1}) H_{k+l} (-q^{-1})$$

$$= 2(-1)^{m+n+1} (q^2 + q) B_m H_n (-q)$$

$$+ q^2 (-1)^{m+n} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} H_{n-l} (-q) H_{k+l} (-q).$$
(2.17)

It is known that $x^n = (1/(n+1)) \sum_{l=0}^n \binom{n+1}{l} B_l(x)$. Let us consider the following fermionic p-adic q-integral on \mathbb{Z}_p :

$$\int_{\mathbb{Z}_{p}} x^{n} d\mu_{-q}(x) = \frac{1}{n+1} \sum_{l=0}^{n} {n+1 \choose l} \int_{\mathbb{Z}_{p}} B_{l}(x) d\mu_{-q}(x).$$

$$= \frac{1}{n+1} \sum_{l=0}^{n} {n+1 \choose l} \sum_{k=0}^{l} {l \choose k} B_{l-k} \int_{\mathbb{Z}_{p}} x^{k} d\mu_{-q}(x)$$

$$= \frac{1}{n+1} \sum_{l=0}^{n} {n+1 \choose l} \sum_{k=0}^{l} {l \choose k} B_{l-k} H_{k} \left(-q^{-1}\right).$$
(2.18)

Therefore by (2.18), we obtain the following theorem.

Theorem 2.5. *For* $n \in \mathbb{Z}_+$ *, one has*

$$H_n(-q^{-1}) = \frac{1}{n+1} \sum_{l=0}^{n} {n+1 \choose l} \sum_{k=0}^{l} {l \choose k} B_{l-k} H_k(-q^{-1}).$$
 (2.19)

From (1.3), we can derive the following:

$$x^{n} = E_{n}(x) + \frac{1}{2} \sum_{l=0}^{n-1} {n \choose l} E_{l}(x).$$
 (2.20)

Let us take the fermionic *p*-adic *q*-integral on \mathbb{Z}_p in (2.20) as follows:

$$\int_{\mathbb{Z}_{p}} x^{n} d\mu_{-q}(x) = \int_{\mathbb{Z}_{p}} E_{n}(x) d\mu_{-q}(x) + \frac{1}{2} \sum_{l=0}^{n-1} {n \choose l} \int_{\mathbb{Z}_{p}} E_{l}(x) d\mu_{-q}(x)
= \sum_{l=0}^{n} {n \choose l} E_{n-l} H_{l}(-q^{-1}) + \frac{1}{2} \sum_{l=0}^{n-1} {n \choose l} \sum_{k=0}^{l} {l \choose k} E_{l-k} H_{k}(-q^{-1}).$$
(2.21)

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.6. *For* $n \in \mathbb{N}$ *, one has*

$$H_n\left(-q^{-1}\right) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} H_l\left(-q^{-1}\right) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^{l} \binom{l}{k} E_{l-k} H_k\left(-q^{-1}\right). \tag{2.22}$$

By Theorems 2.5 and 2.6, we obtain the following corollary.

Corollary 2.7. *For* $n \in \mathbb{N}$ *, one has*

$$\frac{1}{n+1} \sum_{l=0}^{n} {n+1 \choose l} \sum_{k=0}^{l} {l \choose k} B_{l-k} H_k \left(-q^{-1}\right)
= \sum_{l=0}^{n} {n \choose l} E_{n-l} H_l \left(-q^{-1}\right) + \frac{1}{2} \sum_{l=0}^{n-1} {n \choose l} \sum_{k=0}^{l} {l \choose k} E_{l-k} H_k \left(-q^{-1}\right).$$
(2.23)

By (1.3), we easily get $E_n(x) = (-1)^n E_n(1-x)$.

Thus, we have

$$\int_{\mathbb{Z}_{p}} x^{n} d\mu_{-q}(x)
= (-1)^{n} \int_{\mathbb{Z}_{p}} E_{n}(1-x) d\mu_{-q}(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{l} \int_{\mathbb{Z}_{p}} E_{l}(1-x) d\mu_{-q}(x)
= (-1)^{n} \sum_{l=0}^{n} \binom{n}{l} E_{n-l} \int_{\mathbb{Z}_{p}} (1-x)^{l} d\mu_{-q}(x)
+ \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{l} \sum_{k=0}^{l} \binom{l}{k} E_{l-k} \int_{\mathbb{Z}_{p}} (1-x)^{k} d\mu_{-q}(x)
= \sum_{l=0}^{n} \binom{n}{l} E_{n-l} (-1)^{n-l} H_{l} \left(-q^{-1}, -1\right) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^{l} \binom{l}{k} E_{l-k} (-1)^{l-k} H_{k} \left(-q^{-1}, -1\right)
= \sum_{l=0}^{n} \binom{n}{l} E_{n-l} (-1)^{n} H_{l} \left(-q, 2\right) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} \sum_{k=0}^{l} \binom{l}{k} E_{l-k} (-1)^{l} H_{k} \left(-q, 2\right).$$
(2.24)

Therefore, by (2.24), we obtain the following theorem.

Theorem 2.8. *For* $n \in \mathbb{N}$ *, one has*

$$H_{n}(-q^{-1}) = \sum_{l=0}^{n} {n \choose l} E_{n-l}(-1)^{n} H_{l}(-q,2)$$

$$+ \frac{1}{2} \sum_{l=0}^{n-1} {n \choose l} \sum_{k=0}^{l} {l \choose k} E_{l-k}(-1)^{l} H_{k}(-q,2).$$
(2.25)

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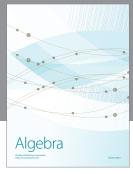
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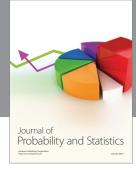
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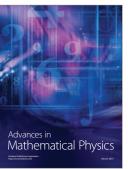




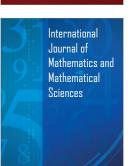


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