# Structural properties of Riordan matrices and extending the matrices ${ }^{\text {s }}$ 

Gi-Sang Cheon *, Sung-Tae Jin<br>Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea

## ARTICLEINFO

## Article history:

Received 8 December 2010
Accepted 24 March 2011
Available online 30 April 2011
Submitted by R.A. Brualdi
AMS classification:
Primary: 05A15
Secondary: 05C38

## Keywords:

Riordan matrix
$A$-sequence
( $a, b$ )-Sequence
Extended Riordan matrix
Companion sequence


#### Abstract

We consider an infinite lower triangular matrix $L=\left[\ell_{n, k}\right]_{n, k \in \mathbf{N}_{\mathbf{0}}}$ and a sequence $\Omega=\left(\omega_{n}\right)_{n \in \mathbf{N}_{0}}$ called the ( $a, b$ )-sequence such that every element $\ell_{n+1, k+1}$ except lying in column 0 can be expressed as


$$
\ell_{n+1, k+1}=\sum_{i=0}^{\lfloor(n-k) / m\rfloor} \omega_{i} \ell_{n-a i, k+b i}, \quad \omega_{0} \neq 0
$$

where $a$ and $b$ are integers with $a+b=m>0$ and $b \geq 0$. This concept generalizes the $A$-sequence of a Riordan matrix. As a result, we explore several structural properties of Riordan matrices by means of $(a, b)$-sequences. In particular, if $b<0$ then this leads to an extended Riordan matrix which is a bilaterally infinite matrix.
© 2011 Elsevier Inc. All rights reserved.

## 1. Introduction

Let $\mathcal{F}=\mathbf{C}[[z]]$ be the ring of formal power series (f.p.s) over the complex field $\mathbf{C}$ and let $\mathbf{N}_{\mathbf{0}}=$ $\{0,1,2, \ldots\}$. For $f=\sum_{n \geq 0} f_{n} z^{n} \in \mathbf{C}[[z]]$, the order of $f$ is the smallest integer $n$ for which $f_{n} \neq 0$ and the set of all f.p.s of order $r$ is denoted by $\mathcal{F}_{r}$. In particular, $\mathcal{F}_{0}$ and $\mathcal{F}_{1}$ are sets of reciprocal and compositional invertible f.p.s, respectively.

Any ordered pair $(g, f) \in \mathcal{F}_{0} \times \mathcal{F}_{1}$ defines an infinite lower triangular matrix $L=\left[\ell_{n, k}\right]_{n, k \in \mathrm{~N}_{0}}$ with nonzero entries on the main diagonal in which $\ell_{n, k}=\left[z^{n}\right] g f^{k}$, where $\left[z^{n}\right]$ is the coefficient operator.

[^0]The matrix $L$ is called the Riordan matrix ${ }^{1}[12,15]$ and denoted by $L=(g(z), f(z))$ or simply $(g, f)$. If we multiply $(g, f)$ by a column vector $\left(c_{0}, c_{1}, \ldots\right)^{T}$ with the generating function (GF) $\Phi$ then the resulting column vector has the $\mathrm{GF} g \Phi(f)$. We call this property the fundamental theorem of a Riordan matrix. One may express the theorem as $(g, f) \Phi=g \Phi(f)$. This also leads to the multiplication of Riordan matrices which can be described in terms of GFs as $(g, f)(h, \ell)=(g h(f), \ell(f))$.

The set of all Riordan matrices, under the above multiplication, forms a group denoted by $\mathcal{R}$ and called the Riordan group [12]. It is easy to show that the identity is $(1, z)$ and the inverse of $(g, f)$ is $(1 / g(\bar{f}), \bar{f})$ where $f$ is the compositional inverse, i.e., $f(\bar{f})=\bar{f}(f)=z$. Particular subgroups of the Riordan group are introduced in [11]. The group structure is of considerable independent interest. One elementary use of the Riordan group is to prove and invert combinatorial identities as discussed by Shapiro et al. [12] and Sprugnoli [15].

A Riordan matrix $L=(g, f)=\left[\ell_{n, k}\right]_{n, k \in \mathbf{N}_{0}}$ is completely characterized [8] by two horizontal sequences, together with $\ell_{0,0} \neq 0$. The sequences are called the $A$-sequence $\left(a_{n}\right)_{n \in \mathbf{N}_{0}}$ and the $Z$ sequence $\left(z_{n}\right)_{n \in \mathbf{N}_{0}}$ such that
(i) $\ell_{n+1, k+1}=\sum_{j \geq 0} a_{j} \ell_{n, k+j}$,
(ii) $\ell_{n+1,0}=\sum_{j \geq 0} z_{j} \ell_{n, j}$.

The $Z$-sequence characterizes column 0 and the $A$-sequence characterizes all the other columns of a Riordan matrix. Since every lower triangular matrix has a unique $Z$-sequence, we can implicitly assume its existence in all the subsequent theorems.

Recently, He and Sprugnoli [6] studied Riordan matrices through their $A$ - and $Z$-sequence characterization. More generally, Merlini et al. [8] determined zones which the generic element $\ell_{n+1, k+1}$ is allowed to linearly depend on. The following theorem gives this characterization of Riordan matrices.

Theorem 1.1 ([8]). A lower triangular matrix $L=\left[\ell_{n, k}\right]_{n, k \in N_{0}}$ is Riordan if and only if there exists the A-matrix $\left[\alpha_{i, j}\right]_{i, j \in \mathbf{N}_{0}}$, with $\alpha_{0,0} \neq 0$, and $r$ sequences $\left(\rho_{j}^{[i]}\right)_{j \in \boldsymbol{N}_{0}}(i=1,2, \ldots, r)$ such that:

$$
\begin{equation*}
\ell_{n+1, k+1}=\sum_{i \geqslant 0} \sum_{j \geqslant 0} \alpha_{i, j} \ell_{n-i, k+j}+\sum_{i=1}^{r} \sum_{j \geqslant 0} \rho_{j}^{[i]} \ell_{n+i, k+i+j+1} . \tag{1}
\end{equation*}
$$

This theorem enables us to find new sequences in a Riordan matrix. This fact is our motivation of this paper. For instance, we note that every element $\ell_{n+1, k+1}$ of $L=(g, f)$ can be expressed as a linear combination:

$$
\ell_{n+1, k+1}=\sum_{i=0}^{n} \alpha_{i, 0} \ell_{n-i, k}, \quad \alpha_{i, 0}=f_{i}=\left[z^{i+1}\right] f
$$

which can be viewed as a weighted "hockey stick identity". The sequence $\left(f_{n}\right)_{n \in \mathbf{N}_{0}}$ is a vertical sequence called the $V$-sequence. It is also essentially the characterization of a Riordan matrix. We now observe that the $A$ - and the $V$-sequence may be regarded as sequences lying on the horizontal and the vertical lines starting from $\ell_{n, k}$, respectively.

In the present paper, more generally we are interested in a Riordan matrix $L=\left[\ell_{n, k}\right]_{n, k \in \mathbf{N}_{0}}$ and a sequence $\Omega=\left(\omega_{n}\right)_{n \in \mathrm{~N}_{0}}$ such that every element $\ell_{n+1, k+1}$ except lying in column 0 can be expressed as a linear combination with coefficients in $\Omega$ of the elements lying on the slanting diagonal obtained by moving $a$ units up or down and $b$ units to the right starting from $\ell_{n, k}$, i.e.,

$$
\begin{equation*}
\ell_{n+1, k+1}=\sum_{i=0}^{\left\lfloor\frac{n-k}{m}\right\rfloor} \omega_{i} \ell_{n-a i, k+b i}, \quad \omega_{0} \neq 0 \tag{2}
\end{equation*}
$$

[^1]

Fig. 1. Example of $(a, b)$-sequences.
where $a$ and $b$ are integers such that $a+b=m \geq 1$ and $b \geq 0$. The conditions on $a, b$ assure that the sums (2) are actually finite. This also happens when $a<0, b>0$ and $-a<b$, which possibly creates an alternative extension to the one described in [8].

In this context, $\Omega=\left(\omega_{n}\right)_{n \in \mathrm{~N}_{0}}$ is said to be the ( $a, b$ )-ray sequence, simply ( $a, b$ )-sequence of a Riordan matrix. We observe explicitly that an ( $a, b$ )-sequence is different from any ( $a p, b p$ )-sequence, when $p>1$. This is obvious, but stresses the fact that the concept does not simply depend on the slope of the ray sequence.

We also note that the $(0,1)$ - and the $(1,0)$-sequence exactly coincide with the $A$ - and the $V$ sequence, respectively. Thus this concept generalizes the $A$ - and the $V$-sequence of Riordan matrices. For example, see Fig. 1.

Even though every Riordan matrix has both the $(0,1)$ - and the $(1,0)$-sequence, it may not have the $(a, b)$-sequence for some $a$ and $b$. As we shall see, the Pascal matrix $(1 /(1-z), z /(1-z))$ does not have the ( 1,1 )-sequence.

From Theorem 1.1, it follows that every infinite lower triangular matrix with a $(a, b)$-sequence is a Riordan matrix. In particular, some ( $a, b$ )-sequences might be expressed in terms of elements of the $A$-matrix and $r$ sequences in Theorem 1.1 as
(i) $\Omega=\left(\alpha_{0,0}, \alpha_{a, b}, \alpha_{2 a, 2 b}, \ldots\right)$ if $a \geq 0$,
(ii) $\Omega=\left(\alpha_{0,0}, \rho_{m-1}^{[-a]}, \rho_{2 m-1}^{[-2 a]}, \ldots\right)$ where $a+b=m$ if $a<0$.

Throughout this paper, we are mainly interested in ray-sequences of a Riordan matrix in the Riordan group. Specifically, in Section 2 we obtain several structural properties of Riordan matrices by means of ray sequences. As an application of $(a, b)$-sequences, the $k$ th weighted $(a, b)$-diagonal sum with a weight sequence formed by moving $a$ unit up or down and $b$ unit to the right starting from $\ell_{k, 0}$ is discussed in Section 3. In Section 4, we will observe other ray sequence called a companion sequence obtained from the reflecting concept of $(a, b)$-sequences. This leads us extending the Riordan matrix which is a bilaterally infinite matrix. Finally, in Section 5 we will examine our results for the (extended) Riordan matrix with ray sequences of $r$-ary numbers (e.g., the Catalan numbers and the ternary numbers). It allows us to derive several combinatorial identities.

## 2. Ray sequences of a Riordan matrix

The concept of ray-sequences of a Riordan matrix plays a very important role in this approach. For a positive integer $m$, let us define the set

$$
\Sigma_{m}=\left\{z g\left(z^{m}\right) \in \mathcal{F}_{1} \mid g(z) \in \mathcal{F}_{0}\right\}
$$

The following theorem will be very useful in our study.

Theorem 2.1. Let $L=(g, f)=\left[\ell_{n, k}\right]_{n, k \in N_{0}}$ be a Riordan matrix and $\Omega(z)$ the GF for the sequence $\left(\omega_{n}\right)_{n \in N_{0}}$ with $\omega_{0} \neq 0$. Then the following are equivalent for any pair $(a, b) \in Z \times Z$ such that $a+b \geq 1$ and $b \geq 0$ :
(i) $L$ has the unique $(a, b)$-sequence $\left(\omega_{n}\right)_{n \in \boldsymbol{N}_{0}}$.
(ii) $f$ is the solution of the functional equation

$$
\begin{equation*}
f=z \Omega\left(z^{a} f^{b}\right) \tag{3}
\end{equation*}
$$

(iii) $f \in \Sigma_{m}$ where $m=a+b$.

Proof. (i) $\Rightarrow$ (ii): By the definition, if $L=(g, f)$ has a $(a, b)$-sequence $\left(\omega_{n}\right)_{\in \mathbf{N}_{0}}$ then we have

$$
g f^{k}=\sum_{n \geq 0} \omega_{n} z^{1+a n} g f^{k-1+b n}, \quad(k \geq 1)
$$

which implies that $f$ satisfies

$$
f=\sum_{n \geq 0} \omega_{n} z^{1+a n} f^{b n}=z \sum_{n \geq 0} \omega_{n}\left(z^{a} f^{b}\right)^{n}=z \Omega\left(z^{a} f^{b}\right) .
$$

(ii) $\Rightarrow$ (iii): Let $f=z \sum_{n \geqslant 0} f_{n} z^{n} \in \mathcal{F}_{1}$ and $a+b=m$. Since $z^{a} f^{b} \in \mathcal{F}_{m}$, it follows from (3) that $f_{k}=\left[z^{k}\right] z \varphi\left(z^{a} f^{b}\right)=0$ for $k$ such that $k \not \equiv 1(\bmod m)$. Hence $f \in \Sigma_{m}$.
(iii) $\Rightarrow$ (i): Let $f \in \Sigma_{m}$. We may assume that $f=\left\langle\widehat{f}\left(z^{m}\right)\right.$ where $\widehat{f}=\sum_{n \geqslant 0} f_{n} z^{n} \in \mathcal{F}_{0}$. For the Riordan matrix $\left(1, z f^{b}\right), b \in \mathbf{N}_{\mathbf{0}}$, let us consider the linear system in matrix form as

$$
\begin{equation*}
\left(1, z \widehat{f}^{b}\right)\left(\omega_{0}, \omega_{1}, \omega_{2}, \ldots\right)^{T}=\left(f_{0}, f_{1}, f_{2}, \ldots\right)^{T} \tag{4}
\end{equation*}
$$

With the GFs we can express (4) as ( $\left.1, z \widehat{f}^{b}\right) \Omega=\widehat{f}$. It follows $\left(1, z^{m} \widehat{f}^{b}\left(z^{m}\right)\right) \Omega=\widehat{f}\left(z^{m}\right)$ for an integer $m=a+b \geq 1$. Hence the system (4) is obviously equivalent to the system :

$$
\begin{equation*}
\left(1, z^{a} f^{b}\right) \Omega=f / z \tag{5}
\end{equation*}
$$

By multiplying both left sides of $(5)$ by $\left(z g f^{k}, z\right)$, and then applying the fundamental theorem we obtain

$$
\begin{equation*}
\left(z g f^{k}, z^{a} f^{b}\right) \Omega=g f^{k+1} \tag{6}
\end{equation*}
$$

Let us compare the coefficients of both sides of (6). Since

$$
\begin{aligned}
{\left[z^{n+1}\right]\left(z g f^{k}, z^{a} f^{b}\right) \Omega } & =\left[z^{n+1}\right] \sum_{i \geq 0} \omega_{i} z^{a i+1} g f^{k+b i}=\sum_{i \geq 0} \omega_{i}\left[z^{n-a i}\right] g f^{k+b i} \\
& =\sum_{i \geqslant 0} \omega_{i} \ell_{n-a i, k+b i}
\end{aligned}
$$

and $\left[z^{n+1}\right] g f^{k+1}=\ell_{n+1, k+1}$, we obtain the Eq. (2) to be the $(a, b)$-sequence of $L$. Further, since ( $1, z \hat{f}^{b}$ ) is invertible, the system (4) has a unique solution

$$
\left(\omega_{0}, \omega_{1}, \omega_{2}, \ldots\right)^{T}=\left(1, z \widehat{f}^{b}\right)^{-1}\left(f_{0}, f_{1}, f_{2}, \ldots\right)^{T}
$$

Therefore, if $f \in \Sigma_{m}$ then $L=(g, f)$ has the unique $(a, b)$-sequence $\left(\omega_{n}\right)_{n \in \mathbf{N}_{0}}$ where $a, b \in Z$ such that $a+b \geq 1$ and $b \geq 0$. This completes the proof.

Let us now consider the set of Riordan matrices defined by

$$
\mathcal{H}_{m}=\left\{(g, f) \in \mathcal{R} \mid g \in \mathcal{F}_{0}, f \in \Sigma_{m}\right\}
$$

Theorem 2.1 asserts that every Riordan matrix in $\mathcal{H}_{m}$ has a $(a, b)$-sequence for any integer pair $(a, b)$ such that $a+b=m \geq 1$ and $b \geq 0$. Thus we see that a Riordan matrix $L=(g, f) \in \mathcal{H}_{m}$ has infinitely many ray sequences of the form $(a-c, b+c)$ for each $c=0,1,2, \ldots$ where $a+b=m$.

Further, since $\ell(f) \in \Sigma_{m}$ and $\bar{f} \in \Sigma_{m}$ for any $f, \ell \in \Sigma_{m}$, it can be readily shown that the set $\mathcal{H}_{m}$ forms a group under the Riordan multiplication, which leads to the following theorem.

Theorem 2.2. For each $m=1,2, \ldots$, the set $\mathcal{H}_{m}$ is a subgroup of the Riordan group. Further, $\mathcal{H}_{k}$ is a subgroup of $\mathcal{H}_{m}$ if and only if $k$ is a multiple of $m$.

By Theorem 2.2, we obtain infinitely many subgroups of the Riordan group by means of (a,b)-sequences. Clearly, $\mathcal{H}_{1}$ is the Riordan group $\mathcal{R}$. Since $\mathcal{H}_{2}=\left\{(g, f) \in \mathcal{R} \mid g \in \mathcal{F}_{0}, f\right.$ is an even function $\}$, we see that $\mathcal{C} \subsetneq \mathcal{H}_{2} \subsetneq \mathcal{R}$ where $\mathcal{C}$ is the checkerboard subgroup $\{(g, f) \in \mathcal{R} \mid g$ is an even, $f$ is an odd function\}.

From now on, the $(a, b)$-sequence and its GF are denoted by $\Omega_{(a, b)}$ and $\Omega_{(a, b)}(z)$, respectively. Since $\Omega_{(0,1)}(z)$ is the GF for the $A$-sequence, we have $f=z A(f)$ from (3). More generally, it follows from $z A(f)=f=z \Omega_{(a, b)}\left(z^{a} f^{b}\right)$ that the $A$-sequence and the ray-sequence are connected by

$$
A(z)=\Omega_{(a, b)}\left(z^{m} / A^{a}\right), \quad m=a+b
$$

Theorem 2.3. For $c \geq 1$, let $\Omega_{(a-c, b+c)}$ be a ray sequence of $L=(g, f) \in \mathcal{H}_{m}$. Then

$$
\Omega_{(a-c, b+c)}(z)=\frac{z}{\overline{z \Omega_{(a-c+1, b+c-1)}(z)}} .
$$

Proof. It suffices to show the case $c=1$. By Theorem 2.1, $L$ has the $(a, b)$-sequence if and only if $L$ has the $(a-1, b+1)$-sequence. Let $\varphi(z)=\Omega_{(a-1, b+1)}(z)$. By (3) we have $z \Omega_{(a, b)}\left(z^{a} f^{b}\right)=f=$ $z \varphi\left(z^{a-1} f^{b+1}\right)$. It follows

$$
\Omega_{(a, b)}\left(z^{a} f^{b}\right)=\varphi\left(z^{a-1} f^{b+1}\right)=\varphi\left(z^{a} f^{b} f / z\right)=\varphi\left(z^{a} f^{b} \Omega_{(a, b)}\left(z^{a} f^{b}\right)\right) .
$$

Replacing $z^{a} f^{b}$ by $w$ and then by setting $h(w)=w \Omega_{(a, b)}(w)$ we obtain

$$
\begin{equation*}
h(w) / w=\varphi(h(w)) \tag{7}
\end{equation*}
$$

Since $w \Omega_{(a, b)}(w) \in \mathcal{F}_{1}$ there exists the compositional inverse $\bar{h}(w)$. Substituting $w=\bar{h}(w)$ into (7) yields $\varphi(w)=w / \bar{h}(w)$. Thus we have

$$
\Omega_{(a-1, b+1)}(w)=w / \overline{w \Omega}_{(a, b)}(w),
$$

as desired.
Theorem 2.4. If $L_{i}=\left(G_{i}, F_{i}\right) \in \mathcal{H}_{m}$ has the ( $a_{i}, b_{i}$ )-sequence for $i=1$, 2 , then the product $L_{1} L_{2} \in \mathcal{H}_{m}$ has the ( $a, b$ )-sequence such that

$$
\begin{equation*}
\Omega_{(a, b)}\left(z^{a} f^{b}\right)=\Omega_{\left(a_{1}, b_{1}\right)}\left(z^{a_{1}} F_{1}^{b_{1}}\right) \Omega_{\left(a_{2}, b_{2}\right)}\left(F_{1}^{a_{2}} f^{b_{2}}\right), \tag{8}
\end{equation*}
$$

where $f=F_{2}\left(F_{1}\right)$.

Proof. Since $f=F_{2}\left(F_{1}\right) \in \Sigma_{m}$, the proof immediately follows from (3) that

$$
\begin{aligned}
z \Omega_{(a, b)}\left(z^{a} f^{b}\right) & =f=F_{2}\left(F_{1}\right)=F_{1} \Omega_{\left(a_{2}, b_{2}\right)}\left(F_{1}^{a_{2}} F_{2}^{b_{2}}\left(F_{1}\right)\right) \\
& =z \Omega_{\left(a_{1}, b_{1}\right)}\left(z^{a_{1}} F_{1}^{b_{1}}\right) \Omega_{\left(a_{2}, b_{2}\right)}\left(F_{1}^{a_{2}} f^{b_{2}}\right),
\end{aligned}
$$

as desired.
In particular, if $A_{i}(z)$ and $A(z)$ are GFs for $A$-sequences of $L_{i}$ and $L_{1} L_{2}$ respectively, then from (8) we obtain

$$
A(f)=A_{1}\left(F_{1}\right) A_{2}(f)=A_{1}\left(f / A_{2}(f)\right) A_{2}(f)
$$

By replacing $z$ by $\bar{f}$ we derive $A(z)=A_{2}(z) A_{1}\left(z / A_{2}(z)\right)$ (also see [6, Theorem 3.3]).
Theorem 2.5. If $L \in \mathcal{H}_{m}$ has the $(a, b)$-sequence with $\Omega_{(a, b)}(z)$ for $a \geq 0$, then $L^{-1} \in \mathcal{H}_{m}$ has the ( $b, a$ )-sequence with the $G F 1 / \Omega_{(a, b)}(z)$.

Proof. Let $L=(g, f)$. Then $L^{-1}=(1 / g(\bar{f}), \bar{f})$. Since $f \in \Sigma_{m}$, by Theorem $2.1 f=z \Omega_{(a, b)}\left(z^{a} f^{b}\right)$. By setting $f(z)=w$ we have $w=\bar{f} \Omega_{(a, b)}\left(\bar{f}^{a} w^{b}\right)$. Hence $\bar{f}=w / \Omega_{(a, b)}\left(w^{b} \bar{f}^{a}\right)$, which implies that $L^{-1}$ has the ( $b, a$ )-sequence whose GF is $1 / \Omega_{(a, b)}$.

Here, it would be interesting to observe Riordan matrices with a (1, 1)-sequence since it leads to a connection with the involutions of the Riordan group. See [1-4,7] for related topics.

Example 2.6. Let us consider the Riordan matrix $L=(f / z, f)$ with the ( 1,1 )-sequence ( $1,2,2, \ldots$ ). Since $\Omega_{(1,1)}(z)=\frac{1+z}{1-z}$, from (3) we have $f=z \frac{1+z f}{1-z f}$. Solving this equation we find that $f=z S\left(z^{2}\right)$ where $S(z)=\left(1-z-\sqrt{1-6 z+z^{2}}\right) / 2 z$ is the GF for the large Schröder numbers $1,2,6,22, \ldots$ (A006318 in [13]). Thus we obtain the Schröder triangle $L=\left(S\left(z^{2}\right), z S\left(z^{2}\right)\right.$ ) of the checkerboard type. If we notice that $1 / \Omega_{(1,1)}(z)=\Omega_{(1,1)}(-z)$, by Theorem 2.5 we see that $\Omega_{(1,1)}(-z)$ is the (1,1)-sequence GF of $L^{-1}$ and thus $L^{-1}=\left(S\left(-z^{2}\right), z S\left(-z^{2}\right)\right)$. In fact,

$$
\left[\begin{array}{cccccccccc}
1 & & & & & & \\
0 & 1 & & & & & \\
2 & 0 & 1 & & & & & \\
0 & 4 & 0 & 1 & & & \\
6 & 0 & 6 & 0 & 1 & & \\
0 & 16 & 0 & 8 & 0 & 1 & \\
22 & 0 & 30 & 0 & 10 & 0 & 1
\end{array}\right]^{-1}=\left[\begin{array}{ccccccc}
1 & & & & & & \\
0 & 1 & & & & & \\
-2 & 0 & 1 & & & & \\
0 & -4 & 0 & 1 & & & \\
6 & 0 & -6 & 0 & 1 & & \\
0 & 16 & 0 & -8 & 0 & 1 & \\
-22 & 0 & 30 & 0 & -10 & 0 & 1 \\
& & & \cdots & & &
\end{array}\right]
$$

A Riordan matrix $Q=\left(g\left(z^{2}\right), z g\left(z^{2}\right)\right), g \in \mathcal{F}_{0}$ is said to be quasi-involution if its inverse is $Q^{-1}=\left(g\left(-z^{2}\right), z g\left(-z^{2}\right)\right)$. Thus quasi-involutions are essentially self-inverse after inserting some minus signs.

We now characterize the quasi-involution.
Theorem 2.7. A Riordan matrix $Q=\left(g\left(z^{2}\right), z g\left(z^{2}\right)\right)$ is quasi-involution if and only if $Q$ has the $(1,1)$ sequence such that $\Omega_{(1,1)}(z)=1 / \Omega_{(1,1)}(-z)$.

Proof. Let $\Omega_{(1,1)}(z)=\sum_{n \geq 0} \omega_{n} z^{n}$, and let $Q^{-1}=\left[q_{n, k}^{\dagger}\right]_{n, k \in \mathbf{N}_{0}}$. Since $Q=\left[q_{n, k}\right]_{n, k \in \mathbf{N}_{0}}$ is an element of the checkerboard subgroup, it is obvious that $Q$ is a quasi involution if and only if $q_{n, k}^{\dagger}=(-1)^{\ell} q_{n, k}$ as $n-k=2 \ell \geq 2$ and $q_{n, k}^{\dagger}=q_{n, k}=0$ otherwise. Hence for any $(n, k)$ such that $n-k=2 \ell$ we have

$$
\begin{aligned}
q_{n+1, k+1}^{\dagger} & =(-1)^{\ell} q_{n+1, k+1}=(-1)^{\ell} \sum_{j \geq 0} \omega_{j} q_{n-j, k+j} \\
& =(-1)^{\ell} \sum_{j \geq 0}(-1)^{(n-k-2 j) / 2} \omega_{j} q_{n-j, k+j}^{\dagger}=\sum_{j \geq 0}(-1)^{j} \omega_{j} q_{n-j, k+j}^{\dagger},
\end{aligned}
$$

which implies that $\Omega_{(1,1)}(-z)$ is the GF for the $(1,1)$-sequence of $Q^{-1}$. By Theorem 2.5, the proof is completed.

Remark. Every Riordan matrix $\left(g\left(z^{2}\right), z g\left(z^{2}\right)\right)$ with the ( 1,1 )-sequence of the form

$$
\Omega_{(1,1)}(z)=\left(\frac{h_{e}+h_{o}}{h_{e}-h_{0}}\right)^{m}, \quad m \in Z
$$

is a quasi involution where $h_{e}$ is an even and $h_{o}$ is an odd function.

## 3. The (a, b)-diagonal sums

One might observe that rising diagonal sums in the Pascal matrix establish the Fibonacci sequence $\left(F_{n}\right)_{n \in \mathbf{N}_{0}}$ with $F_{0}=F_{1}=1$. The rising diagonal sums are formed by moving 1 unit up and 1 unit to the right.

We are now interested in the $(a, b)$-diagonal sums formed by moving $a$ unit up or down and $b$ unit to the right. First, we look at (2) when $k=0$ for each $n=0,1,2, \ldots$. This leads to the concept of weighted diagonal sums of a Riordan matrix.

For integers $a, b$ such that $a+b>0, b \geqslant 0$, we define the $k$ th weighted $(a, b)$-diagonal sum with a weight sequence $\left(\pi_{i}\right)_{i \geq 0}$ of a Riordan matrix $L=\left[\ell_{n, k}\right]_{n, k \in \mathbf{N}_{0}}$ by

$$
\delta_{k}^{(a, b)}:=\pi_{0} \ell_{k, 0}+\pi_{1} \ell_{k-a, b}+\pi_{2} \ell_{k-2 a, 2 b}+\cdots=\sum_{i \geq 0} \pi_{i} \ell_{k-i a, i b}
$$

where $\ell_{k-i a, i b}=0$ if $k-i a<0$. The corresponding GF is denoted by $\Delta^{(a, b)}(z)$, i.e., $\Delta^{(a, b)}(z)=$ $\sum_{k \geq 0} \delta_{k}^{(a, b)} z^{k}$.

Theorem 3.1. Let $L=(g, f)$ be a Riordan matrix and $\phi(z)$ be the GF for a weight sequence $\left(\pi_{i}\right)_{i \geq 0}$. Then

$$
\begin{equation*}
\Delta^{(a, b)}(z)=g \phi\left(z^{a} f^{b}\right) \tag{9}
\end{equation*}
$$

Proof. It immediately follows from

$$
\Delta^{(a, b)}(z)=\sum_{i \geq 0} \pi_{i} z^{i a} g f^{i b}=g \sum_{i \geq 0} \pi_{i}\left(z^{a} f^{b}\right)^{i}=g \phi\left(z^{a} f^{b}\right)
$$

Using $\phi(z)=1 /(1-z)$ we obtain the GF for $(a, b)$-diagonal sums with weights all 1 given by

$$
\begin{equation*}
\Delta^{(a, b)}(z)=\frac{g}{1-z^{a} f^{b}} . \tag{10}
\end{equation*}
$$

In particular, $\Delta^{(1,1)}(z)$ and $\Delta^{(0,1)}(z)$ represent the GFs for rising diagonal sums and row sums of a Riordan matrix, respectively. Also see [10].

Example 3.2. Let us consider the well-known identities

$$
B=\frac{1+z C^{2}}{1-z C^{2}} \text { and } C=\frac{1}{1-z C},
$$

where $B=1 / \sqrt{1-4 z}$ and $C=(1-\sqrt{1-4 z}) / 2 z$ are the GFs for the central binomial numbers $B_{n}=\binom{2 n}{n}$ and the Catalan numbers $C_{n}=\frac{1}{n+1}\binom{2 n}{n}$, respectively.

We can interpret the first identity as follows. Take $\phi(z)=\frac{1+z}{1-z}=1+2 z+2 z^{2}+\cdots$. Since

$$
B=\frac{1+z C^{2}}{1-z C^{2}}=\phi\left(z C^{2}\right)=\phi\left(z^{-1} \cdot(z C)^{2}\right)=\Delta^{(-1,2)}(z)
$$

it follows from (9) that the $n$th central binomial numbers $\binom{2 n}{n}$ are the same as weighted $(-1,2)$ diagonal sums with the weight sequence $(1,2,2, \ldots)$ of the associated Catalan matrix $(1, z C)$. Similarly, the second identity tells us that the $n$th row sum of $(1, z C)$ is the $n$th Catalan number $\frac{1}{n+1}\binom{2 n}{n}$ by (10).

In addition, we can also find other identity $B=C /\left(1-z C^{2}\right)$ connecting $C$ with $B$ from the nonweighted ( $-1,2$ )-diagonal sums of the Catalan matrix $(C, z C)$.

## 4. Extending the Riordan matrix

As previously noted, the concept of a $(a, b)$-sequence $\Omega=\left(\omega_{n}\right)_{n \in \mathbf{N}_{0}}$ for a Riordan matrix $L=$ [ $\left.\ell_{n, k}\right]_{n, k \in \mathbf{N}_{0}}$ may be viewed as every element $\ell_{n+1, k+1}$ can be expressed as a linear combination with coefficients in $\Omega$ of the elements in $L$ lying on the line $t_{1}$ with the slope $a / b(1 / 0$ means $\infty)$ starting from $\ell_{n, k}$.

In this section, we will observe other ray sequence obtained from reflecting the line $t_{1}$ about the line with the slope 1 that passes through the element $\ell_{n+1, k+1}$. This concept leads to an extended Riordan matrix which is a bilaterally infinite matrix.

We begin by defining a finite Riordan matrix. A matrix $L_{n}=(g, f)_{n}$ is said to be Riordan matrix of order $n$ if it is the $n \times n$ principal submatrix of $L=(g, f)$. The flip-transpose $L_{n}^{F}$ of $L_{n}$ is defined by $L_{n}^{F}=E L_{n}^{T} E$ where $E$ is the $n \times n$ backward identity matrix, i.e.,

$$
E=\left[\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
& \cdots & & \\
1 & 0 & 0 & 0
\end{array}\right]
$$

Theorem 4.1. Let $L=(g, f)$ be a Riordan matrix. Then $L_{n}^{F}$ is the invertible Riordan matrix of order $n$ given by $\left(g(\bar{f}) \cdot \bar{f}^{\prime} \cdot(z / \bar{f})^{n}, \bar{f}\right)_{n}$.

Proof. Let $L=\left[\ell_{i, j}\right]_{i, j \in \mathbf{N}_{0}}$ and let $L_{n}^{F}=\left[\ell_{i, j}^{F}\right]_{0 \leq i, j<n}$. Applying Theorem 3 in $[9]$ to $L^{-1}=(1 / g(\bar{f}), \bar{f})$, we obtain

$$
\ell_{i, j}=\left[z^{i-j}\right] g(\bar{f}) \cdot \bar{f}^{\prime} \cdot(z / \bar{f})^{i+1}
$$

Thus

$$
\begin{aligned}
\ell_{i, j}^{F} & =\ell_{n-j-1, n-i-1}=\left[z^{i-j}\right] g(\bar{f}) \cdot \bar{f}^{\prime} \cdot(z / \bar{f})^{n-j}=\left[z^{i}\right] g(\bar{f}) \cdot \bar{f}^{\prime} \cdot(z / \bar{f})^{n-j} z^{j} \\
& =\left[z^{i}\right]\left(g(\bar{f}) \cdot \bar{f}^{\prime} \cdot(z / \bar{f})^{n}\right) \bar{f}^{j}
\end{aligned}
$$

It follows $L_{n}^{F}=\left(g(\bar{f}) \cdot \bar{f}^{\prime} \cdot(z / \bar{f})^{n}, \bar{f}\right)_{n}$, as desired. Further, $L_{n}^{F}$ is invertible since the first GF of $L_{n}^{F}$ belongs to $\mathcal{F}_{0}$ and $\bar{f} \in \mathcal{F}_{1}$. Hence the proof is completed.

The following is an immediate consequence of Theorem 2.1.
Lemma 4.2. If $f \in \Sigma_{m}$ then an $n \times n$ Riordan matrix $(g, f)_{n}$ has $a(a, b)$-sequence satisfying (2) for nonnegative integer pair $(a, b)$ such that $a+b=m \geq 1$ and $b \geq 0$.

Theorem 4.3. Let $L=\left[\ell_{i, j}\right]_{i, j \geq 0} \in \mathcal{H}_{m}$ be a Riordan matrix. Then there exists a unique sequence $\Omega^{*}=$ $\left(\omega_{n}^{*}\right)_{n \in \mathrm{~N}_{0}}$ with $\omega_{0}^{*} \neq 0$ such that every element $\ell_{n+1, k+1}$ except lying in column 0 can be expressed as a linear combination with coefficients in $\Omega^{*}$ of the elements lying on the line with the slope b/a starting from $\ell_{n+2, k+2}$ where $a, b \geq 0$ are integers such that $a+b=m \geq 1$, i.e.,

$$
\begin{equation*}
\ell_{n+1, k+1}=\sum_{i=0}^{\left\lfloor\frac{n-k}{m}\right\rfloor} \omega_{i}^{*} \ell_{n+2-b i, k+2+a i} \quad\left(n, k \in \mathrm{~N}_{0}\right) \tag{11}
\end{equation*}
$$

Proof. For a sufficiently large $n$, let us consider $L_{n}=(g, f)_{n}=\left[\ell_{i, j}\right]_{i, j \in N_{0}}$ where $f \in \Sigma_{m}$. By Theorem 4.1, $L_{n}^{F}=\left[\ell_{i, j}^{F}\right]_{0 \leq i, j<n}$ is an $n \times n$ Riordan matrix. Further, since $\bar{f} \in \Sigma_{m}$, by Lemma 4.2 it has some ( $a, b$ )-sequence $\left(\omega_{n}^{*}\right)_{n \in \mathrm{~N}_{0}}$ where $a, b \geq 0$ are integers such that $a+b=m \geq 1$. Thus

$$
\begin{aligned}
\ell_{n-j-2, n-i-2} & =\ell_{i+1, j+1}^{F}=\sum_{k \geq 0} \omega_{k}^{*} \ell_{i-a k, j+b k}^{F}=\sum_{k \geq 0} \omega_{k}^{*} \ell_{n-(j+b k)-1, n-(i-a k)-1} \\
& =\sum_{k \geq 0} \omega_{k}^{*} \ell_{n-j-b k-1, n-i+a k-1 .}
\end{aligned}
$$

Substituting $s=n-j-3, t=n-i-3$ yields

$$
\ell_{s+1, t+1}=\sum_{k \geq 0} \omega_{k}^{*} \ell_{s+2-b k, t+2+a k},
$$

as desired when $n \rightarrow \infty$. The uniqueness follows from the uniqueness of the $(a, b)$-sequence of a Riordan matrix in $\mathcal{H}_{m}$.

In Lemma 2.4.1 of [8], it is shown that every element $\ell_{n+1, k+1}$ of a Riordan matrix can be expressed as a horizontal sequence starting from $\ell_{n+2, k+2}$. The sequence is called the $B$-sequence of the Riordan matrix and its generating function is $B(z)=A(z)^{-1}$. If $a=0, b=1$ in Theorem 4.3 then we have the following corollary. It asserts that every element $\ell_{n+1, k+1}$ of a Riordan matrix can be expressed as a vertical sequence starting from $\ell_{n+2, k+2}$.

Corollary 4.4. An infinite lower triangular matrix $L=\left[\ell_{i, j}\right]_{i, j \geq 0}$ is a Riordan matrix if and only if there exists a unique sequence $\Omega^{*}=\left(\omega_{n}^{*}\right)_{n \in \mathrm{~N}_{0}}$ with $\omega_{0}^{*} \neq 0$ such that every element $\ell_{n+1, k+1}$ except lying in column 0 can be expressed as a linear combination with coefficients in $\Omega^{*}$ of the elements in the next column starting from the next row, i.e.,

$$
\begin{equation*}
\ell_{n+1, k+1}=\sum_{i=0}^{n-k} \omega_{i}^{*} \ell_{n+2-i, k+2} \quad\left(n, k \in \mathrm{~N}_{0}\right) \tag{12}
\end{equation*}
$$

Further, $\Omega^{*}(z)=\frac{z}{f(z)}$, which coincides with the $A$-sequence of $L^{-1}$.
Proof. Let $L$ be an infinite lower triangular matrix with the sequence $\Omega^{*}$ satisfying (12). By the definition, $\Omega^{*}$ coincides with the $A$-sequence of $L_{n}^{F}$ for a sufficiently large $n$. Hence $L_{n}=\left(L_{n}^{F}\right)^{F}$ is a Riordan matrix of order $n$ and so is $L$ when $n \rightarrow \infty$. Further, we have $\bar{f}=z \Omega^{*}(\bar{f})$. Thus $\Omega^{*}(z)=\frac{z}{f(z)}$, which is obviously the $A$-sequence of $L^{-1}$. The converse follows from Theorem 4.3. It completes the proof.


Fig. 2. Two kinds of ray-sequences.

Remark. If $\left(\omega_{m}\right)_{m \in \mathbf{N}_{\mathbf{0}}}$ is a ray sequence of $(g, f)^{-1}$ then $\left(\omega_{m}\right)_{m \in \mathbf{N}_{\mathbf{0}}}$ is also ray sequence of $(g, f)_{n}^{F}$ as $n \rightarrow \infty$.

By Theorems 2.1 and 4.3, we see that if $L=(g, f) \in \mathcal{H}_{m}$ then for any nonnegative integer pair $(a, b)$ such that $a+b=m \geq 1, L$ has both the $(a, b)$-sequence $\left(\omega_{n}\right)_{n \in \mathbf{N}_{0}}$ starting from $\ell_{n, k}$ and the sequence $\left(\omega_{n}^{*}\right)_{n \in \mathrm{~N}_{0}}$ starting from $\ell_{n+2, k+2}$. We note that it may be viewed as all the elements $\ell_{n+2-b i, k+2+a i}$ for $i=0,1,2, \ldots$ are lying on the line $t_{1}^{*}$ with the slope $b / a$ starting from $\ell_{n+2, k+2}$, which is obtained from reflecting the line $t_{1}$ with the slope $a / b$ starting at $\ell_{n, k}$ about the line $t_{0}$ with the slope 1 that passes through the element $\ell_{n+1, k+1}$, see Fig. 2.

Let us now consider a $(a, b)$-sequence of $L=\left[\ell_{n, k}\right] \in \mathcal{H}_{m}$ when $a<0$. In this case, we are interested in the elements $\ell_{n+2-b i, k+2+a i}$ for $i=0,1,2, \ldots$ lying on the line $t_{1}^{*}$ which satisfy (11). If we allow extending the matrix $L$ to negative columns, then (11) is valid for any integer pair ( $a, b$ ) with $a+b=m \geq 1$, and some elements might be located on negative columns. Therefore we can define $\ell_{n, k}$ for all integers $n$ and $k$ to be (11) where $\ell_{n, k}=0$ when $n<k$. By the extension of Taylor expansion, it can be shown that such extended matrix agrees with an extended Riordan matrix ${ }^{2}\langle g, f\rangle:=\left[\ell_{n, k}\right]_{n, k \in \mathbb{Z}}$ defined by for all integers $n$ and $k$,

$$
\ell_{n, k}=\left[z^{n}\right] g f^{k}
$$

where $g \in \mathcal{F}_{0}$ and $f \in \Sigma_{m}$.
We note that for an extended Riordan matrix $L_{E}:=\langle g, f\rangle$, both Theorems 2.1 and 4.3 are valid for any pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $a+b \geq 1$. Thus let us still denote a ( $a, b$ )-sequence of $L_{E}$ by $\Omega_{(a, b)}$ for given integers $a, b$ such that $a+b=m \geq 1$.

From now on, we will call the sequence $\left(\omega_{n}^{*}\right)_{n \in N_{0}}$ in Theorem 4.3 the companion sequence of $\Omega_{(a, b)}$. It is denoted by $\Omega_{(a, b)}^{c}$. In particular, the companion sequences of $A$ - and $V$-sequence are denoted by $A^{c}$ and $V^{c}$ respectively.

Theorem 4.5. Let $f \in \Sigma_{m}$. For each $(a, b)$-sequence $\Omega_{(a, b)}$ of an extended Riordan matrix $L_{E}=\langle g, f\rangle$, $L_{E}$ has the corresponding companion sequence $\Omega_{(a, b)}^{c}$.

Proof. By Theorem 4.3, it suffices to show that two cases: (i) $a<0$ and $b \geq 0$, (ii) $a \geq 0$ and $b<0$. First assume that (i) $a<0$ and $b \geq 0$. For a fixed integer pair $(n, k)$ such that $n>k$, let us define the $\gamma \times \gamma$ submatrix $L_{\gamma}:=L[\alpha, \ldots, \beta]$ obtained from $L_{E}$ by taking the elements of $L_{E}$ lying in both row and columns $\alpha, \alpha+1, \ldots, \beta$ where $\gamma=\beta-\alpha+1$ and $\alpha=\left\lfloor\frac{a n+b k}{m}\right\rfloor+1, \beta=\left\lceil\frac{a k+b n}{m}\right\rceil+1$. Since $\alpha \leq k+2+a i$ and $\alpha \leq n+2-b i$, it suffices to consider a $(a, b)$-sequence of $L_{\gamma}$. First we observe

[^2]that $L_{\gamma}$ may be viewed as a proper Riordan matrix $(\hat{\mathrm{g}}, f)_{\gamma}$ where $\hat{\mathrm{g}}=z^{-\alpha} \mathrm{g} f^{\alpha}$. Thus by Theorem 4.1, $L_{\gamma}^{F}$ is also finite Riordan matrix given by $\left(\hat{g}^{*}, \bar{f}\right)_{\gamma}$ where $\hat{g}^{*}=\hat{g}(\bar{f}) \cdot \bar{f}^{\prime} \cdot(z / \bar{f})^{\gamma}$.

Let us denote $L_{E}=\left[\ell_{i, j}\right]_{i, j \in \mathbb{Z}}, L_{\gamma}=\left[\hat{\ell}_{i, j}\right]_{o \leq i, j<\gamma}$ and $L_{\gamma}^{F}=\left[\hat{\ell}_{i, j}^{*}\right]_{o \leq i, j<\gamma}$. Since $\ell_{i, j}=\hat{\ell}_{i-\alpha, j-\alpha}$ and $\hat{\ell}_{i, j}=\hat{\ell}_{\gamma-j-1, \gamma-i-1}^{*}$, we have

$$
\begin{equation*}
\ell_{n+1, k+1}=\hat{\ell}_{n-\alpha+1, k-\alpha+1}=\hat{\ell}_{\beta-k-1, \beta-n-1}^{*} . \tag{13}
\end{equation*}
$$

Since $\bar{f} \in \Sigma_{m}$, by Theorem 2.1 there exists some ( $a, b$ )-sequence $\left(\omega_{i}^{*}\right)_{i \in \mathbf{N}_{\mathbf{0}}}$ such that

$$
\begin{equation*}
\hat{\ell}_{\beta-k-1, \beta-n-1}^{*}=\sum_{i=0}^{\left\lfloor\frac{n-k}{m}\right\rfloor} \omega_{i}^{*} \hat{\ell}_{\beta-k-2-a i, \beta-n-2+b i}^{*} \tag{14}
\end{equation*}
$$

We will now show that $\left(\omega_{i}^{*}\right)_{i \in \mathbf{N}_{0}}$ is the companion sequence of $\Omega_{(a, b)}$. It follows from (13) and (14) that

$$
\begin{aligned}
\ell_{n+1, k+1} & =\sum_{i=0}^{N} \omega_{i}^{*} \hat{\ell}_{\beta-k-2-a i, \beta-n-2+b i}^{*}=\sum_{i=0}^{N} \omega_{i}^{*} \hat{\ell}_{\gamma-(\beta-n-2+b i)-1, \gamma-(\beta-k-2-a i)-1} \\
& =\sum_{i=0}^{N} \omega_{i}^{*} \hat{\ell}_{n-\alpha+2-b i, k-\alpha+2+a i}=\sum_{i=0}^{N} \omega_{i}^{*} \ell_{n+2-b i, k+2+a i} .
\end{aligned}
$$

Hence from (11), it follows that $\left(\omega_{i}^{*}\right)_{i \in \mathbf{N}_{0}}$ is the companion sequence of $\Omega_{(a, b)}$, i.e., $\Omega_{(a, b)}^{c}=\left(\omega_{i}^{*}\right)_{i \in \mathbf{N}_{\mathbf{0}}}$.
By a similar argument, if we take $\alpha=\left\lfloor\frac{a k+b n}{m}\right\rfloor-1$ and $\beta=\left\lceil\frac{a n+b k}{m}\right\rceil+3$, one can show that the case (ii).

We note that if $f \in \Sigma_{m}$ then $L_{E}=\langle g, f\rangle$ has both $(a, b)$ - and $(b, a)$-sequence because of $a+b=$ $m \geq 1$.

Theorem 4.6. Let $f \in \Sigma_{m}$ and $a+b=m \geq 1$. Then the GF for the companion sequence of $\Omega_{(a, b)}$ of $L_{E}=\langle g, f\rangle$ is given by

$$
\begin{equation*}
\Omega_{(a, b)}^{c}(z)=\frac{1}{\Omega_{(b, a)}(z)} . \tag{15}
\end{equation*}
$$

Proof. Since $f \in \Sigma_{m}$ and $a+b=m$, applying Theorem 2.1 to the extended Riordan matrix there exists a unique $(b, a)$-sequence $\Omega_{(b, a)}$ such that $f=z \Omega_{(b, a)}\left(z^{b} f^{a}\right)$. Further, $\Omega_{(a, b)}^{c}$ coincides with the $(a, b)$-sequence of $L_{\gamma}^{F}=\left(\hat{g}^{*}, \bar{f}\right)$. By Theorem 2.1 again, we have $\bar{f}=z \Omega_{(a, b)}^{c}\left(z^{a} \bar{f}^{b}\right)$ and thus $f=z / \Omega_{(a, b)}^{c}\left(z^{b} f^{a}\right)$. Hence $f=z \Omega_{(b, a)}\left(z^{b} f^{a}\right)=z / \Omega_{(a, b)}^{c}\left(z^{b} f^{a}\right)$, which implies (15).

Corollary 4.7. Let $A(z)$ and $V(z)$ be GFs of the $A$ - and $V$-sequence of a Riordan matrix $L=(g, f)$, respectively. Then

$$
\begin{equation*}
A^{c}(z)=\frac{1}{V(z)} \text { and } V^{c}(z)=\frac{1}{A(z)} \tag{16}
\end{equation*}
$$

Theorem 4.8. Let $L_{E}=\langle g, f\rangle=\left[\ell_{n, k}\right]_{n, k \in \mathbb{Z}}$ be an extended Riordan matrix where $f \in \Sigma_{m}$. For each integer s there exists a s-sequence $\left(\omega_{n, s}\right)_{n \in N_{0}}$ with the $G F \Omega_{(a, b)}^{s}(z)$ such that

$$
\begin{equation*}
\ell_{n+1, k+1}=\sum_{i=0}^{\left\lfloor\frac{n-k}{m}\right\rfloor} \omega_{i, s} \ell_{n+1-s-a i, k+1-s+b i}, s \in \mathbb{Z} \tag{17}
\end{equation*}
$$

Proof. First let $s \geq 0$. We proceed by induction on $s$. Since $\left(\omega_{n, 0}\right)_{n \in N_{0}}=(1,0,0, \ldots)$, it holds for $s=0$. Let $s \geq 1$. Assume that there exists a $s$-sequence $\left(\omega_{n, s}\right)_{n \in \mathrm{~N}_{0}}$ with the GF $\Omega_{(a, b)}^{s}(z)$. Since ( $a, b$ )-sequence is $\left(\omega_{n, 1}\right)_{n \in \mathrm{~N}_{0}}$, applying to $\ell_{n+1-s-a i, k+1-s+b i}$ we have

$$
\begin{aligned}
\ell_{n+1, k+1} & =\sum_{i \geq 0} \omega_{i, s} \ell_{n+1-s-a i, k+1-s+b i} \\
& =\sum_{i \geq 0} \omega_{i, s} \sum_{u \geq 0} \omega_{u, 1} \ell_{(n+1-s-a i)-1-a u,(k+1-s+b i)-1+b u} \\
& =\sum_{i \geq 0} \sum_{u \geq 0} \omega_{i, s} \omega_{u, 1} \ell_{n-s-a(i+u), k-s+b(i+u)} .
\end{aligned}
$$

By setting $v=i+u$, we obtain

$$
\ell_{n+1, k+1}=\sum_{v \geq 0}\left(\sum_{i=0}^{v} \omega_{i, s} \omega_{v-i, 1}\right) \ell_{n-s-a v, k-s+b v}
$$

Thus by induction, $\left(\omega_{n, s+1}\right)_{n \in \mathrm{~N}_{0}}$ is the $(s+1)$-sequence of $L_{E}$ where

$$
\begin{equation*}
\omega_{n, s+1}=\sum_{i=0}^{n} \omega_{i, s} \omega_{n-i, 1} . \tag{18}
\end{equation*}
$$

Moreover, it follows from (18) that

$$
\sum_{n \geq 0} \omega_{n, s+1} z^{n}=\Omega_{(a, b)}^{s}(z) \Omega_{(a, b)}(z)=\Omega_{(a, b)}^{s+1}(z) .
$$

By a similar argument, one can show that the case $s<0$. It completes the proof.

## 5. Examples and combinatorial identities

Several kinds of Riordan matrices are related to the $r$-ary numbers given by $\mathfrak{b}_{n}^{(r)}:=\frac{1}{(r-1) n+1}\binom{c n}{n}$, e.g., the Pascal and the Catalan matrices. The $G F \mathfrak{B}_{r}(z)=\sum_{n \geq 0} \mathfrak{b}_{n}^{(r)} z^{n}$ satisfies the functional equation $\mathfrak{B}_{r}(z)=1+z \mathfrak{B}_{r}^{r}(z),(r \in \mathbb{Z})$. It can be shown [5] that the following identity is valid for all real numbers $s$ :

$$
\begin{equation*}
\mathfrak{B}_{r}^{s}(z)=\sum_{n \geq 0} \frac{s}{r n+s}\binom{r n+s}{n} z^{n} . \tag{19}
\end{equation*}
$$

In this section, we will examine our previous results for the Riordan matrix with a ray sequence of $r$-ary numbers. It allows us to derive several combinatorial identities.

Theorem 5.1. For an integer pair $(a, b)$ such that $a+b=m \geq 1$, let $\Omega_{(a, b)}(z)$ be the $G F$ for the $(a, b)$ sequence of $L_{E}=\langle g, f\rangle$ where $f \in \Sigma_{m}$. Then $\Omega_{(a, b)}(z)=\mathfrak{B}_{r}(z)$ if and only if $\Omega_{(a-1, b+1)}(z)=\mathfrak{B}_{r-1}(z)$.

Proof. Let $\varphi=z / \bar{h}$ where $h=z \mathfrak{B}_{r}(z)$. Since $\mathfrak{B}_{r}(z)=1+z \mathfrak{B}_{r}^{r}(z)$, we have $h=z+h^{r} / z^{r-2}$. By replacing $z$ by $\bar{h}$, we obtain $z=\bar{h}+z^{r} / \bar{h}^{r-2}$, i.e., $z / \bar{h}=1+z\left(z^{r-1} / \bar{h}^{r-1}\right)$. Hence $\varphi=1+z \varphi^{r-1}$, which implies $\varphi=\mathfrak{B}_{r-1}(z)$. By Theorem 2.3 , the proof is completed.

The next corollary follows from Theorems 5.1 and 4.6.
Corollary 5.2. Let $A(z)$ be the $G F$ for the $A$-sequence of $L_{E}=\langle g, f\rangle$. For each integer $i$ we have:
(i) $A(z)=\mathfrak{B}_{r}(z)$ if and only if $\Omega_{(i, 1-i)}(z)=\mathfrak{B}_{i+r}(z)$;
(ii) $A(z)=\mathfrak{B}_{r}(z)$ if and only if $\Omega_{(i, 1-i)}^{c}(z)=1 / \mathfrak{B}_{1-i+r}(z)$.

Example 5.3. Let us consider the extended Catalan matrix $L_{E}=\langle C, z C\rangle=\left[\ell_{n, k}\right]_{n, k \in \mathbb{Z}}$ which is a bilaterally infinite matrix, where $\ell_{n, k}=\frac{k+1}{n+1}\binom{2 n-k}{n}$ if $n \neq-1$, and $\ell_{-1,-1}=1, \ell_{-1, k}=-1$ for $k<-1$. The matrix is given by


Since $A(z)=\frac{1}{1-z}=\mathfrak{B}_{1}(z)$, from Corollary 5.2 and Theorem 4.8, it follows that $L_{E}$ has (i,1-$i)$-sequences and the corresponding companion sequences such that $\Omega_{(i, 1-i)}(z)=\mathfrak{B}_{1+i}(z)$ and $\Omega_{(i, 1-i)}^{c}(z)=1 / \mathfrak{B}_{2-i}(z)=\mathfrak{B}_{i-1}(-z)$ for $i \in \mathbb{Z}$.

For examples, let $i=2$. Since

$$
\begin{aligned}
& \Omega_{(2,-1)}(z)=\mathfrak{B}_{3}(z)=1+z+3 z^{2}+12 z^{3}+55 z^{4}+\cdots, \\
& \Omega_{(2,-1)}^{c}(z)=\mathfrak{B}_{1}(-z)=1-z+z^{2}-z^{3}+z^{4}-\cdots,
\end{aligned}
$$

the boxed element $\ell_{4,1}=C_{4}=14$ may be expressed by means of the $(2,-1)$-sequence and its companion sequence, respectively:

$$
\begin{aligned}
& 14=1 \cdot 5+1 \cdot 0+3 \cdot(-1)+12 \cdot 1, \\
& 14=1 \cdot 28+(-1) \cdot 20+1 \cdot 7+(-1) \cdot 1 .
\end{aligned}
$$

Further, $L_{E}$ has $s$-sequences with the $\operatorname{GFs} \Omega_{(i, 1-i)}^{s}(z)=\mathfrak{B}_{1+i}^{s}(z)$ for $(i, s) \in \mathbb{Z} \times \mathbb{Z}$. Thus $\ell_{n+1, k+1}$ can be expressed in terms of $r$-ary numbers as

$$
\ell_{n+1, k+1}=\sum_{j=0}^{n-k}\left[z^{j}\right] \mathfrak{B}_{1+i}^{s}(z) \ell_{n+1-s-i j, k+1-s+(1-i) j} .
$$

In particular, if $k=0$ then for any pair $(i, s) \in \mathbb{Z} \times \mathbb{Z}$, the $n$th Catalan number can be expressed as

$$
C_{n+1}=\sum_{j=0}^{n} \frac{s(j-i j-s+2)}{(j+i j+s)(n-i j-s+2)}\binom{j+i j+s}{j}\binom{2 n-j-i j-s+1}{n-j} .
$$

From this expression one can derive several combinatorial identities for the Catalan numbers.

## Acknowledgments

We thank the referee(s) for valuable comments and suggestions that improved the presentation of this paper. In addition, we thank professor L. Shapiro for helpful comments and reading this paper carefully.

## References

[1] N.T. Cameron, A. Nkwanta, On some (pseudo) involutions in the Riordan group, J. Integer Seq. 8 (2005), Article 05.3.7.
[2] G.-S. Cheon, H. Kim, Simple proofs of open problems on the involution of the Riordan group, Linear Algebra Appl. 428 (2008) 930-940.
[3] G.-S. Cheon, H. Kim, L.W. Shapiro, Riordan group involutions, Linear Algebra Appl. 428 (2008) 941-952.
[4] G.-S. Cheon, S.-T. Jin, H. Kim, L.W. Shapiro, Riordan group involutions and the $\Delta$-sequence, Discrete Appl. Math. 157 (2009) 1696-1701.
[5] R. Graham, D. Knuth, O. Patashnik, Section 5.4 of Concrete Mathematics, second ed., Addison-Wesley, 1994.
[6] T.-X. He, R. Sprugnoli, Sequence characterization of Riordan arrays, Discrete Math. 309 (2009) 3962-3974.
[7] A. Luzón, A. Morón, Riordan matrices in the reciprocation of quadratic polynomials, Linear Algebra Appl. 430 (2009) $2254-2270$.
[8] D. Merlini, D.G. Rogers, R. Sprugnoli, M.C. Verri, On some alternative characterizations of Riordan arrays, Canad. J. Math. 49 (2) (1997) 301-320.
[9] D. Merlini, R. Sprugnoli, M.C. Verri, Combinatorial sums and implicit Riordan arrays, Discrete Math. 309 (2009) 475-486.
[10] A. Nkwanta, L.W. Shapiro, Pell walks and Riordan matrices, Fibonacci Quart. 43 (2005) 170-180.
[11] L.W. Shapiro, Bijections and the Riordan group, Theoret. Comput. Sci. 307 (2003) 403-413.
[12] L.W. Shapiro, S. Getu, W.-J. Woan, L. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229-239.
[13] N.J.A. Sloane, The On-line Encyclopedia of Integer Sequences. Available from: <http://www.research.att.com/~njas/ sequences/>.
[14] R. Sprugnoli, Combinatorial sums through Riordan arrays, in: Abstracts of Lectures and Talks in Combinatorics 2010, Italy, 2010, pp. 80-88.
[15] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132 (1994) 267-290.


[^0]:    ${ }^{1}$ This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-20100015796).

    * Corresponding author.

    E-mail addresses: gscheon@skku.edu (G.-S. Cheon), anieshe@skku.edu (S.-T. Jin).
    0024-3795/\$ - see front matter © 2011 Elsevier Inc. All rights reserved.
    doi:10.1016/j.laa.2011.04.001

[^1]:    ${ }^{1}$ Sometimes it is called the proper Riordan array, see [15].

[^2]:    2 The concept of an extended Riordan matrix has been presented by R. Sprugnoli, in the invited talk at the conference "Combinatorics 2010" held in Verbania (Italy) from June 28 to July 3, 2010, see [14].

