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# An invariant p-adic q-integral associated with q-Euler numbers and polynomials

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# An invariant p-adic q-integral associated with q-Euler numbers and polynomials

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#### Abstract

The purpose of this paper is to consider q-Euler numbers and polynomials which are q-extensions of ordinary Euler numbers and polynomials by the computations of the p-adic q-integrals due to T. Kim, cf. [1, 3, 6, 12], and to derive the "complete sums for q-Euler polynomials" which are evaluated by using multivariate p-adic q-integrals. These sums help us to study the relationships between p-adic q-integrals and non-archimedean combinatorial analysis.

## 1 Introduction

Let p be a fixed odd prime, and let  $\mathbb{C}_p$  denote the p-adic completion of the algebraic closure of  $\mathbb{Q}_p$ . For d a fixed positive integer with (p, d) = 1, let

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{a,p = 1}}^{0 < a < dp} a + dp\mathbb{Z}_p,$$

 $a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},\$ 

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where  $a \in \mathbb{Z}$  lies in  $0 \le a < dp^N$ , (cf. [1], [2], [14]).

The *p*-adic absolute value in  $\mathbb{C}_p$  is normalized so that  $|p|_p = \frac{1}{p}$ . Let *q* be variously considered as an indeterminate a complex number  $q \in \mathbb{C}$ , or a *p*-adic number  $q \in \mathbb{C}_p$ . If  $q \in \mathbb{C}$ , we always assume |q| < 1. If  $q \in \mathbb{C}_p$ , we always assume  $|q - 1|_p < p^{-\frac{1}{p-1}}$ , so that  $q^x = \exp(x \log q)$  for  $|x|_p \leq 1$ . Throughout this paper, we use the following notation :

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}.$$

We say that f is a uniformly differentiable function at a point  $a \in \mathbb{Z}_{p^{-}}$  and denote this property by  $f \in UD(\mathbb{Z}_{p})$ - if the difference quotients

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

have a limit l = f'(a) as  $(x, y) \to (a, a)$ , [1, 11, 12]. For  $f \in UD(\mathbb{Z}_p)$ , let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \le j < p^N} q^j f(j) = \sum_{0 \le j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p), \text{ cf. } [2, 4],$$

representing a q-analogue of Riemann sums for f.

The integral of f on  $\mathbb{Z}_p$  will be defined as limit  $(n \to \infty)$  of these sums, when it exists. An invariant *p*-adic *q*-integral of a function  $f \in UD(\mathbb{Z}_p)$  on  $\mathbb{Z}_p$  is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{0 \le j < p^N} f(j) q^j.$$

Note that if  $f_n \to f$  in  $UD(Z_p)$ ; then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x).$$

It was well known that the ordinary Euler numbers are defined by

$$F(t) = \frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

where we use the technique method notation by replacing  $E^m$  by  $E_m$  ( $m \ge 0$ ), symbolically, cf.[2, 3, 6, 12]. In this paper, we consider q-Euler numbers and polynomials which are qextensions of ordinary Euler numbers and polynomials by the computations of the p-adic q-integrals, and derive the "complete sums for q-Euler polynomials" which are evaluated by using multivariate p-adic q-integrals. These sums help us to study the relationships between p-adic q-integrals and non-archimedean combinatorial analysis.

## 2 *q*-Euler and Genocchi numbers associated with *p*-adic *q*integral

The Euler polynomials are defined by means of the following generating function:  $\frac{2}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$ . Note that  $E_n(0) = E_n$ . From these Euler polynomials, we can evaluate the value of the following alternating sums of powers of consecutive integers [1, 2, 3, 11]:

$$-1^{m} + 2^{m} - 3^{m} + \dots + (-1)^{m-1} (n-1)^{m} = \frac{1}{2} \left( (-1)^{n+1} E_{m}(n) - E_{m} \right).$$
 (2.1)

In a fermionic sense, we now consider the following p-adic q-integrals:

$$\int_{X_f} [x]_q^k d\mu_{-q}(x) = \int_{\mathbb{Z}_p} [x]_q^k d\mu_{-q}(x) = E_{k,q} \quad \text{for} \quad k, f \in \mathbb{N}.$$
(2.2)

From the computation of this p-adic q-integral, we derive the following Eq.(3):

$$E_{k,q} = [2]_q \left(\frac{1}{1-q}\right)^k \sum_{l=0}^k \binom{k}{l} (-1)^l \frac{1}{1+q^{l+1}},$$
(2.3)

where  $\binom{k}{i}$  is the binomial coefficient. Note that  $\lim_{q\to 1} E_{k,q} = E_k$ . Hence,  $E_{k,q}$  is a q-extension of Euler numbers which are called q-Euler numbers. Let  $F_q(t) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}$  be the generating function of these q-Euler numbers. Then we easily see that [6, 8, 9, 10]

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{[2]_q}{[2]_{q^{j+1}}} \left(\frac{1}{q-1}\right)^j \frac{t^j}{j!} = [2]_q \sum_{l=0}^{\infty} (-q)^l e^{[l]_q t}.$$
(2.4)

By using an invariant *p*-adic *q*-integral on  $\mathbb{Z}_p$ , we can also consider a *q*-extension of ordinary Euler polynomials which are called *q*-Euler polynomials[3,8,12]. For  $x \in \mathbb{Z}_p$ , we define *q*-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) = E_{k,q}(x).$$
(2.5)

By (5), we easily see that

$$E_{k,q}(x) = \sum_{n=0}^{k} \binom{k}{n} [x]_{q}^{k-n} q^{nx} E_{n,q}.$$

In Eq.(5), it is easy to see that

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{k=0}^n \binom{n}{k} (-1)^k q^{kk} \frac{1}{1+q^{k+1}}.$$

By using the definition of Eq.(5), we will give the distribution of q-Euler polynomials. From the definition of a p-adic q-integral, we derive the below formula:

$$\int_{X_m} [x+y]_q^n d\mu_{-q}(y) = \frac{[m]_q^m}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a \int_{\mathbb{Z}_p} [\frac{a+x}{m} + y]_{q^m}^n d\mu_{-q^m}(y), \text{ if } m \text{ is odd.}$$

Thus, if m is an odd integer, then we have

$$E_{n,q}(x) = \frac{[m]_q^n}{[m]_{-q}} \sum_{a=0}^{m-1} (-1)^a q^a E_{n,q^m}(\frac{a+x}{m}).$$

From the definition of the q-Euler polynomials, we note that

$$F_q(x,t) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t}$$

As is well know, the Genocchi numbers are also defined by

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}.$$

Thus, we easily see that  $G_n = \sum_{l=0}^{n-1} {n \choose l} 2^l B_l$ , where  $B_l$  are ordinary Bernoulli numbers. We now define a *q*-extension of Genocchi number which are called *q*-Genocchi numbers as follows:

$$F_q^*(t) = [2]_q t \sum_{l=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \text{ see } [8].$$
(2.6)

From Eq. (2.6), we can derive the following, see Refs. [8, 12]

$$G_{n,q} = n \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{[2]_{q^{l+1}}}, \text{ when } m \text{ is odd }.$$
(2.7)

From Eq. (2.6), we can also recover the defining relation for the definition of q-Genocchi polynomials as follows:

$$F_q^*(x,t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}, \quad \text{when } n \text{ is odd, (see [8]).}$$
(2.8)

Let  $a_1, a_2, \dots, a_k$  be positive integers. For  $w \in \mathbb{Z}_p$ , we define multiple Daehee q-Euler polynomials by using the invariant *p*-adic *q*-integrals as follows, cf. [7, 8, 12]:

$$E_n^{(k)}(w,q|a_1,a_2,\cdots,a_k) = \int_{\mathbb{Z}_p^k} [w + \sum_{j=1}^k a_j x_j]^n d\mu_{-q}(x),$$
(2.9)

and

$$E_n^{(k)}(q|a_1,\cdots,a_k) = \int_{\mathbb{Z}_p^k} [\sum_{j=1}^k a_j x_j]^n d\mu_{-q}(x),$$

where

$$\int_{\mathbb{Z}_p^k} f(x) d\mu_{-q}(x) = \underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} f(x) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r).$$

From Eq. (2.9), we can derive the following theorem:

**Theorem 1.** Let  $a_1, a_2, \dots, a_k$  be positive integers. Then we have

$$E_n^{(k)}(w,q|a_1,\cdots,a_k) = \frac{[2]_q^k}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \prod_{j=1}^k \left(\frac{1}{[2]_q^{1+ra_j}}\right).$$
 (2.10)

Given elements  $\alpha_1, \dots, \alpha_m \in \mathbb{C}_p$  and positive integers  $N_1, \dots, N_m, n$ , it is easy to see that [1, 6]

$$[N_1(x_1 + \alpha_1) + \dots + N_m(x_m + \alpha_m)]^n$$
(2.11)

$$=\sum_{\substack{i_1,\cdots,i_m\geq 0\\i_1+\cdots+i_m=n}}\sum_{k_1=0}^{n-i_1}\sum_{k_2=0}^{n-i_1-i_2}\cdots\sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}}$$
(2.12)

$$\times \begin{pmatrix} n \\ i_1, \cdots, i_m \end{pmatrix} \begin{pmatrix} n-i_1 \\ k_1 \end{pmatrix} \begin{pmatrix} n-i_1-i_2 \\ k_2 \end{pmatrix} \cdots \begin{pmatrix} n-i_1-i_2-\cdots-i_{m-1} \\ k_{m-1} \end{pmatrix}$$
(2.13)

$$\times (q-1)^{k_1 + \dots + k_{m-1}} [N_1]^{i_1 + k_1} \cdots [N_{m-1}]^{i_{m-1} + k_{m-1}} [N_m]^{i_m}$$
(2.14)

$$\times [x_1 + \alpha_1 : q^{N_1}]^{k_1 + i_1} \cdots [x_{m-1} + \alpha_{m-1} : q^{N_{m-1}}]^{k_{m-1} + i_{m-1}} [x_m + \alpha_m : q^{N_m}]^{i_m}, \quad (2.15)$$

Hence, we have

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [N_1(x_1 + \alpha_1) + \dots + N_m(x_m + \alpha_m)]^n d\mu_{-q^{N_1}}(x_1) \cdots d\mu_{-q^{N_m}}(x_m)}_{I} \quad (2.16)$$

m times

$$=\sum_{\substack{i_1,\cdots,i_m\geq 0\\i_1+\cdots+i_m=n}}\sum_{k_1=0}^{n-i_1}\sum_{k_2=0}^{n-i_1-i_2}\cdots\sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}}$$
(2.17)

$$\times \begin{pmatrix} n\\i_1,\cdots,i_m \end{pmatrix} \begin{pmatrix} n-i_1\\k_1 \end{pmatrix} \begin{pmatrix} n-i_1-i_2\\k_2 \end{pmatrix} \cdots \begin{pmatrix} n-i_1-i_2-\cdots-i_{m-1}\\k_{m-1} \end{pmatrix}$$
(2.18)

$$\times (q-1)^{k_1 + \dots + k_{m-1}} [N_1]^{i_1 + k_1} \cdots [N_{m-1}]^{i_{m-1} + k_{m-1}} [N_m]^{i_m}$$
(2.19)

$$\times E_{k_1+i_1}(\alpha_1, q^{N_1}) \cdots E_{k_{m-1}+i_{m-1}}(\alpha_{m-1}, q^{N_{m-1}}) E_{i_m}(\alpha_m, q^{N_m}).$$
(2.20)

From (2.9), (2.10), (2.11) and (2.16), we can derive the following theorem:

**Theorem 2.** (Complete sum for multiple Daehee q-Euler polynomials) Given elements  $\alpha_1, \dots, \alpha_m \in \mathbb{C}_p$  and positive integers  $N_1, \dots, N_m, n$ ,

$$\sum_{\substack{i_1,\cdots,i_m\geq 0\\ i_1+\cdots+i_m=n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \cdots \sum_{k_{m-1}=0}^{n-i_1-\cdots-i_{m-1}} \times \binom{n}{i_1,\cdots,i_m} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \cdots \binom{n-i_1-i_2-\cdots-i_{m-1}}{k_{m-1}} \times (q-1)^{k_1+\cdots+k_{m-1}} [N_1]^{i_1+k_1} \cdots [N_{m-1}]^{i_{m-1}+k_{m-1}} [N_m]^{i_m} \times E_{k_1+i_1}(\alpha_1,q^{N_1}) \cdots E_{k_{m-1}+i_{m-1}}(\alpha_{m-1},q^{N_{m-1}}) E_{i_m}(\alpha_m,q^{N_m}) = E_n^{(m)}(N_1\alpha_1+\cdots+N_m\alpha_m,q|N_1,\cdots,N_m).$$

### **3** Further Remarks and Observations

In this section, we assume that  $q \in \mathbb{C}$  with |q| < 1. Let  $\Gamma(s)$  be the ordinary gamma function given by  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ ,  $s \in \mathbb{C}$ . From (8) and complex integration, we can derive the following formula:

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q^*(x, -t) dt = [2]_q \sum_{n=0}^\infty \frac{(-1)^{n+1} q^{n+x}}{[n+x]_q}, \quad \text{for } s \in \mathbb{C}.$$
(3.1)

For  $s \in \mathbb{C}$ , we define the (Hurwitz's type) q-Genocchi zeta function as follows [3, 12]:

$$\zeta_{q,G}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{x+n}}{[n+x]_q^s}, \text{ where } x \in \mathbb{R} \text{ with } 0 < x < 1.$$
(3.2)

By (2.8), (3.1) and (3.2), we can see that

$$\zeta_{q,G}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F_q^*(x,-t) dt = \sum_{n=0}^\infty \frac{G_{n,q}(x)}{n!} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{n+s-2} dt\right).$$
(3.3)

By using Laurent series in Eq. (3.3), we easily see that (see Refs. [3, 12, 13])

$$\zeta_{q,G}(1-n,x) = \frac{(-1)^{n-1}}{n} G_{n,q}(x), \ n \in \mathbb{N}$$

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