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# EULER-BOOLE SUMMATION REVISITED 

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#### Abstract

We study a connection between Euler-MacLaurin Summation and Boole Summation suggested in an AMM note from 1960, which explains them as two cases of a general approach to approximation. Herein we give details and extensions of this idea.


## 1. Introduction

The Euler-Maclaurin summation formula [9, 2.11.1],

$$
\begin{align*}
\sum_{j=a}^{n-1} f(j)=\int_{a}^{n} f(x) d x & +\sum_{k=1}^{m} \frac{B_{k}}{k!}\left(f^{(k-1)}(n)-f^{(k-1)}(a)\right)  \tag{1}\\
& +\frac{(-1)^{m+1}}{m!} \int_{a}^{n} \widetilde{B}_{m}(y) f^{(m)}(y) d y
\end{align*}
$$

is a well-known formula from classical analysis giving a relation between the finite sum of values of a function $f$, whose first $m$ derivatives are absolutely integrable on $[a, n]$, and its integral, for $a, m, n \in \mathbb{N}, a<n$. This elementary formula appears often in introductory texts [2, 16], usually with mention to a particular application - Stirling's asymptotic formula. However, general approaches to such formulae are not often mentioned in the same context.

In the formula above, the $B_{l}$ are the Bernoulli numbers and the $\widetilde{B}_{l}(x)$ are the periodic Bernoulli polynomials. The Bernoulli polynomials are most succinctly characterized by a generating function [1, 23.1.1]:

$$
\begin{equation*}
\frac{t e^{x t}}{e^{t}-1}=\sum_{n=0}^{\infty} B_{n}(x) \frac{t^{n}}{n!} . \tag{2}
\end{equation*}
$$

The periodic Bernoulli polynomials are defined by taking only the fractional part of $x$ : $\widetilde{B}_{n}(x):=$ $B_{n}(\{x\})$ [9, 24.2.11-12]. Evaluating at the point $x=0$ gives the Bernoulli numbers [1, 23.1.2]: $B_{l}:=B_{l}(0)$.

A similar formula comes from starting with a different set of polynomials. The Euler polynomials $E_{n}(x)$ are given by the generating function [1, 23.1.1]

$$
\begin{equation*}
\frac{2 e^{x t}}{e^{t}+1}=\sum_{n=0}^{\infty} E_{n}(x) \frac{t^{n}}{n!} . \tag{3}
\end{equation*}
$$

Let the periodic Euler polynomials $\widetilde{E}_{n}(x)$ be defined by $\widetilde{E}_{n}(x+1)=-\widetilde{E}_{n}(x)$ and $\widetilde{E}_{n}(x)=E_{n}(x)$ for $0 \leq x<1$ [9, 24.2.11-12]. Unlike the periodic Bernoulli, which have period 1, the $\widetilde{E}_{n}(x)$ have period 2, and exhibit an even (vs. odd) symmetry about zero. Thirdly, define the Euler numbers by $E_{n}:=2^{n} E_{n}(1 / 2)[1,23.1 .2]$; i.e. [9, 24.2.6],

$$
\begin{equation*}
\frac{2 e^{t}}{e^{2 t}+1}=\sum_{n=0}^{\infty} E_{n} \frac{t^{n}}{n!} . \tag{4}
\end{equation*}
$$

The alternating version of (1), using Euler polynomials, is the Boole summation formula [9, 24.17.1-2],: Let $a, m, n \in \mathbb{N}, a<n$. If $f(t)$ is a function on $t \in[a, n]$ with $m$ continuous absolutely
integrable derivatives, then for $0<h<1$,

$$
\begin{array}{r}
\sum_{j=a}^{n-1}(-1)^{j} f(j+h)=\frac{1}{2} \sum_{k=0}^{m-1} \frac{E_{k}(h)}{k!}\left((-1)^{n-1} f^{(k)}(n)+(-1)^{a} f^{(k)}(a)\right) \\
+\frac{1}{2(m-1)!} \int_{a}^{n} f^{(m)}(x) \widetilde{E}_{m-1}(h-x) d x \tag{5}
\end{array}
$$

The first appearance of this form is due to Boole [4]; a similar formula is believed [8] to have been known to Euler as well. Mention of this beautiful formula in the literature is regrettably scarce in comparison to that of (1), although in [5] it is used to explain a curious property of truncated alternating series for $\pi$ and $\log 2$-for which Boole summation is better suited than Euler-Maclaurin.

In his 1960 note, Strodt [14] indicated a unified operator-theoretic approach to proving both of these formulae, but with very few of the details given. In the years since, excellent generalizations of Euler-MacLaurin Summation have appeared [3], yet not using this particular approach. The main goals of this note are to fill in Strodt's details, and to demonstrate the extent to which his ideas can be extended to other cases.

We begin with a characterization of the operators suggested by Strodt and their associated polynomials. Section 2 ends with a couple of main theorems about these polynomials, which lead to corollaries in both the Euler and Bernoulli polynomials-this is the content of Section 3. The following section describes the polynomials arising from well-known probability densities. In Section 5 we detail Strodt's unified development of the summation formulae of Euler and Boole. Section 6 describes a generalized development of inversion formulae involving the Euler and Bernoulli numbers. We conclude our discussion with a conjecture of a general asymptotic formula in Section 7.

## 2. Strodt's Operators and polynomials

We introduce a class of operators on functions of a finite real interval. Let $C[a, b]$ denote the continuous functions on the finite interval $[a, b]$, in the supremum norm. Define, for $n \in \mathbb{N}$, the uniform interpolation Strodt operators $\mathcal{S}_{n}: C[x, x+1] \mapsto C[x, x+1]$ by

$$
\begin{equation*}
\mathcal{S}_{n}(f)(x):=\sum_{j=0}^{n} \frac{1}{n} \cdot f(x+j / n) \tag{6}
\end{equation*}
$$

The operators used by Strodt to prove Euler-MacLaurin summation and Boole summation are both covered under this definition, as we shall demonstrate. Since the operators defined by (6) are positive, they are necessarily bounded linear operators.

We will show that an operator in this class will send a polynomial in $x$ to another one of the same degree. Furthermore, we claim they are automorphisms on the ring of degree- $k$ polynomials. This fact will be used to define both Euler and Bernoulli polynomials as inverses of $x^{k}$ under different operators. There is evidence that this approach is known, at least informally, in the Bernoulli case [17], but not for Euler polynomials.

We begin by noting an important property of any interpolation operator on polynomials.
Proposition 2.1. For each $k \in \mathbb{N}$, let $P_{k}:=\left\{\sum_{i=0}^{k} a_{i} x^{i}: a_{i} \in \mathbb{R}\right\} \cong \mathbb{R}^{k+1}$. Then for all $n \in \mathbb{N}$, given $A \in P_{k}$ there is a unique $B \in P_{k}$ such that $\mathcal{S}_{n}(B)=A$.

This property will be proven for a more general class of operators. Given a finite set of points $\left\{x_{i}\right\}_{i=1}^{N} \subset[0,1]$ and a probability weight function

$$
w:\left\{x_{i}\right\}_{i=1}^{N} \mapsto(0,1) \text { so that } \sum_{i=1}^{N} w\left(x_{i}\right)=1
$$

we define the corresponding finite Strodt operator

$$
\begin{equation*}
\mathcal{S}_{w}(f)(x):=\sum_{i=1}^{N} f\left(x+x_{i}\right) w\left(x_{i}\right) \tag{7}
\end{equation*}
$$

that generalizes $\mathcal{S}_{n}$. Even more broadly, define the generalized Strodt operators

$$
\begin{equation*}
\mathcal{S}_{\mu}(f)(x):=\int f(x+u) d \mu \tag{8}
\end{equation*}
$$

where the integral is Lebesgue over a measurable subset of $u \in \mathbb{R}$ and $\mu$ is a Lebesgue measure. We will often consider the special case $d \mu=g(u) d u$ where $g(u)$ is any absolutely continuous probability weight function with finite moments:

$$
\begin{equation*}
\int g(u) d u=1 \quad \text { and } \quad \int|u|^{k} g(u) d u<\infty \quad \text { for all } k \in \mathbb{N} \text {. } \tag{9}
\end{equation*}
$$

Here and from this point forward we will use the convention $\int=\int_{-\infty}^{\infty}$, where the support of $g$ is used to effectively limit the interval.

Hence (7) can be seen to be an instance of (8) where we formally write $g$ as a Dirac delta "function:"

$$
\begin{equation*}
\mathcal{S}_{\mu}=\mathcal{S}_{\omega}=\mathcal{S}_{g}, \quad \text { where } \quad g(u)=\sum_{i=1}^{N} w\left(x_{i}\right) \delta_{x_{i}}(u) \tag{10}
\end{equation*}
$$

More precisely, we would be identifying $\mathcal{S}_{\mu}$ and $\mathcal{S}_{g}(f)$ with the Lebesgue-Stieltjes integral $\int f(x+$ u) $d G$ where $d G=g(u) d u$, see [15, pp. 282-284]. Thus, the original class $\left\{\mathcal{S}_{n}, n \in \mathbb{N}\right\}$ is also covered by this definition. We will prove that a version of Proposition 2.1 holds for even this most general class of operators, which we continue to denote $\mathcal{S}_{g}$.

Now suppose $f \in P_{k}$; that is, $f:=\sum_{n=0}^{k} f_{n} x^{n}$, for $f_{n} \in \mathbb{R}$. By definition,

$$
\begin{equation*}
h(x):=\mathcal{S}_{g}(f)=\int \sum_{n=0}^{k} f_{n}(x+u)^{n} g(u) d u \tag{11}
\end{equation*}
$$

and we see via the binomial theorem that $h(x)$ is again a polynomial of degree $k$ :

$$
\begin{equation*}
h(x)=\sum_{j=0}^{k} h_{j} x^{j} \tag{12}
\end{equation*}
$$

where $h_{j}=\sum_{n=j}^{k} f_{n}\binom{n}{j} M_{n-j}$ and

$$
\begin{equation*}
M_{l}:=\int u^{l} d G(u) \tag{13}
\end{equation*}
$$

Regarding degree- $k$ polynomials as vectors of length $k+1$ via the natural isomorphism, we see that the restriction of the operator $\mathcal{S}_{g}$ to $P_{k} \cong \mathbb{R}^{k+1}$ can be represented by a $(k+1)^{2}$ matrix. The coefficients of this matrix are read directly from (12):

$$
\left.\mathcal{S}_{g}\right|_{P_{k}}[i, j]:=\left\{\begin{array}{ll}
\binom{j-1}{i-1} M_{j-i} & \text { for } 1 \leq i \leq j \leq k+1  \tag{14}\\
0 & \text { otherwise }
\end{array} .\right.
$$

This allows us to prove that this operator is invertible.
Proposition 2.2. Let $g$ be a probability density function whose absolute moments exist. For all $h \in P_{k}$, there is a unique $f \in P_{k}$ so that $\mathcal{S}_{g}(f)=h$.

Proof. We see from (14) that $\left.\mathcal{S}_{g}\right|_{P_{k}}$ is an upper-triangular matrix whose determinant is

$$
\begin{equation*}
\operatorname{det}\left(\left.\mathcal{S}_{g}\right|_{P_{k}}\right)=\prod_{t=0}^{k}\binom{t}{t} X_{t, t}=1 \tag{15}
\end{equation*}
$$

and so $h(x)$ is the unique solution of a linear system of equations:

$$
\begin{equation*}
h=\mathcal{S}_{g}^{(-1)} f \tag{16}
\end{equation*}
$$

The reader will note that Proposition 2.1 follows from this as a special case.
We can recover both the Euler polynomials and the Bernoulli polynomials using this uniqueness property with different weight functions. Define the Euler operator as

$$
\begin{equation*}
\mathcal{S}_{E}(f)(x):=\frac{f(x)+f(x+1)}{2} ; \quad \text { i.e., } g(u):=\delta_{0}(u) / 2+\delta_{1}(u) / 2 \tag{17}
\end{equation*}
$$

and the Bernoulli operator by

$$
\begin{equation*}
\mathcal{S}_{B}(f)(x):=\int_{0}^{1} f(x+u) d u ; \quad \text { i.e., } g(u):=\chi_{[0,1]} \tag{18}
\end{equation*}
$$

We now claim that the set of Euler polynomials $E_{n}(x)$ is the unique set of polynomials that satisfy

$$
\begin{equation*}
\mathcal{S}_{E}\left(E_{n}(x)\right)=x^{n} \text { for all } n \in \mathbb{N}_{0} \tag{19}
\end{equation*}
$$

while the Bernoulli polynomials are the unique polynomials which satisfy

$$
\begin{equation*}
\mathcal{S}_{B}\left(B_{n}(x)\right)=x^{n} \text { for all } n \in \mathbb{N}_{0} \tag{20}
\end{equation*}
$$

Remark 1. One now sees the motivation behind the definition of $\mathcal{S}_{n}$ in (6). The Euler operator is $\mathcal{S}_{2}$. The Bernoulli operator corresponds via a Riemann sum to the limit of $\mathcal{S}_{n}$ as $n \rightarrow \infty$. Hence, the range of $n$ provides an interpolation of sorts between these two developments.

We shall prove the generating functions of the Bernoulli and Euler polynomials follow from (19) and (20). Thus these formulae comprise sufficent definitions of the polynomials. Unlike the conventional recursive definitions of $B_{n}(x)$ and $E_{n}(x)$, no extra conditions, such as initial values, are necessary [14]. Furthermore, they each take the form $\mathcal{S}_{g}^{-1}\left(x^{n}\right)$ for different weight functions. This suggests a natural generalization of Bernoulli and Euler polynomials which we will call Strodt polynomials. They will be denoted $P_{n}^{g} \equiv P_{n}, n \in \mathbb{N}_{0}$ (we will typically suppress the $g$ in the notation).

Theorem 2.3. For each $n$ in $\mathbb{N}_{0}$, let $P_{n}^{g}(x)$ be the Strodt polynomials associated with a given density $g(x)$; that is, for all $x \in \mathbb{R}, P_{n}^{g}(x)$ is defined implicitly by the relation

$$
\begin{equation*}
\mathcal{S}_{g}\left(P_{n}^{g}(x)\right)=x^{n} \text { for all } n \in \mathbb{N}_{0} \tag{21}
\end{equation*}
$$

where $\mathcal{S}_{g}$ is a Strodt operator. Then

$$
\begin{equation*}
\frac{d}{d x} P_{n}^{g}(x)=n P_{n-1}^{g}(x) \text { for all } n \in \mathbb{N} \tag{22}
\end{equation*}
$$

Proof. By the Lebesgue Dominated Convergence Theorem [12], we have that

$$
\begin{aligned}
\frac{d}{d x} \int P_{n}(x+u) g(u) d u & =\lim _{\nu \rightarrow \infty} \int \nu\left(P_{n}(x+1 / \nu+u)-P_{n}(x+u)\right) d G(u) \\
& =\int \lim _{\nu \rightarrow \infty} \nu\left(P_{n}(x+1 / \nu+u)-P_{n}(x+u)\right) d G(u) \\
& =\int \frac{d}{d x} P_{n}(x+u) d G(u)
\end{aligned}
$$

Therefore, in view of (21), we have that

$$
\begin{equation*}
\mathcal{S}_{g}\left(\frac{d}{d x} P_{n}-n P_{n-1}\right)=\frac{d}{d x}\left(x^{n}\right)-n x^{n-1}=0 \tag{23}
\end{equation*}
$$

But $\mathcal{S}_{g}$ is one-to-one on polynomials by Proposition 2.2, hence the desired conclusion.
We note that the class of Strodt polynomials comprises a subset of Appell sequences [11]. These are the polynomial sequences given by generating functions are of the form

$$
\begin{equation*}
\sum_{n=0}^{\infty} A_{n}(x) \frac{x^{n}}{n!}=\frac{e^{x t}}{G(t)} \tag{24}
\end{equation*}
$$

Here $G(t)$ is a function defined formally by the coefficient sequence of its series in $t$, where the only restriction is that the leading term must not be zero. This is in fact equivalent to (22) [11].

The next theorem clarifies the relation between Strodt and Appell.
Theorem 2.4. Suppose a class of polynomials $\left\{C_{n}(x)\right\}_{n \geq 0}$ with real coefficients has an exponential generating function of the form

$$
\begin{equation*}
\sum_{n=0}^{\infty} C_{n}(x) \frac{t^{n}}{n!}=e^{x t} R(t) \tag{25}
\end{equation*}
$$

where $R(t)$ is a continuous function on the real line. Then the exponential generating function of the polynomial sequence which is the image of the $C_{n}$ under the operator $\mathcal{S}_{g}$ is given by

$$
\begin{equation*}
\sum_{n=0}^{\infty} \mathcal{S}_{g}\left(C_{n}(x)\right) \frac{t^{n}}{n!}=e^{x t} R(t) Q_{g}(t) \tag{26}
\end{equation*}
$$

where

$$
\begin{equation*}
Q_{g}(t):=\int e^{u t} d G(u) \tag{27}
\end{equation*}
$$

Proof. Assume that the parameter $t$ is within the radius of uniform convergence of the formal power series (25) for an arbitrary fixed value of $x$. Then we can integrate the series (25) termwise to produce:

$$
\begin{equation*}
\sum_{n=0}^{\infty} \int C_{n}(x+u) g(u) d u \frac{t^{n}}{n!}=\int e^{(x+u) t} R(t) g(u) d u \tag{28}
\end{equation*}
$$

This is equivalent to (26).
As in the case of the $P_{n}^{g}(x)=P_{n}(x)$, the functions $Q_{g}(t)=Q(t)$ are implicitly dependent on the weight function $g(u)$, but this will be suppressed in the notation.

Expanding the exponential integrand of (27) in a Taylor series, and then integrating termwise, we see that

$$
\begin{equation*}
Q(t)=\sum_{n=0}^{\infty} M_{n} \frac{t^{n}}{n} \tag{29}
\end{equation*}
$$

where $M_{n}$ is, as in (13), the $n$th moment of the cumulative distribution function of the density $g(u)$. Therefore $Q(t)$ is the moment generating function of $g$. See [6] for more information on moment generating functions.

Theorem 2.4 has the following consequence on the generating function of Strodt polynomials:

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n}(x) \frac{t^{n}}{n!}=\frac{e^{x t}}{Q(t)} \tag{30}
\end{equation*}
$$

To see why this must be true, apply (26) with $C_{n}(x)=P_{n}(x)$, and conclude that $R(t)$ must equal $1 / Q(t)$.

Remark 2. Now we see the precise connection to Appell sequences: The Appell sequences with respect to a function $Q(t)$ are Strodt exactly when $Q(t)$ is the moment generating function of some cumulative distribution function.

We can now verify our claim regarding the definition of Bernoulli and Euler polynomials.
Corollary 2.5. Formulae (19) and (20) are sufficient definitions of the classical Euler polynomials and Bernoulli polynomials, respectively.

Proof. We show that the conditions (19) and (20) imply the generating functions of each class of polynomials. Start with the Euler polynomials, whose generating function is given by [9, 24.2.8] as

$$
\begin{equation*}
\sum_{n=0}^{\infty} E_{n}(x) \frac{t^{n}}{n!}=\frac{2 e^{x t}}{e^{t}+1} \tag{31}
\end{equation*}
$$

Assume that for each integer $n \geq 0, E_{n}(x)$ is a polynomial of degree $n$ that satisfies (19); that is,

$$
\begin{equation*}
\mathcal{S}_{E}\left(E_{n}(x)\right)=x^{n} \text { for all } n \in \mathbb{N}_{0} \tag{32}
\end{equation*}
$$

Proposition 2.2 assures us that the $E_{n}(x)$ are well-defined. Now if the $E_{n}(x)$ do have an exponential generating function, then by Theorem 2.4 it must be of the form

$$
\begin{equation*}
\sum_{n=0}^{\infty} E_{n}(x) \frac{t^{n}}{n!}=\frac{e^{x t}}{Q(t)} \tag{33}
\end{equation*}
$$

We verify that

$$
\begin{equation*}
Q(t)=\int e^{u t}\left(\delta_{0}(u) / 2+\delta_{1}(u) / 2\right) d u=\frac{e^{t}+1}{2} \tag{34}
\end{equation*}
$$

thereby matching the generating function in (31) to that in (33). Therefore we conclude by analytic continuation that the formula (19) implies that the $E_{n}(x)$ are the Euler polynomials.

The argument for Bernoulli polynomials is similar. Again, let $B_{n}(x)$ be the unique polynomials satisfying

$$
\begin{equation*}
\mathcal{S}_{B}\left(B_{n}(x)\right)=x^{n} \text { for all } n \in \mathbb{N}_{0} \tag{35}
\end{equation*}
$$

If the generating function of $B_{n}(x)$ exists, Theorem 2.4 implies that it is of the form

$$
\begin{equation*}
\sum_{n=0}^{\infty} B_{n}(x) \frac{t^{n}}{n!}=\frac{e^{x t}}{Q(t)} \tag{36}
\end{equation*}
$$

where this time

$$
\begin{equation*}
Q(t)=\int_{0}^{1} e^{u t} d u=\frac{e^{t}-1}{t} \tag{37}
\end{equation*}
$$

We combine these two to produce

$$
\begin{equation*}
\sum_{n=0}^{\infty} B_{n}(x) \frac{t^{n}}{n!}=\frac{t e^{x t}}{e^{t}-1} \tag{38}
\end{equation*}
$$

which indeed is the generating function for Bernoulli polynomials. Thus the formula (20) implies that the $B_{n}(x)$ are the Bernoulli polynomials by analytic continuation.

## 3. First Consequences

We now show a few interesting special cases of Strodt polynomials and add to the list of properties that are directly implied by Theorems 2.3 and 2.4. In addition to what is shown here, we believe that most properties of Bernoulli and Euler polynomials that appear as formulas in a reference such as [1] or [9] are true in general for Strodt polynomials.

The Bernoulli polynomials of the kth order (or type, but not to be confused with the Bernoulli polynomials of the $k$ th kind, which are altogether different!) are given by the generating function [9, 24.16.1]

$$
\begin{equation*}
\sum_{n=0}^{\infty} B_{n}^{(k)}(x) \frac{t^{n}}{n!}=\left(\frac{t}{e^{t}-1}\right)^{k} e^{x t} \tag{39}
\end{equation*}
$$

We see that, for $k=1$, this generating function agrees with (2). Therefore Bernoulli polynomials of the first order are the same as regular Bernoulli polynomials: $B_{n}^{(1)}(x)=B_{n}(x)$. Since the generating function of $k$ th-order Bernoulli is just $e^{x t}$ divided by a power of $Q(t)$, we can also define these polynomials by iterations of $\mathcal{S}_{B}$, the Bernoulli operator. Henceforth we will let $\mathcal{S}_{g}^{(k)}$ denote the $k$-fold composition of the Strodt operator.

Corollary 3.1. For each positive integer $k$, if a degree-n polynomial $B_{n}^{(k)}(x)$ satisfies

$$
\begin{equation*}
\mathcal{S}_{B}^{(k)}\left[B_{n}^{(k)}(x)\right]=x^{n} \quad \text { for } n \in \mathbb{N}_{0} \tag{40}
\end{equation*}
$$

where $\mathcal{S}_{B}$ is the Bernoulli operator given in (18), then $B_{n}^{(k)}(x)$ is a Bernoulli polynomial of the kth order.

Proof. We emulate the proof of Corollary 2.5 for Bernoulli polynomials. Fix $k \geq 1$, and assume that for each $n \geq 1, B_{n}^{(k)}(x)$ is a degree-n polynomial satisfying (40). Proposition 2.2 implies that the $B_{n}^{(k)}(x)$ are well-defined. Using Theorem 2.4 inductively, we see that the generating function of these polynomials must be

$$
\begin{equation*}
\sum_{n=0}^{\infty} B_{n}^{(k)}(x) \frac{t^{n}}{n!}=\frac{e^{x t}}{[Q(t)]^{k}} \tag{41}
\end{equation*}
$$

Since

$$
\begin{equation*}
[Q(t)]^{k}=\left(\int_{0}^{1} e^{u t} d u\right)^{k}=\left(\frac{e^{t}-1}{t}\right)^{k} \tag{42}
\end{equation*}
$$

the right-hand sides of (41) and (39) match. We conclude by analytic continuation that the $B_{n}^{(k)}(x)$ are indeed the Bernoulli polynomials of the $k$ th order.

A similar property holds for Euler polynomials of the $k$ th order.
Corollary 3.2. Let $0 \leq k, n \in \mathbb{N}$. If a degree-n polynomial $E_{n}^{(k)}(x)$ satisfies

$$
\begin{equation*}
\mathcal{S}_{E}^{(k)}\left[E_{n}^{(k)}(x)\right]=x^{n} \tag{43}
\end{equation*}
$$

where $\mathcal{S}_{E}$ is the Bernoulli operator given in (17), then $E_{n}^{(k)}(x)$ is an Euler polynomial of the $k$ th order.

Proof. The proof is similar to that of the Bernoulli polynomials. Let $k \geq 1$, and assume that each $E_{n}^{(k)}(x)$ is a degree- $n$ polynomial satisfying (43). Proposition 2.2 implies that $E_{n}^{(k)}(x)$ is well-defined. Using Theorem 2.4 inductively, we see that the generating function of these polynomials must be

$$
\begin{equation*}
\sum_{n=0}^{\infty} E_{n}^{(k)}(x) \frac{t^{n}}{n!}=\frac{e^{x t}}{[Q(t)]^{k}} \tag{44}
\end{equation*}
$$

Since

$$
\begin{equation*}
[Q(t)]^{k}=\left(\frac{e^{t}+1}{2}\right)^{k} \tag{45}
\end{equation*}
$$

the generating function of the $E_{n}^{(k)}(x)$ must be

$$
\begin{equation*}
\sum_{n=0}^{\infty} E_{n}^{(k)}(x) \frac{t^{n}}{n!}=\left(\frac{2}{e^{t}+1}\right)^{k} e^{x t} \tag{46}
\end{equation*}
$$

This is exactly the generating function of Euler polynomials of the $k$ th order [9, 24.16.2], and so the corollary follows by analytic continuation.

We are motivated to create a new definition: For positive integers $n$ and $k$, the Strodt polynomials of the $k$ th order, denoted $P_{n}^{(k)}(x)$, are the polynomials which satisfy

$$
\begin{equation*}
\mathcal{S}_{g}^{(k)}\left(P_{n}^{(k)}(x)\right)=x^{n} \tag{47}
\end{equation*}
$$

By Proposition 2.2 they are well-defined, and Theorem 2.4 implies that their generating function is of the form

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n}^{(k)}(x) \frac{t^{n}}{n!}=\frac{e^{x t}}{[Q(t)]^{k}} \tag{48}
\end{equation*}
$$

We now construct a binomial recurrence formula for Strodt polynomials. This is meant to generalize entry 23.1.7 in [1], which states that

$$
\begin{equation*}
B_{n}(x+h)=\sum_{k=0}^{n}\binom{n}{k} B_{k}(x) h^{n-k} \tag{49}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{n}(x+h)=\sum_{k=0}^{n}\binom{n}{k} E_{k}(x) h^{n-k} \tag{50}
\end{equation*}
$$

This property is an equivalent definition of Appell sequences [11].
Corollary 3.3. For $n \in \mathbb{N}_{0}$, let $P_{n}(x)$ be a Strodt polynomial for a given weight function $g(u)$; i.e., for each $n \geq 0, P_{n}(x)$ is the unique polynomial satisfying $\mathcal{S}_{g}\left(P_{n}(x)\right)=x^{n}$. Then

$$
\begin{equation*}
P_{n}(x+h)=\sum_{k=0}^{n}\binom{n}{k} P_{k}(x) h^{n-k} \tag{51}
\end{equation*}
$$

Proof. Since $\mathcal{S}_{g}\left(P_{n}(x)\right)=x^{n}$, with $x=x+h$ we have

$$
\begin{equation*}
\mathcal{S}_{g}\left(P_{n}(x+h)\right)=(x+h)^{n} . \tag{52}
\end{equation*}
$$

Now use the binomial theorem to expand the right hand side in powers of $x$ and $h$. But each power $x^{k}$ is equal to $\mathcal{S}_{g}\left(P_{k}(x)\right)$, again by definition. Therefore, we have

$$
\begin{equation*}
\mathcal{S}_{g}\left(P_{n}(x+h)\right)=\mathcal{S}_{g}\left(\sum_{k=0}^{n}\binom{n}{k} P_{k}(x) h^{n-k}\right) \tag{53}
\end{equation*}
$$

by linearity of $\mathcal{S}_{g}$. Now take $\mathcal{S}_{g}^{(-1)}$ to both sides, invoking Proposition 2.2, to arrive at (51).
The known recurrence formulae for Bernoulli and Euler polynomials can be derived directly from here. For example, when $h=1$, we have

$$
\begin{equation*}
P_{n}(x+1)-P_{n}(x)=\sum_{k=0}^{n-1}\binom{n}{k} P_{k}(x) \tag{54}
\end{equation*}
$$

Rewriting the left hand side as an integral yields

$$
\begin{equation*}
P_{n}(x+1)-P_{n}(x)=\int_{0}^{1} P_{n}^{\prime}(x+u) d u=n \int_{0}^{1} P_{n-1}(x+u) d u \tag{55}
\end{equation*}
$$

Now suppose that $g(u):=\chi_{[0,1]}$, so that the $P_{n}(x)$ are Bernoulli polynomials. Then we have that

$$
\begin{equation*}
n \int_{0}^{1} B_{n-1}(x+u) d u=n \mathcal{S}_{B}\left(B_{n-1}(x)\right)=n x^{n-1} \tag{56}
\end{equation*}
$$

We have thus derived a standard binomial relation for Bernoulli polynomials [9, 24.5.1]:

$$
\begin{equation*}
n x^{n-1}=\sum_{k=0}^{n-1}\binom{n}{k} B_{k}(x) \tag{57}
\end{equation*}
$$

On the other hand, we can rewrite (54) as

$$
\begin{equation*}
P_{n}(x+1)+P_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} P_{k}(x)+P_{n}(x) . \tag{58}
\end{equation*}
$$

The left hand side of this equation is equal to $2 \mathcal{S}_{E}\left(P_{n}(x)\right)$, and so in the case that $P_{n}(x)=E_{n}(x)$, this leads to the formula $[9,24.5 .2]$

$$
\begin{equation*}
\sum_{k=0}^{n}\binom{n}{k} E_{k}(x)+E_{n}(x)=2 x^{n} \tag{59}
\end{equation*}
$$

We can also let $h=-1$, this yields alternating versions of the formulae above. The result in (51) becomes

$$
\begin{equation*}
P_{n}(x-1)-P_{n}(x)=\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k} P_{k}(x) . \tag{60}
\end{equation*}
$$

We repeat the derivation above to obtain an alternating version of the Bernoulli recurrence. We write

$$
\begin{equation*}
P_{n}(x-1)-P_{n}(x)=-\int_{0}^{1} P_{n}^{\prime}(x-1+u) d u=-n \int_{0}^{1} P_{n-1}(x-1+u) d u \tag{61}
\end{equation*}
$$

Now let $g(u):=\chi_{[0, \underline{1}}$, so that the $P_{n}(x)$ are Bernoulli polynomials. Since

$$
\begin{equation*}
-n \int_{0}^{1} B_{n-1}(x-1+u) d u=-n \mathcal{S}_{B}\left(B_{n-1}(x-1)\right)=-n(x-1)^{n-1} \tag{62}
\end{equation*}
$$

we can conclude that

$$
\begin{equation*}
n(x-1)^{n-1}=\sum_{k=0}^{n-1}(-1)^{n-k+1}\binom{n}{k} B_{k}(x) \tag{63}
\end{equation*}
$$

Alternatively, rewrite (54) as

$$
\begin{equation*}
P_{n}(x-1)+P_{n}(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} P_{k}(x)+P_{n}(x) \tag{64}
\end{equation*}
$$

The left hand side of this equation is equal to $2 \mathcal{S}_{E}\left(P_{n}(x-1)\right)$. Now suppose that we are in the Euler polynomial case, so that $P_{n}(x)=E_{n}(x)$. We thus obtain an alternating recurrence formula for the Euler polynomials:

$$
\begin{equation*}
\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} E_{k}(x)+E_{n}(x)=2(x-1)^{n} \tag{65}
\end{equation*}
$$

## 4. Strodt polynomials of Probability Densities

We have already seen how the generating functions and additive properties of Euler and Bernoulli polynomials can be recovered if one begins by defining them as Strodt polynomials. In fact, one could choose any density function and, proceeding in the same manner, obtain a class of polynomials with similar properties. In this section we will discover the Strodt polynomials associated with each of a selection of well-known probability distributions; the reader is encouraged to discover the consequences arising from other densities.

Example 4.1. Gaussian Density Function.
Here we take $g(u)=\frac{1}{\sqrt{\pi}} e^{-u^{2} / 2},-\infty<u<\infty$. We calculate the moment generating function of this distribution:

$$
\begin{equation*}
Q(t)=\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{u t} e^{-u^{2} / 2} d u=e^{t^{2} / 4} \tag{66}
\end{equation*}
$$

The generating function for the Strodt polynomials in this case is thus

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n}(x) \frac{t^{n}}{n!}=e^{x t} / Q(t)=e^{x t-t^{2} / 4} \tag{67}
\end{equation*}
$$

At this point a similarity to the Hermite polynomials is evident; their generating function is known to be $e^{2 x t-t^{2}}$ [1, 22.9.17]. Thus it follows that the Strodt polynomials in this case are scalings of the Hermite polynomials, as

$$
\begin{equation*}
2^{n} P_{n}(x)=H_{n}(x) \quad \text { for all } n \in \mathbb{N} \tag{68}
\end{equation*}
$$

We now have the following immediate corollary to Theorem 2.4 and 2.3.
Corollary 4.2. The Hermite polynomials $H_{n}(x)$ are scaled Strodt polynomials for the Gaussian density function. They satisfy

$$
\begin{equation*}
\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} H_{n}(x+u) e^{-u^{2} / 2} d u=(2 x)^{n}, \text { for all } n \in \mathbb{N}_{0} \tag{69}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d x} H_{n}(x)=2 n H_{n-1}(x), \text { for all } n \in \mathbb{N} \tag{70}
\end{equation*}
$$

Proof. The $P_{n}(x)$ in (67) were contrived to satisfy the generating function that makes them the Strodt polynomials of the Gaussian distribution. Then they satisfy

$$
\begin{equation*}
S_{g}\left(P_{n}(x)\right)=x^{n} \tag{71}
\end{equation*}
$$

which leads to (69), after one divides by a power of 2 . Similarly, (70) follows from Theorem 2.3.
The symmetry of the function $e^{-u^{2} / 2}$ about the origin leads to an additional property of the Strodt operator for this distribution.
Proposition 4.3. For $g(u)=\frac{1}{\sqrt{2 \pi}} e^{-u^{2} / 2}$, the operator $\mathcal{S}_{g}$ sends even polynomials to even polynomials and odd polynomials to odd polynomials.

Proof. Examine the image of the monomial $x^{n}$ under the Strodt operator. We expand $(x+u)^{n}$ in the integrand via the binomial theorem, yielding

$$
\begin{equation*}
\mathcal{S}_{g}\left(x^{n}\right)=\frac{1}{\sqrt{2 \pi}} \sum_{j=0}^{n}\binom{n}{j} x^{j} \int_{-\infty}^{\infty} u^{n-j} e^{-u^{2} / 2} d u \tag{72}
\end{equation*}
$$

We see that the integrals in this finite sum vanish if $n-j$ is an odd integer, for that causes the integrand to be an odd function. Therefore, the resulting polynomial must have the same parity
as the original function, which is the parity of $n$ itself. Since any even (odd) polynomial is a linear combination of even (odd) monomials, the proposition follows by linearity of the operator.

Remark 3. The Hermite polynomials are the only Appell sequence which is orthogonal; see [13] for a nice discussion of this topic. Since Strodt polynomials are a subset of Appell sequences, we know that Hermite polynomials are the only orthogonal Strodt polynomials.

Example 4.4. Poisson Distribution.
The Poisson distribution $X$ for a real parameter $\lambda$ is given by the probability function

$$
\begin{equation*}
P(X=j)=e^{-\lambda} \frac{\lambda^{j}}{j!} \quad \text { for } j \in \mathbb{N}_{0} \tag{73}
\end{equation*}
$$

This corresponds to the weight function

$$
\begin{equation*}
g(u):=\sum_{j=0}^{\infty} \delta_{j}(u) e^{-\lambda} \frac{\lambda^{u}}{\Gamma(u+1)} . \tag{74}
\end{equation*}
$$

We will develop the main properties using a general value of $\lambda$ as far as this is possible.
As before, we construct the Strodt polynomials for this distribution, using Theorem 2.4, which specifies the form of their exponential generating function. Since the moment generating function is

$$
\begin{equation*}
Q(t)=e^{-\lambda} \sum_{j=0}^{\infty} e^{j t} \frac{\lambda^{j}}{j!}=e^{\lambda\left(e^{t}-1\right)} \tag{75}
\end{equation*}
$$

this generating function is given by

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n, \lambda}(x) \frac{t^{n}}{n!}=e^{x t} e^{-\lambda\left(e^{t}-1\right)} \tag{76}
\end{equation*}
$$

where the polynomials also depend on the value of the real parameter $\lambda$. We expand the righthand side of this equation in much the same way as the intermediate step of (75) and bring the factor $e^{x t}$ within the sum. Then we expand in a Taylor series with respect to $t$ to obtain

$$
\begin{equation*}
e^{x t} e^{-\lambda\left(e^{t}-1\right)}=\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{(j+x)^{n} t^{n}}{n!} e^{\lambda} \frac{(-\lambda)^{j}}{j!} \tag{77}
\end{equation*}
$$

The sums on $j$ and $n$ are independent, and we switch their order. This yields

$$
\begin{equation*}
e^{x t} e^{-\lambda\left(e^{t}-1\right)}=\sum_{n=0}^{\infty} \frac{t^{n}}{n!} e^{\lambda} \sum_{j=0}^{\infty} \frac{(j+x)^{n}(-\lambda)^{j}}{j!} \tag{78}
\end{equation*}
$$

Now we expand $(j+x)^{n}$ using the binomial theorem, and rearrange to produce

$$
\begin{equation*}
e^{x t} e^{-\lambda\left(e^{t}-1\right)}=\sum_{n=0}^{\infty} \frac{t^{n}}{n!} \sum_{m=0}^{n}\binom{n}{m} x^{n-m} e^{-\lambda} \sum_{j=0}^{\infty} \frac{j^{m}(-\lambda)^{j}}{j!} . \tag{79}
\end{equation*}
$$

In the inner sum, replace $j^{m}$ using the following well-known formula [1, 24.1.4 B]:

$$
\begin{equation*}
j^{m}=\sum_{k=0}^{m} S(m, k)(j)_{m}=\sum_{k=0}^{m} S(m, k) j(j-1) \cdots(j-m+1) \tag{80}
\end{equation*}
$$

where the $S(m, k)$ are the Stirling numbers of the second kind. These are defined as the number of ways to partition a set of $m$ elements into $k$ nonempty subsets; [10] describes their use in combinatorics. The above identity allows us to reach a closed form for the inner sum.

$$
\begin{equation*}
e^{\lambda} \sum_{j=0}^{\infty} \frac{j^{m}(-\lambda)^{j}}{j!}=e^{\lambda} \sum_{k=0}^{m} S(m, k)(-\lambda)^{m} \sum_{j=m}^{\infty} \frac{(-\lambda)^{j-m}}{(j-m)!}=\sum_{j=0}^{m} S(m, j)(-\lambda)^{j} . \tag{81}
\end{equation*}
$$

The polynomial on the right is identified in [10] as the $m$ th Bell polynomial $B_{m}(-\lambda)$, and includes a similar derivation. We will, however, eschew this notation in order to avoid confusion with the Bernoulli polynomials. Finally, we have achieved a closed form for the Strodt polynomials:

$$
\begin{equation*}
P_{n, \lambda}(x)=\sum_{m=0}^{n}\binom{n}{m} \sum_{j=0}^{m} S(m, j)(-\lambda)^{j} x^{n-m} \tag{82}
\end{equation*}
$$

In the special cases $\lambda=-1$ and 1 , the inner sum will reduce to the $m$ th Bell number and complementary Bell number [10], respectively.

Again, we have a corollary of Theorems 2.4 and 2.3.
Corollary 4.5. Let $P_{n, \lambda}(x)$ be defined by (82) for $n \in \mathbb{N}_{0}$ and fixed $\lambda \in \mathbb{R}$. Then

$$
\begin{equation*}
e^{-\lambda} \sum_{j=0}^{\infty} P_{n, \lambda}(x+j) \frac{\lambda^{j}}{j!}=x^{n} \tag{83}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d x} P_{n, \lambda}(x)=n P_{n-1, \lambda}(x) \tag{84}
\end{equation*}
$$

Proof. We derived (82) from the power series

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n, \lambda}(x) \frac{t^{n}}{n!}=e^{x t-\lambda\left(e^{t}-1\right)}=e^{x t} / Q(t) \tag{85}
\end{equation*}
$$

thus by Theorem 2.4 the $P_{n, \lambda}(x)$ are the Strodt polynomials for the Poisson distribution with fixed parameter $\lambda$. Then (83), which reads $\mathcal{S}_{g}\left(P_{n, \lambda}(x)\right)=x^{n}$, follows by definition of Strodt polynomials and (84) is a consequence of Theorem 2.3.

## Example 4.6. Exponential Distribution.

For $\lambda>0$ we take the density function associated with the exponential distribution to be

$$
g(u):= \begin{cases}\lambda e^{-\lambda u} & \text { for } u>0  \tag{86}\\ 0 & u \leq 0\end{cases}
$$

and we calculate the moment generating function

$$
\begin{equation*}
Q(t)=\lambda \int_{0}^{\infty} e^{u t-\lambda u} d u=\frac{\lambda}{\lambda-t} \tag{87}
\end{equation*}
$$

for real $t<\lambda$. Therefore the generating function for the polynomials given by $\mathcal{S}_{g}\left(x^{n}\right)$, with regard to Theorem 2.4, is equal to $e^{x t} \frac{\lambda}{\lambda-t}$. Introduce the scalings $x^{\prime}:=\lambda x$ and $t^{\prime}:=t / \lambda$. This yields the formula

$$
\begin{equation*}
\sum_{n=0}^{\infty} \mathcal{S}_{g}\left(\left(\frac{x^{\prime}}{\lambda}\right)^{n}\right) \frac{\lambda^{n}\left(t^{\prime}\right)^{n}}{n!}=\frac{e^{x^{\prime} t^{\prime}}}{1-t^{\prime}} \tag{88}
\end{equation*}
$$

We thus have the following property of the images of $x^{n}$ under the Strodt operator for the exponential distribution.

Proposition 4.7. When $g(u)$ is the exponential density (86), the Strodt operator $\mathcal{S}_{g}$ applied to the monomial $x^{n}$ gives a scaled Taylor series truncation of the exponential function:

$$
\begin{equation*}
\mathcal{S}_{g}\left(x^{n}\right)=n!\lambda^{-n} \sum_{m=0}^{n} \frac{(\lambda x)^{m}}{m!}, \text { for all } n \in \mathbb{N}_{0}, \lambda>0 \tag{89}
\end{equation*}
$$

Proof. On the right hand side of (88), expand $e^{x^{\prime} t^{\prime}}$ in a Taylor series and $\left(1-t^{\prime}\right)^{-1}$ in a geometric series. Then match $\left(t^{\prime}\right)^{n}$ coefficients. Theorem 2.4 implies that

$$
\begin{equation*}
\mathcal{S}_{g}\left(\left(\frac{x^{\prime}}{\lambda}\right)^{n}\right) \frac{\lambda^{n}}{n!}=\sum_{m=0}^{n} \frac{\left(x^{\prime}\right)^{m}}{m!} \tag{90}
\end{equation*}
$$

Finally, replace $x^{\prime}$ with $\lambda x$.
We claim that the Strodt polynomials for the exponential distribution are given by

$$
\begin{equation*}
P_{0, \lambda}(x)=1, P_{n, \lambda}(x)=\frac{x^{n}}{n!}-\frac{x^{n-1}}{(n-1)!\lambda} \text { for } n \in \mathbb{N} . \tag{91}
\end{equation*}
$$

Indeed,

$$
\begin{align*}
\mathcal{S}_{g}\left(\frac{x^{n}}{n!}-\frac{x^{n-1}}{(n-1)!\lambda}\right) & =\frac{\mathcal{S}_{g}\left(x^{n}\right)}{n!}-\frac{\mathcal{S}_{g}\left(x^{n-1}\right)}{(n-1)!\lambda}  \tag{92}\\
& =\lambda^{-n} \sum_{m=0}^{n} \frac{(\lambda x)^{m}}{m!}-\lambda^{-n} \sum_{m=0}^{n-1} \frac{(\lambda x)^{m}}{m!}  \tag{93}\\
& =x^{n} \tag{94}
\end{align*}
$$

for $n \geq 1$, and $\mathcal{S}_{g}(1)=1$. The justification for (92) is the linearity of the Strodt operator, and (93) follows from (89).

## 5. Summation Formulae

The focus of Strodt's brief note [14] was to compare the summation formulae of Euler-MacLaurin and Boole. We now fill the details for a general formula and show how it can be specified to obtain either formula.

We begin the argument for a general density function. Let $z=a+h$ for $0<h<1$. For a fixed integer $m \geq 0$, define the remainder as

$$
\begin{equation*}
R_{m}(z):=f(z)-\sum_{k=0}^{m} \frac{\mathcal{S}_{g}\left(f^{(k)}(a)\right)}{k!} P_{k}(h) \tag{95}
\end{equation*}
$$

for a sufficiently smooth function $f$. The process of deriving a summation formula in general essentially reduces to finding an expression for $R_{m}(z)$ as an integral involving the Strodt polynomials corresponding to the operator $\mathcal{S}_{g}$.

Start with $m=0$. Since $P_{0}(z)=1$ and $\int g(u) d u=1$, we have

$$
\begin{equation*}
R_{0}(z)=f(a+h)-\int f(a+u) g(u) d u=\int[f(a+h)-f(a+u)] g(u) d u \tag{96}
\end{equation*}
$$

We rewrite the integrand in the right hand side as

$$
\begin{equation*}
f(a+h)-f(a+u)=\int_{u}^{h} f^{\prime}(a+s) d s \tag{97}
\end{equation*}
$$

assuming that $f$ has a continuous derivative. Using Fubini's theorem, we switch the order of integration. This yields

$$
\begin{equation*}
R_{0}(z)=\iint_{u}^{h} f^{\prime}(a+s) d s g(u) d u=\int V(s, h) f^{\prime}(a+s) d s \tag{98}
\end{equation*}
$$

where we define the piecewise function

$$
V(s, h):=\left\{\begin{array}{ll}
\int_{-\infty}^{s} g(u) d u & \text { for } s<h  \tag{99}\\
\int_{-\infty}^{s} g(u) d u-1 & \text { for } s \geq h
\end{array} .\right.
$$

At this point we separate the development into Euler-MacLaurin and Boole cases.
(a) For Boole Summation, $P_{n}(x)=E_{n}(x)$ and $g(u)=(\delta(u)+\delta(u+1)) / 2$. We calculate that

$$
2 \cdot V(s, h)=\left\{\begin{array}{ll}
1 & \text { for } 0 \leq s<h  \tag{100}\\
-1 & \text { for } h \leq s \leq 1 \\
0 & \text { otherwise }
\end{array}=\widetilde{E}_{0}(h-s) \chi_{[0,1]}(s)\right.
$$

where $\widetilde{E}_{0}(x)$ is the periodic Euler polynomial on $[0,1]$. Thus we have

$$
\begin{equation*}
f(a+h)=\mathcal{S}_{E}(f(a))+\frac{1}{2} \int_{0}^{1} f^{\prime}(a+s) \widetilde{E}_{0}(h-s) d s \tag{101}
\end{equation*}
$$

which corresponds to (5) in the special case $m=1$ and $n=a+1$. We view this as the core formula in Boole Summation, and the rest of (5) can be fleshed out by summing and integrating by parts.

We begin by rewriting (101) using the change of variables $x:=a+s$ in the integrand. Since $a \in \mathbb{N}, \widetilde{E}_{0}(a+h-s)=(-1)^{a} \widetilde{E}_{0}(h-s)$ and

$$
\begin{equation*}
f(a+h)=\frac{1}{2}(f(a)+f(a+1))+\frac{(-1)^{a}}{2} \int_{a}^{a+1} f^{\prime}(x) \widetilde{E}_{0}(h-x) d x \tag{102}
\end{equation*}
$$

Now take the alternating sum of both sides as $j$ ranges from $a$ to $n-1$. The sum telescopes and we combine the intervals of integration to obtain a single integral on $[a, n]$. This gives us

$$
\begin{equation*}
\sum_{j=a}^{n-1}(-1)^{j} f(j+h)=\frac{1}{2}\left((-1)^{a} f(a)+(-1)^{n-1} f(n)\right)+\frac{1}{2} \int_{a}^{n} f^{\prime}(x) \widetilde{E}_{0}(h-x) d x \tag{103}
\end{equation*}
$$

This is exactly (5) with $m=1$.
To complete (5) for a general positive integer $m \geq 1$ requires a proof by induction. An integration by parts confirms that

$$
\begin{align*}
\int_{a}^{n} f^{(k)}(x) \widetilde{E}_{k-1}(h-x) d s & =\frac{\widetilde{E}_{k}(h)}{k}\left((-1)^{n-1} f^{(k)}(n)+(-1)^{a} f^{(k)}(a)\right)  \tag{104}\\
& +\frac{1}{k} \int_{a}^{n} f^{(k+1)} \widetilde{E}_{k+1}(h-x) f^{(k+1)}(x) d x
\end{align*}
$$

for $k \geq 0$, which supplies the induction step.


$$
V(s, h)= \begin{cases}s & \text { for } 0 \leq s<h  \tag{105}\\ s-1 & \text { for } h \leq s \leq 1 \\ 0 & \text { otherwise }\end{cases}
$$

Therefore,

$$
\begin{align*}
R_{1}(z) & =\int_{0}^{1} V(s, h) f^{\prime}(a+s) d s-\int_{0}^{1} f^{\prime}(a+s) d s(h-1 / 2)  \tag{106}\\
& =\int_{0}^{1} \widetilde{B}_{1}(s-h) f^{\prime}(a+s) d s \tag{107}
\end{align*}
$$

This is owing to the observation $V(s, h)-(h-1 / 2)=\widetilde{B}_{1}(s-h) g(s)$. Hence we have

$$
\begin{align*}
f(a+h) & =\int_{0}^{1} f(s+a+h) d s+B_{1}(h)(f(a+1)-f(a))  \tag{108}\\
& +\int_{0}^{1} f^{\prime}(a+s) \widetilde{B}_{1}(s-h) d s
\end{align*}
$$

As in the Boole case, we are now essentially done. To completely recover (1), we simply integrate by parts and sum over consecutive integers.

Summing over the integers within the interval $[a, n-1]$ yields

$$
\begin{equation*}
\sum_{j=a}^{n-1} f(j+h)=\int_{a}^{n} f(x+h) d x+B_{1}(h)(f(n)-f(a))+\int_{a}^{n} f^{\prime}(x) \widetilde{B}_{1}(x-h) d x \tag{109}
\end{equation*}
$$

Here we have also shifted both intervals integration via $x:=a+s$. This formula is the initial case for an induction proof. Also,

$$
\begin{equation*}
\sum_{n=0}^{\infty} B_{n}(1-x) \frac{t^{n}}{n!}=\frac{t e^{(1-x) t}}{e^{t}-1}=\frac{t e^{-x t}}{1-e^{-t}}=\sum_{n=0}^{\infty} B_{n}(x) \frac{(-t)^{n}}{n!} \tag{110}
\end{equation*}
$$

whence we derive by analytic continuation the well-known fact $\widetilde{B}_{n}(-h)=B_{n}(1-h)=(-1)^{n} B_{n}(h)$ for all $n \geq 0$ [1, 23.1.8]. Now an integration by parts yields

$$
\begin{align*}
\int_{n}^{a} f^{(k)}(x) \widetilde{B}_{k}(x-h) d x & =\frac{(-1)^{k+1} B_{k+1}(h)}{k+1}\left(f^{(k)}(n)-f^{(k)}(a)\right)  \tag{111}\\
& -\frac{1}{k+1} \int_{n}^{a} f^{(k+1)}(x) \widetilde{B}_{k+1}(x) d x
\end{align*}
$$

for all $k \geq 1$, which provides the induction step. The result is

$$
\begin{align*}
\sum_{j=a}^{n-1} f(j+h) & =\int_{a}^{n} f(x+h) d x+\sum_{k=1}^{m} \frac{B_{k}(h)}{k!}\left(f^{(k-1)}(n)-f^{(k-1)}(a)\right)  \tag{112}\\
& +\frac{(-1)^{m+1}}{m!} \int_{a}^{n} f^{(m)}(x) \widetilde{B}_{m}(x-h) d x
\end{align*}
$$

This is a generalized version of (1), as one can see by taking the limit $h \rightarrow 0$.
(c) Taylor series approximation can be seen as another case in this general approach. Let $g(u)=$ $\delta(u)$, so that $\mathcal{S}_{g}\left(x^{n}\right)=x^{n}$. This means that $P_{n}(x)=x^{n}$. We will call this operator $\mathcal{S}_{1}$, to compare to (6) as well as to indicate that this is the identity operator. In this case,

$$
V(s, h):= \begin{cases}1 & \text { for } 0<s<h  \tag{113}\\ 0 & \text { otherwise }\end{cases}
$$

Thus we have

$$
\begin{equation*}
f(a+h)=f(a)+\int_{a}^{a+h} f^{\prime}(x) d x \tag{114}
\end{equation*}
$$

Here we have substituted $x:=a+s$ in the integral. We then use integration by parts to verify that

$$
\begin{equation*}
\int_{a}^{a+h}(x-a-h)^{k-1} f^{(k)}(x) d x=-\frac{(-h)^{k}}{k} f^{(k)}(a)-\int_{a}^{a+h}(x-a-h)^{k} f^{(k+1)}(x) d x \tag{115}
\end{equation*}
$$

for all $k \geq 1$. Therefore, by induction, for all $m \geq 0$,

$$
\begin{equation*}
f(a+h)=\sum_{k=0}^{m} f^{(k)}(a) \frac{h^{k}}{k!}+\frac{(-1)^{m}}{m!} \int_{a}^{a+h}(x-a-h)^{m} f^{(m+1)}(x) d x \tag{116}
\end{equation*}
$$

The derivations of Euler-MacLaurin and Boole summation formulae, seen in this way, appear essentially similar to that of Taylor series approximation. This comparison has been made in [7] in the Euler-MacLaurin and Taylor cases.

Remark 4. In each of the above cases, the proof relies on the fact that the function $V(s, h)$ can be related back to a polynomial from the original sequence. At present, we do not know how to extend this idea to a general Strodt polynomial.

## 6. Inversion Formulae

In this section we define the concept of a Strodt number and use it to generalize inversion formulae that exist for Bernoulli and Euler numbers. This generalization also allows us to recover such number sequences as the Bell numbers.

We begin with a definition. For $n \in \mathbb{N}_{0}$, define the $n$th Strodt number as the constant coefficient of the $n$th Strodt polynomial; that is,

$$
\begin{equation*}
P_{n}^{(g)}:=P_{n}^{(g)}(0) \tag{117}
\end{equation*}
$$

As before, we will routinely omit the superscript $g$ in the notation. As a result of (48), the generating function for this sequence is

$$
\begin{equation*}
\sum_{n=0}^{\infty} P_{n} \frac{t^{n}}{n!}=\frac{1}{Q(t)} \tag{118}
\end{equation*}
$$

where $Q(t)$ is as defined in (26).
We offer, as a motivation for these definitions, the apparent structure in the inversion formulas below [9, 24.5.9-10]. In each of the pairs of equations

$$
\begin{align*}
& a_{n}=\sum_{k=0}^{n}\binom{n}{k} \frac{b_{n-k}}{k+1}, \quad b_{n}=\sum_{k=0}^{n}\binom{n}{k} B_{k} a_{n-k} ;  \tag{119}\\
& a_{n}=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2 k} b_{n-2 k}, \quad b_{n}=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2 k} E_{2 k} a_{n-2 k} ; \tag{120}
\end{align*}
$$

the set of equations on the left taken for all $n \in \mathbb{N}_{0}$ will imply those on the right for all $n$, and vice versa. The $B_{n}$ and $E_{n}$ denote the $n$th Bernoulli and Euler numbers, respectively. We can derive both these formulae result from a general property of Strodt numbers.

Theorem 6.1. Let $P_{n}$ be the Strodt numbers associated with a probability measure $g(u)$ and $M_{k}$ its $k$ th moment as defined in (13). Then, each formula of the pair

$$
\begin{equation*}
a_{n}=\sum_{k=0}^{n}\binom{n}{k} b_{n-k} M_{k}, \quad b_{n}=\sum_{k=0}^{n}\binom{n}{k} a_{n-k} P_{k} \tag{121}
\end{equation*}
$$

taken for all $0 \leq n \leq m$, implies the other for all $0 \leq n \leq m$.
Proof. We show that each of the equations in (121) results from applying the Strodt operator to a polynomial.

For fixed $n \geq 0$, let $B(x):=\sum_{k=0}^{n}\binom{n}{k} b_{n-k} x^{k}$, and $A(x)$ be defined by $A:=\mathcal{S}_{g}(B)$. By Proposition 2.2, $A(x)$ is a degree- $n$ polynomial which can be represented by $A(x)=\sum_{k=0}^{n}\binom{n}{k} a_{n-k} x^{k}$, where the $a_{n-k}$ are implicitly defined. We now solve for them. Recalling (12), if $f(x)=\sum_{j=0}^{n} f_{j} x^{j}$ and $h=\mathcal{S}_{g}(f)$, then

$$
\begin{equation*}
h(x)=\sum_{j=0}^{k} h_{j} x^{j} \tag{122}
\end{equation*}
$$

where $h_{j}=\sum_{n=j}^{k} f_{n}\binom{n}{j} M_{n-j}$ and

$$
\begin{equation*}
M_{l}:=\int u^{l} d G \tag{123}
\end{equation*}
$$

With $B$ and $A$ in the place of $f$ and $g$, the result is

$$
\begin{equation*}
\sum_{j=0}^{n}\binom{n}{j} a_{n-j} x^{j}=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\binom{n}{k} b_{n-k}\binom{k}{j} M_{k-j}\right) x^{j} \tag{124}
\end{equation*}
$$

Solving for the $x^{j}$ coefficients, there are actually $n+1$ linear equations for the $a_{n-k}$ in terms of the $b_{n-k}$ that result. Supposing that an equation of this sort holds for $0 \leq n \leq m$, there are $\frac{1}{2}(m+1)(m+2)$ total equations for only $m+1$ unknowns, and we must check that they are consistent. Taking $\frac{d}{d x}$ of both sides of (124), we have

$$
\begin{equation*}
n \sum_{j=0}^{n-1}\binom{n-1}{j} a_{n-1-j} x^{j}=n \sum_{j=0}^{n-1}\left(\sum_{k=j}^{n-1}\binom{n-1}{k} b_{n-1-k}\binom{k}{j} M_{k-j}\right) x^{j} \tag{125}
\end{equation*}
$$

which is, after dividing by $n$, exactly the same equation with $n$ replaced by $n-1$. Thus the equations are consistent and any linearly independent subset of $m+1$ of them is equivalent to the entire set. In particular, evaluating (124) at $x=0$, we get a set of equations such that the $n$th contains $a_{n}$ but the $(n-1)$ st does not, for all $0 \leq n \leq m$. Therefore,

$$
\begin{equation*}
a_{n}=\sum_{k=0}^{n}\binom{n}{k} b_{n-k} M_{k}, \quad 0 \leq n \leq m \tag{126}
\end{equation*}
$$

is equivalent to $A=\mathcal{S}_{g}(B)$, where $A(x)=\sum_{k=0}^{n}\binom{n}{k} a_{n-k} x^{k}$ and $B(x):=\sum_{k=0}^{n}\binom{n}{k} b_{n-k} x^{k}$ for all $0 \leq n \leq m$. This is the left hand equation of (121).

On the other hand, we can derive another equivalent set of $m+1$ equations by writing out $B=\mathcal{S}_{g}^{(-1)}(A)$ in detail. We have

$$
\begin{equation*}
\sum_{k=0}^{n}\binom{n}{k} b_{n-k} x^{k}=\sum_{k=0}^{n}\binom{n}{k} a_{n-k} P_{k}(x) \tag{127}
\end{equation*}
$$

where the $P_{k}(x)$ are Strodt polynomials, for all $0 \leq n \leq m$. Set $x=0$ to obtain the left hand equation of (121):

$$
\begin{equation*}
b_{n}=\sum_{k=0}^{n}\binom{n}{k} a_{n-k} P_{k} \tag{128}
\end{equation*}
$$

We conclude that the set of these equations for $0 \leq n \leq m$ is equivalent to (126) via $A=$ $\mathcal{S}_{g}(B)$.

The Bernoulli numbers are given in $[1,23.1 .2]$ by the evaluation $B_{l}:=B_{l}(0)$. Thus they are the Strodt numbers for the Bernoulli operator. The moment $M_{k}$ is thus calculated as

$$
\begin{equation*}
M_{k}=\int_{0}^{1} u^{k} d u=\frac{1}{k+1} \tag{129}
\end{equation*}
$$

and (119) is recovered immediately from this and (121).
To recover the corresponding equation for Euler numbers, we consider the polynomials $\mathcal{E}_{n}(x)$ that are generated by the distribution $g(u)=\frac{1}{2}\left(\delta_{-1}(u)+\delta_{1}(u)\right)$. This distribution is a scaling of the distribution in the Euler polynomial case, as it averages the points -1 and 1 as opposed to 0 and 1. The generating function of the $\mathcal{E}_{n}(x)$ is

$$
\begin{equation*}
\sum_{n=0}^{\infty} \mathcal{E}_{n}(x) \frac{t^{n}}{n!}=\frac{2 e^{x t}}{e^{-t}+e^{t}} \tag{130}
\end{equation*}
$$

as one can check by using Theorem 2.4 and calculating

$$
\begin{equation*}
Q(t)=\frac{e^{(-1) \cdot t}+e^{1 \cdot t}}{2} \tag{131}
\end{equation*}
$$

The Strodt numbers corresponding to this distribution, $\mathcal{E}_{n}:=\mathcal{E}_{n}(0)$, are generated by

$$
\begin{equation*}
\sum_{n=0}^{\infty} \mathcal{E}_{n} \frac{t^{n}}{n!}=\frac{2}{e^{-t}+e^{t}}=\operatorname{sech}(t) \tag{132}
\end{equation*}
$$

This, however, is exactly the generating function for Euler numbers [9, 24.2.6] , and so $\mathcal{E}_{n}=E_{n}$ for all $n \geq 0$. At this point we also see (by the evenness of the hyperbolic secant function) that the odd Euler numbers are equal to zero. The $k$ th moment of this function is calculated as

$$
\begin{equation*}
\int u^{k} g(u) d u=\frac{1^{k}+(-1)^{k}}{2} \tag{133}
\end{equation*}
$$

and the pair in (121) become

$$
\begin{equation*}
a_{n}=\sum_{k=0}^{n}\binom{n}{k} b_{n-k}\left(\frac{1^{k}+(-1)^{k}}{2}\right) \quad \text { and } \quad b_{n}=\sum_{k=0}^{n}\binom{n}{k} a_{n-k} E_{k} . \tag{134}
\end{equation*}
$$

Now we verify that the odd terms in both of the above sums are equal to zero, the left because of vanishing odd moments, and the right because of vanishing odd Euler numbers. The result is (120).

Now that we have seen a uniform derivation of the inversion formulae for both Euler and Bernoulli numbers, we will develop a less obviously connected formula that is nonetheless a consequence of Proposition 6.1. We return to Example 4.4 in Section 4, where we calculated

$$
\begin{equation*}
P_{n, \lambda}(x)=\sum_{m=0}^{n}\binom{n}{m} \sum_{j=0}^{m} S(m, j)(-\lambda)^{j} x^{n-m} \tag{135}
\end{equation*}
$$

the Strodt polynomials generated by the Poisson measure, $g(u)=\sum_{j=0}^{\infty} \delta_{j}(u) e^{-\lambda} \frac{\lambda^{u}}{\Gamma(u+1)}$. Evaluating at $x=0$, we find that the Strodt number sequence is given by

$$
\begin{equation*}
P_{n}(\lambda)=P_{n, \lambda}(0)=\sum_{j=0}^{n} S(n, j)(-\lambda)^{j} \tag{136}
\end{equation*}
$$

which are Bell polynomials $\underline{10]}$. Computing the $k$ th moment of the Poisson distribution we obtain Bell polynomials again:

$$
\begin{equation*}
M_{k}(\lambda)=e^{\lambda} \sum_{j=0}^{\infty} \frac{j^{k} \lambda^{j}}{j!}=\sum_{j=0}^{k} S(n, j) \lambda^{j} \tag{137}
\end{equation*}
$$

The derivation of this identity is contained in Example 4.4. Now, empolying (121), we derive

$$
\begin{equation*}
a_{n}=\sum_{k=0}^{n}\binom{n}{k} b_{n-k}\left(\sum_{j=0}^{k} S(n, j) \lambda^{j}\right) \quad \text { and } \quad b_{n}=\sum_{k=0}^{n}\binom{n}{k} a_{n-k}\left(\sum_{j=0}^{k} S(n, j)(-\lambda)^{j}\right) \tag{138}
\end{equation*}
$$

as a new pair of inversion formulae. When $\lambda=1$, the moment becomes a sum across a row in the triangle of Stirling numbers, and the Strodt numbers become an alternating sum of the same. These are known $\underline{10]}$ as Bell numbers, $\mathcal{B}_{n}$, and complementary Bell numbers, $\widetilde{\mathcal{B}}_{n}$, respectively. Therefore, each of the pair

$$
\begin{equation*}
a_{n}=\sum_{k=0}^{n}\binom{n}{k} b_{n-k} \mathcal{B}_{k} \quad \text { and } \quad b_{n}=\sum_{k=0}^{n}\binom{n}{k} a_{n-k} \widetilde{\mathcal{B}}_{k} \tag{139}
\end{equation*}
$$

implies the other.


Figure 1. Graph of $\cos \frac{4 \pi}{3} x$ versus $(-1)^{\lfloor n / 2\rfloor-1} \frac{2^{2 n} \pi^{n+1}}{n!3^{n+3 / 2}} P_{n}(x-1 / 4)$ for $n=20$ and $n=40$

## 7. Asymptotic Properties and Conclusions

In [9] we find asymptotic formulae for Bernoulli and Euler polynomials as $n \rightarrow \infty$. Specifically,

$$
(-1)^{\lfloor n / 2\rfloor-1} \frac{(2 \pi)^{n}}{2(n!)} B_{n}(x) \rightarrow \begin{cases}\cos (2 \pi x), & n \text { even }  \tag{140}\\ \sin (2 \pi x), & n \text { odd }\end{cases}
$$

and

$$
(-1)^{\mathrm{L}(n+1) / 2\rfloor} \frac{\pi^{n+1}}{4(n!)} E_{n}(x) \rightarrow \begin{cases}\sin (\pi x), & n \text { even },  \tag{141}\\ \cos (\pi x), & n \text { odd },\end{cases}
$$

appear as 24.11 .5 and 24.11.6. The convergence is uniform in $x$ on compact subsets of $\mathbb{C}$.
Experimenting with plots for real $x$ as $n$ becomes increasingly large suggests that a similar asymptotic property is true for any Strodt uniform interpolation polynomial $P_{n}(x)$. As an example of our experimental results, Figure 1 displays the Strodt polynomials $P_{n}(x)$ for $g(u)=$ $\frac{1}{3}(\delta(u)+\delta(u-1 / 2)+\delta(u-1))$, which is the 3-point mean. We have displayed $n=20$ and $n=40$ for the Strodt polynomials, multiplied by a conjectured scaling factor $(-1)^{\lfloor n / 2\rfloor-1} \frac{2^{2 n} \pi^{n+1}}{n!3^{n+3 / 2}}$ and horizontally offset by $1 / 4$. They clearly appear to be converging to $\cos \frac{4 \pi}{3} x$.

Remark 5. We have not attempted proof of the asymptotics in this case. Nor do we conjecture the precise asymptotic formula for general interpolating polynomials, let alone for general Strodt polynomials. This is all the target of a 2007 Clemson Research Experience for Undergraduates project.

Our project started with the goal of elucidating Strodt's idea of using integral operators to derive the Euler-MacLaurin and Boole Summation formulae. Using Strodt operators to define the Bernoulli and Euler polynomials, we have ultimately suggested, largely by example, the variety of properties that can be recovered. This lends some important perspective. We see their properties not just as two isolated cases, but as part of a general context. They are Strodt polynomials, which we have defined as a special class of Appell sequences that satisfy a property involving Strodt operators. We see the worth of this additional property in the proofs of the summation and inversion formulae, and it may well assist in describing the asymptotics. Overall, the description of these polynomials by their images under integral operators enhances the connectivity of seemingly unrelated table identities.

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