

Contents lists available at ScienceDirect

# European Journal of Combinatorics



journal homepage: www.elsevier.com/locate/ejc

# A systematic approach to matrix forms of the Pascal triangle: The twelve triangular matrix forms and relations

Babiga Birregah<sup>a</sup>, Prosper K. Doh<sup>b</sup>, Kondo H. Adjallah<sup>c</sup>

<sup>a</sup> Institute Charles Delaunay, University of Technology of Troyes, France

<sup>b</sup> University of Nancy 2, France

<sup>c</sup> INRIA-COSTEAM, LGIPM, ENIM, France

# ARTICLE INFO

Article history: Received 25 May 2009 Accepted 10 September 2009 Available online 29 October 2009

# ABSTRACT

This work initiates a systematic investigation into the matrix forms of the Pascal triangle as mathematical objects in their own right. The present paper is especially devoted to the so-called G-matrices, i.e. the set of the twelve  $(n + 1) \times (n + 1)$  triangular matrix forms that can be derived from the Pascal triangle expanded to the level  $n \ (2 \le n \in \mathbb{N})$ . For n = 1, the G-matrix set reduces to a set of four distinct matrices. The twelve G-matrices are defined and the classic Pascal recursion is reformulated for each of the twelve G-matrices. Three sets of matrix transformations are then introduced to highlight different relations between the twelve G-matrices and for generating them from appropriately chosen subsets.

© 2009 Elsevier Ltd. All rights reserved.

# 1. Introduction

Various matrix forms of the Pascal triangle are encountered in the current literature. Doh [10] proposes the following matrix formulation for the classic binomial theorem:

$$(1 \quad (1+t) \quad \dots \quad (1+t)^n) = (1 \quad t \quad \dots \quad t^n) \underbrace{\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \dots & \dots & \begin{pmatrix} n \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \dots & \dots & \begin{pmatrix} n \\ 1 \end{pmatrix} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \begin{pmatrix} n \\ n \end{pmatrix} \end{pmatrix}}_{G_{1,n}}$$
(1)

E-mail address: adjallah@univ-metz.fr (K.H. Adjallah).

<sup>0195-6698/\$ –</sup> see front matter  $\textcircled{\sc 0}$  2009 Elsevier Ltd. All rights reserved. doi:10.1016/j.ejc.2009.10.009

Simple transposition of the above gives

$$\begin{pmatrix} 1\\(1+t)\\\vdots\\(1+t)^{n-1}\\(1+t)^n \end{pmatrix} = \underbrace{\begin{pmatrix} \begin{pmatrix} 0\\0 \end{pmatrix} & 0 & \cdots & 0\\ \\ \begin{pmatrix} 0\\1 \end{pmatrix} & \begin{pmatrix} 1\\1 \end{pmatrix} & 0 & \cdots & 0\\\vdots & \ddots & \ddots & \ddots & \vdots\\ \begin{pmatrix} n-1\\0 \end{pmatrix} & \begin{pmatrix} n-1\\1 \end{pmatrix} & \cdots & \begin{pmatrix} n-1\\n-1 \end{pmatrix} & 0\\ \begin{pmatrix} n\\0 \end{pmatrix} & \begin{pmatrix} n\\1 \end{pmatrix} & \cdots & \begin{pmatrix} n\\n-1 \end{pmatrix} & 0\\ \begin{pmatrix} n\\0 \end{pmatrix} & \begin{pmatrix} n\\1 \end{pmatrix} & \cdots & \begin{pmatrix} n\\n \end{pmatrix} \end{pmatrix}}_{G_{11,n}}$$
(2)

The foregoing exhibits two examples of triangular matrix forms of the Pascal triangle, namely  $G_{1,n}$  and  $G_{11,n}$  as defined in Table 1. Other examples are encountered in [10,18,8,3,9,11,4,13]. In general, authors refer to the particular form encountered simply as the Pascal matrix. Authors in [7,12,15,16, 2,17] refer to  $G_{1,n}$  and  $G_{11,n}$  respectively as the upper and lower triangular Pascal matrices.

In the sequel, the generic binomial coefficient  $\binom{a}{b}$  is defined by the relation

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!} & \text{if } b \le a\\ 0 & \text{otherwise} \end{cases}$$

We now turn to the definition of the twelve  $(n + 1) \times (n + 1)$  G-matrices

$$\mathcal{G}_n = \left\{ G_{k,n} | (1 \le k \le 12), (2 \le n \in \mathbb{N}) \right\}$$

and illustrate them for n = 3. We shall write  $[G_{k,n}]_{ij}$  to denote the entry at the intersection of row i and column j. In the sequel, row and column subscripts are assumed to run from 0 to n and the Pascal triangle to comprise levels  $0, \ldots, n$ .

**Definition 1.** The Table 1 below presents the definitions of the twelve G-matrices in  $g_n$ .

#### Table 1

Definitions of the generic entries of the twelve G-matrices.

$\left[G_{1,n}\right]_{ij} = \binom{j}{i}$	$\left[G_{2,n}\right]_{ij} = \binom{j}{n-i}$	$\left[G_{3,n}\right]_{ij} =$
for $i, j = 0,, n$	for $i, j = 0,, n$	$\begin{cases} \binom{n+j-i}{j} & \text{if } i \ge j \\ 0 & \text{otherwise} \end{cases}$
$\left[G_{4,n}\right]_{ij} = \binom{n-i}{j}$	$\left[G_{5,n}\right]_{ij} = \binom{n-i}{n-j}$	$\left[G_{6,n}\right]_{ij} =$
for $i, j = 0,, n$	for $i, j = 0,, n$	$\begin{cases} \binom{2n-i-j}{n-j} & \text{if } i+j \ge n\\ 0 & \text{otherwise} \end{cases}$
$\left[G_{7,n}\right]_{ij} = \binom{n-j}{n-i}$	$\left[G_{8,n}\right]_{ij} = \binom{n-j}{i}$	$\left[G_{9,n}\right]_{ij} =$
for $i, j = 0,, n$	for $i, j = 0,, n$	$\begin{cases} \binom{n+i-j}{i} & \text{if } i \le j \\ 0 & \text{otherwise} \end{cases}$
$\begin{bmatrix} G_{10,n} \end{bmatrix}_{ij} = \begin{pmatrix} i \\ n-j \end{pmatrix}$	$\left[G_{11,n}\right]_{ij} = \binom{i}{j}$	$\left[G_{12,n}\right]_{ij} =$
for $i, j = 0,, n$	for $i, j = 0,, n$	$\begin{cases} \binom{i+j}{i} & \text{if } i+j \le n \\ 0 & \text{otherwise} \end{cases}$

1206

Explicit application of the above definitions for n = 3 yields the twelve  $4 \times 4$  G-matrices as follows

$$\begin{split} G_{1,3} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad G_{2,3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad G_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ G_{4,3} &= \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad G_{5,3} = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad G_{6,3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ G_{7,3} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad G_{8,3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad G_{9,3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ G_{10,3} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{pmatrix} \quad G_{11,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \quad G_{12,3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{split}$$

In [18,14],  $G_{4,n}$ , is referred to as the binomial matrix. It is clear from the above definitions that  $G_{4,n}$  is just one of the three possible upper-left triangular matrix forms of the Pascal triangle. In [1],  $G_{7,n}$  and  $G_{11,n}$  are rightly characterized as image pair by reflection in the anti-diagonal.

Starting with  $G_{1,n}$  and  $G_{12,n}$ , for instance, one can generate the eleven remaining G-matrices through matrix transposition combined with left and right actions of the permutation matrix R as in Definition 2. These matrix forms correspond to some reordering of the components of the polynomial power basis vector, and hence of the polynomial space [10]. Forty-eight polynomial relations similar to those in (1) and (2) can be associated with twelve G-matrices.

The paper is organized as follows. Section 2 reformulates the Pascal recursion

$$\binom{i}{j} = \binom{i-1}{j-1} + \binom{i-1}{j}$$

in terms of each of the twelve G-matrices along with associated Pascal-like lattices to show the directions of propagation induced by the Pascal recursion. Section 3 introduces three sets of matrix transformations: the T-group, the set of circulant transformations and the set of median symmetry transformations. The T-group turns out to be a matrix transcription of the dihedral group  $D_4$ . It is shown that the action of the T-group on any of the G-matrix pairs in

$${G_{k,n}|k \in \{1, 2, 4, 5, 7, 8, 10, 11\}} \times {G_{k,n}|k \in \{3, 6, 9, 12\}}$$

partitions  $g_n$  into two orbits of four and eight vertices (Fig. 2). The four circulant transformations are referred to in the sequel as  $\alpha$ -,  $\beta$ -,  $\delta$ - and  $\gamma$ -circulant operators. Three iterations of any of the compositions in  $\{\alpha\gamma\delta\beta, \beta\alpha\gamma\delta, \delta\beta\alpha\gamma, \gamma\delta\beta\alpha\}$  on the appropriate initial G-matrix is shown to generate all the twelve matrices (see cyclic diagram in Fig. 4). The expansion of  $(\alpha\gamma\delta\beta)^3 G_{1,n}$  in particular generates the twelve G-matrices in the order in which we number them. Every square matrix has four characteristic triangular bipartitions (Fig. 5) each of which is invariant under three of the median symmetry transformations (including reflections in the main diagonal and in the antidiagonal). Circulant transformations carry a given triangular bipartition into a specific bipartition. Concluding Section 4 presents some of our findings and ongoing works.

# 2. Pascal recursion

Relations (3)–(14) below explicitly express the classic Pascal recursion in terms of the twelve G-matrices.

$$\mathcal{R}_{1,n}: \left[G_{1,n}\right]_{i,j} = \left[G_{1,n}\right]_{i-1,j-1} + \left[G_{1,n}\right]_{i,j-1} \tag{3}$$

$$\mathcal{R}_{2,n}: \left[G_{2,n}\right]_{i,j} = \left[G_{2,n}\right]_{i,j-1} + \left[G_{2,n}\right]_{i+1,j-1} \tag{4}$$

$$\mathcal{R}_{3,n}: \left[G_{3,n}\right]_{i,j} = \left[G_{3,n}\right]_{i,j-1} + \left[G_{3,n}\right]_{i+1,j}$$
(5)

$$\mathcal{R}_{4,n} : \left[ G_{4,n} \right]_{i,j} = \left[ G_{4,n} \right]_{i+1,j-1} + \left[ G_{4,n} \right]_{i+1,j} \tag{6}$$

$$\mathcal{R}_{5,n}: \left[G_{5,n}\right]_{i,j} = \left[G_{5,n}\right]_{i+1,j} + \left[G_{5,n}\right]_{i+1,j+1} \tag{7}$$

$$\mathcal{R}_{6,n}: \left[G_{6,n}\right]_{i,j} = \left[G_{6,n}\right]_{i+1,j} + \left[G_{6,n}\right]_{i,j+1} \tag{8}$$

$$\mathcal{R}_{7,n}: \left[G_{7,n}\right]_{i,j} = \left[G_{7,n}\right]_{i+1,j+1} + \left[G_{7,n}\right]_{i,j+1}$$
(9)

$$\mathcal{R}_{8,n}: \left[G_{8,n}\right]_{i,j} = \left[G_{8,n}\right]_{i-1,j+1} + \left[G_{8,n}\right]_{i,j+1} \tag{10}$$

$$\mathcal{R}_{9,n}: \left[G_{9,n}\right]_{i,j} = \left[G_{9,n}\right]_{i-1,j} + \left[G_{9,n}\right]_{i,j+1}$$
(11)

$$\mathcal{R}_{10,n}: \left[G_{10,n}\right]_{i,j} = \left[G_{10,n}\right]_{i-1,j} + \left[G_{10,n}\right]_{i-1,j+1}$$
(12)

$$\mathcal{R}_{11,n}: \left[G_{11,n}\right]_{i,j} = \left[G_{11,n}\right]_{i-1,j-1} + \left[G_{11,n}\right]_{i-1,j}$$
(13)

$$\mathcal{R}_{12,n}: \left[G_{12,n}\right]_{i,j} = \left[G_{12,n}\right]_{i-1,j} + \left[G_{12,n}\right]_{i,j-1}.$$
(14)

The twelve Pascal-like lattices of Fig. 1, where a circle stands for an entry of a G-matrix, highlight the direction of propagation of the Pascal recursion for each of the twelve G-matrices ([6]). The four columns of the figure also show the four characteristic bipartitions of a given square matrix into complementary triangular sub-blocks. The connected sub-blocks represent the major triangular sub-blocks and the unconnected sub-blocks the complementary minor sub-blocks.

# 3. Three sets of G-matrix transformations

# 3.1. The T-group

# 3.1.1. Basics

**Definition 2** (*The Reflection Matrix*). A reflection matrix of order (n + 1) is the  $(n + 1) \times (n + 1)$  matrix *R* defined by

 $[R]_{i,j} = \delta_{n-i}^j \qquad i, j = 0, \dots, n$ 

where  $\delta_{n-i}^{j}$  is the Kronecker symbol.

One can easily verify that *R* is a permutation matrix. Now, if the matrix *A* is of the same dimension as *R* then the generic components of the two matrices *AR* and *RA* are related to those of *A* as follows

$$[AR]_{i,j} = [A]_{i,n-j}$$
  $[RA]_{i,j} = [A]_{n-i,j}$ .

Referring to the table of G-matrix definitions, it takes little algebra to show in particular that

 $G_{2,n} = RG_{1,n},$   $G_{7,n} = RG_{1,n}R,$  $G_{8,n} = G_{1,n}R.$ 

# 3.1.2. Definition of the T-group

**Definition 3.** Let  $\mathcal{M}_{n+1}$  be the set of  $(n + 1) \times (n + 1)$  matrices and R the  $(n + 1) \times (n + 1)$  reflection matrix. The T-group is the set of the eight matrix transformations  $\mathcal{T} = \{T_k, k = 0, ..., 7\}$  defined on  $\mathcal{M}_{n+1}$  as follows.  $\forall A \in \mathcal{M}_{n+1}$ 

B. Birregah et al. / European Journal of Combinatorics 31 (2010) 1205-1216



Fig. 1. Twelve Pascal-like lattices associated to the 12 matrix forms.

$$T_0(A) = A$$
  

$$T_1(A) = RA$$
  

$$T_2(A) = RAR$$
  

$$T_3(A) = AR$$
  

$$T_4(A) = A^T$$
  

$$T_5(A) = A^T R$$
  

$$T_6(A) = RA^T R$$
  

$$T_7(A) = RA^T$$

where  $A^T$  is the transpose of A.

**Theorem 3.1.**  $(\mathcal{T}, \circ)$  is a group, where the symbol  $\circ$  denotes function composition.

**Proof.** This is easily proved by constructing the Cayley table of  $(\mathcal{T}, \circ)$ . For example:  $T_1 \circ T_2(A) = T_1(RAR) = R(RAR) = R^2AR = AR = T_3(A)$ 

**Theorem 3.2** ( $g_n$  orbits under T-action). The T-group partitions the set  $g_n$  into two orbits:

$$\mathcal{G}_n/\mathcal{T} = \left\{\mathcal{T}_{G_{1,n}}, \mathcal{T}_{G_{3,n}}\right\}$$

where  $\mathcal{T}_{G_{i,n}}$  denotes the orbit of  $G_{i,n}$  under the action of  $\mathcal{T}$ . More explicitly,

 $\mathcal{T}_{G_{i,n}} = \left\{ T_0 G_{i,n}, T_1 G_{i,n}, T_2 G_{i,n}, T_3 G_{i,n}, T_4 G_{i,n}, T_5 G_{i,n}, T_6 G_{i,n}, T_7 G_{i,n} \right\}.$ 

Proof. Direct calculation gives

$$\mathcal{T}_{G_{1,n}} = \left\{ G_{1,n}, G_{2,n}, G_{7,n}, G_{8,n}, G_{11,n}, G_{10,n}, G_{5,n}, G_{4,n} \right\}$$
$$\mathcal{T}_{G_{3,n}} = \left\{ G_{3,n}, G_{12,n}, G_{9,n}, G_{6,n} \right\}. \quad \blacksquare \tag{15}$$

As should be expected,

 $\mathcal{G}_n = \mathcal{T}_{G_{1,n}} \cup \mathcal{T}_{G_{3,n}}$  and  $\mathcal{T}_{G_{1,n}} \cap \mathcal{T}_{G_{3,n}} = \emptyset$ .



Fig. 2. Graph of the two orbits.

The T-group is simply a matrix transcription of the dihedral group  $D_4$ . Barbé [5] applied the same group using a different set of notations to formalize geometric transformations on binary difference patterns.

Fig. 2 shows the two orbits determined by the action of the T-group on  $g_n$ . Evidently with the T-group, one cannot attain all the twelve G-matrices starting from any particular one. Judicious compositions of the circulant transformations defined in Section 3.2 achieve this.

# 3.2. The circulant transformations

We define the circulant operators in terms of transformations of the generic matrix subscript vector (i, j) ( $0 \le i, j \le n$ ).

**Definition 4.** The  $\alpha$ -circulant operator

 $(i, j) \xrightarrow{\alpha} (i, i+j) \pmod{n+1}$ 

**Definition 5.** The  $\beta$ -circulant operator

 $(i, j) \xrightarrow{\beta} (i - j - 1, j) \pmod{n + 1}$ 

**Definition 6.** The  $\delta$ -circulant operator

 $(i,j) \xrightarrow{\delta} (i,i+j+1) \pmod{n+1}$ 

**Definition 7.** The  $\gamma$ -circulant operator

 $(i, j) \xrightarrow{\gamma} (i - j, j) \pmod{n+1}$ 

Fig. 3 illustrates the action of the  $\alpha$ -circulant operator on a square matrix.

**Theorem 3.3.** Action of the circulant transformations on  $\mathcal{G}_n$ 

If  $i \in \{1, 5, 9\}$  then  $\beta G_{i,n} = G_{i+1,n}$ If  $i \in \{2, 6, 10\}$  then  $\delta G_{i,n} = G_{i+1,n}$ If  $i \in \{3, 7, 11\}$  then  $\gamma G_{i,n} = G_{i+1,n}$ If  $i \in \{4, 8, 12\}$  then  $\alpha G_{i,n} = G_{i+1,n}$ where by convention  $G_{13} = G_1$ .



**Fig. 3.** Action of the  $\alpha$ -circulant operator.



**Fig. 4.** Circulant transformations and  $g_n$ .



Fig. 5. The four triangular bipartitions of a square matrix.

The diagram in Fig. 4 presents a dynamic application of Theorem 3.3 to the generation of the twelve G-matrices starting from any particular one. For example, starting with any  $G_{k,n} \in \{G_{1,n}, G_{5,n}, G_{9,n}\}$ , it takes a step-by-step expansion of  $(\alpha \gamma \delta \beta)^3 G_{k,n}$  which exhibits all the intermediate matrices to obtain the twelve G-matrices in their cyclic order.

The circulant transformations provide a useful tool for relating other square matrices derived from a Pascal-like Triangle whose entries are related by a generalized Pascal recursion of the form  $f_{i,j} = af_{i-1,j-1} + bf_{i-1,j}$  ([10], p. 91).

#### 3.3. Median symmetry transformations

The third set of transformations of interest formalizes reflections in the midpoints of sub-diagonals, sub-rows and sub-columns of the complementary triangular sub-blocks of the four triangular bipartitions of a given square matrix shown in Fig. 5. Since the midpoints are on the medians of the relevant triangular sub-blocks, we refer to them for short as median symmetry transformations. As reflections, median symmetry transformations are by definition self-inverse and fall into three categories as follows.

#### B. Birregah et al. / European Journal of Combinatorics 31 (2010) 1205–1216

#### 3.3.1. Diagonal median symmetry transformations

The first category of midpoint transformations is assigned the collective generic symbol  $\sigma$ . They are reflections in the midpoints of sub-diagonals of triangular sub-blocks and therefore preserve the diagonals of entry positions. They coincide with the T-group transformations  $T_4$  and  $T_6$  as follows

(i) 
$$\sigma_{Se} = \sigma_{Nw} = T_4$$
  
(ii)  $\sigma_{Ne} = \sigma_{Sw} = T_6$ .

For this reason, little will be said about them in the sequel.

#### 3.3.2. Row median symmetry transformations

These are reflections in the midpoints of the sub-rows of the two triangular sub-blocks of a given bipartition (cf. Section 3.3.3). There are four such reflections all of which globally preserve the row of matrix entries. They are defined as follows.

**Definition 8.** The  $\rho_{Se}$  operator

$$(i,j) \xrightarrow{\rho_{Se}} (i,-1+i-j) \pmod{n+1}.$$

**Definition 9.** The  $\rho_{Nw}$  operator

 $(i, j) \xrightarrow{\rho_{Nw}} (i, i-j) \pmod{n+1}.$ 

**Definition 10.** The  $\rho_{Sw}$  operator

$$(i,j) \xrightarrow{\rho_{Sw}} (i,-1-i-j) \pmod{n+1}.$$

**Definition 11.** The  $\rho_{Ne}$  operator

 $(i, j) \xrightarrow{\rho_{Ne}} (i, -2 - i - j) \pmod{n+1}.$ 

#### 3.3.3. Illustrating $\rho_{Se}$ : Application to 4 $\times$ 4 matrices

The definition of  $\rho_{Se}$  is based on the NE/sw bipartitioning of the initial 4 × 4 matrix A as shown below. The entries of the major sub-block are in bold characters while those of the minor sub-block are in normal characters.

<i>A</i> =	$(a_{00})$	<b>a</b> 01	a <sub>02</sub>	a <sub>03</sub> \	(*	a <sub>03</sub>	a <sub>02</sub>	<b>a</b> 01	a <sub>00</sub> \
	<i>a</i> <sub>10</sub>	a <sub>11</sub>	a <sub>12</sub>	a <sub>13</sub>	$\alpha$ (A) $-$	a <sub>10</sub>	a <sub>13</sub>	a <sub>12</sub>	a <sub>11</sub>
	a <sub>20</sub>	<i>a</i> <sub>21</sub>	<b>a</b> <sub>22</sub>	<b>a</b> 23	$\rho_{Se}(n) = 0$	a <sub>21</sub>	<i>a</i> <sub>20</sub>	<b>a</b> 23	a <sub>22</sub>
	$a_{30}$	<i>a</i> <sub>31</sub>	a <sub>32</sub>	a <sub>33</sub> /		a <sub>32</sub>	<i>a</i> <sub>31</sub>	a <sub>30</sub>	a <sub>33</sub> /

The sub-row midpoints for the  $\rho_{Se}$  reflections in the major *NE* triangular sub-block are on the median originating from the South-east entry position  $a_{33}$  and passing through the position  $a_{12}$ . The sub-row midpoints for the  $\rho_{Se}$  reflections in the minor *sw* triangular sub-block are on the median through the entry positions  $a_{10}$  and  $a_{31}$ .

#### 3.3.4. Column median symmetry transformations

These are reflections in the midpoints of the sub-columns of the two triangular sub-blocks of a given bipartition (see illustration in Fig. 6). There are four such reflections all of which globally preserve the column of matrix entries. They are defined as follows.

1212



**Fig. 6.** Axis of reflection for the  $\kappa_{Nw}$  operator.



**Fig. 7.** Graph of the actions of the three MST  $\kappa_{Nw}$ ,  $\rho_{Se}$  and  $\sigma_{Ne}$  on the NE triangular G-matrices: the NE/sw triangular bipartition is preserved.

**Definition 12.** The  $\kappa_{Nw}$  operator

 $(i,j) \xrightarrow{\kappa_{Nw}} (-i+j,j) \pmod{n+1}.$ 

**Definition 13.** The  $\kappa_{Ne}$  operator

$$(i,j) \xrightarrow{\kappa_{Ne}} (-1-i-j,j) \pmod{n+1}.$$

**Definition 14.** The  $\kappa_{Sw}$  operator

$$(i,j) \xrightarrow{\kappa_{SW}} (-2-i-j,j) \pmod{n+1}$$
.

**Definition 15.** The  $\kappa_{Se}$  operator

$$(i,j) \xrightarrow{\kappa_{Se}} (-1-i+j,j) \pmod{n+1}.$$

#### 3.3.5. Median symmetry transformations and canonical bipartitions of $g_n$

The major triangular sub-block of the canonical bipartition of a G-matrix coincides with the Pascal triangle while the minor sub-block entries are all zero. Inspection shows that the definition of any G-matrix coincides with that of its canonical bipartition. By definition every median symmetry transformation (MST) carries a particular triangular bipartition into itself. Figs. 7–10 highlight the partitioning of  $g_n$  according to canonical bipartitions along with the median symmetry transformations that preserve canonical bipartitions of the same structure. Fig. 7 for example shows that  $\kappa_{Nw}$ ,  $\rho_{Se}$  and  $\sigma_{Ne}$  carry a NE/sw canonical bipartition into another or the same NE/sw canonical bipartition. The three remaining figures show the three median symmetry transformations that carry a SE/nw, a SW/ne or NW/se canonical bipartition into another or the same SE/nw, SW/ne or NW/se canonical bipartition.



**Fig. 8.** Graph of the actions of the three MST  $\kappa_{Sw}$ ,  $\rho_{Ne}$  and  $\sigma_{Se}$  on the SE triangular G-matrices: the SE/nw triangular bipartition is preserved.



**Fig. 9.** Graph of the actions of the three MST  $\kappa_{Se}$ ,  $\rho_{Nw}$  and  $\sigma_{Sw}$  on the SW triangular G-matrices: the SW/ne triangular bipartition is preserved.



**Fig. 10.** Graph of the actions of the three MST  $\kappa_{Ne}$ ,  $\rho_{Sw}$  and  $\sigma_{Nw}$  on the NW triangular G-matrices: *the NW*/*se triangular bipartition is preserved.* 

The two T-group transformations  $T_1$  and  $T_3$  provide the link between the four graphs of Figs. 7–10 as shown in Fig. 11. The G-matrix at the vertex of a given triangle is invariant under the action of the MST labeling the opposite edge, e.g.,  $\kappa_{Nw}(G_1) = G_1$ .

### 4. Conclusion

This work initiates a systematic investigation into matrix forms of the Pascal triangle as mathematical objects in their own right. The present study focused on the G-matrix set comprising



Fig. 11. The T-group linkage of the four graphs of Figs. 7–10.

the twelve triangular matrix forms that can be derived from the Pascal triangle expanded to the level  $n \ge 2$ . Three sets of transformations were introduced to capture and formalize different ways the G-matrices relate to each other, transformations that readily extend to the space of arbitrary square matrices.

The T-group was shown to structure the space of square matrices of a given dimension into orbits of eight vertices. One interesting theoretical implication is that any isolated matrix identity is in fact a pointer to an eight-vertex orbit of identities. In addition, both the set of circulant transformations and of the median symmetry transformations can be shown to be closed under T-group conjugation.

Ongoing works include individual characterization and interpretation of the twelve G-matrices, systematization of two-factor product calculations, full matrix forms of the Pascal triangle and generalizations. Application areas of the mathematical tools developed include computational geometry, automata and combinatorics on formal compositions of the transformations presented in this paper.

#### References

- Lowell Abrams, Donniell E. Fishkind, Silvia Valdes-Leon, Reflecting the Pascal matrix about its main antidiagonal, Linear and Multilinear Algebra 47 (2000) 129–136.
- [2] Lidia Aceto, Donato Trigiante, The matrices of Pascal and other greats, American Mathematical Monthly 108 (3) (2001) 232–245.
- [3] Teijo Arponen, A matrix approach to polynomials, Linear Algebra and its Applications 359 (2002) 181–196.
- [4] Teijo Arponen, Matrix approach to polynomials 2, Linear Algebra and its Applications 394 (2005) 257–276.
- [5] André Barbé, Symmetric patterns in the cellular automaton that generates Pascal's triangle modulo 2, Discrete Applied Mathematics 105 (1–3) (2000) 1–38.
- [6] Babiga Birregah, Transformations géométriques sur les formes matricielles du triangle de pascal modèles d'automates à états finis et réseaux, Ph.D. Thesis Université de Lomé, 2007.
- [7] Geoff Boyd, Charles A. Micchelli, Gilbert Strang, Ding-Xuan Zhou, Binomial matrices, Advances in Computational Mathematics 14 (4) (2001) 379–391.
- [8] Gregory S. Call, Daniel J. Velleman, Pascal's matrices, The American Mathematical Monthly 100 (4) (1993) 372–376.
- [9] Gi-Sang Cheon, Suk-Geun Hwang, Seog-Hoon Rim, Seok-Zun Song, Matrices determined by a linear recurrence relation among entries, Linear Algebra and its Applications 373 (2003) 89–99.
- [10] Prosper K. Doh, Courbes paramétriques polynomiales et formes matricielles du théorème binomial: Nouveaux outils fondamentaux pour la conception et fabrication assistée par ordinateur, Thèse de doctorat d'État ès Sciences Mathématiques, Université de Nancy 1, France, 1988.
- [11] Alan Edelman, Gilbert Strang, Pascal's matrices, The American Mathematical Monthly 111 (3) (2004) 189-197.
- [12] Moawwad E.A. El-Mikkawy, On solving linear systems of the Pascal type, Applied Mathematics and Computation 136 (2003) 195–202.

- [13] Johannes Gmainer, Pascal's triangle, normal rational curves, and their invariant subspaces, European Journal of Combinatorics 22 (1) (2001) 37–49.
- [14] Eric Liverance, John Pitsenberger, Diagonalization of the binomial matrix, Fibonacci Quarterly 34 (1) (1996) 55–67.
- [15] Wang Xiang, Zhou Jituan, A fast eigenvalue algorithm for Pascal matrices, Applied Mathematics and Computation 183 (2) (2006) 711–716.
   [16] Wang Xiang, Lu Linzhang, A fast algorithm for solving linear systems of the Pascal type, Applied Mathematics and
- Computation 175 (2006) 441–451.
- [17] Sheng-liang Yang, Zhong-kui Liu, Explicit inverse of the Pascal matrix plus one, International Journal of Mathematics and Mathematical Sciences 2006 (2006) 1–7.
- [18] Zhizhen Zhang, Tianming Wang, Generalized Pascal matrix and recurrence sequences, Linear Algebra and its Applications 283 (1998) 289–299.