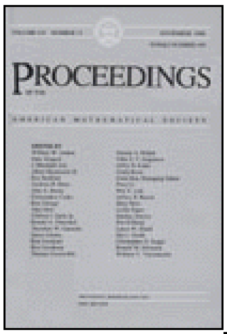

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JACOBI'S GENERATING FUNCTION FOR JACOBI POLYNOMIALS

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ABSTRACT. An idea of Hermite is used to give a simple proof of Jacobi's generating function for Jacobi polynomials.

One of the standard ways to prove the orthogonality of Legendre polynomials is to take their generating function

$$(1 - 2xr + r^2)^{-1/2} = \sum^{\infty} P_n(x) r^n \tag{1}$$

$n=0$

and show that the integral

$$\int_{-1}^{1}(1-2xr+r^2)^{-1/2}(1-2xs+s^2)^{-1/2}\,dx$$

is a function of the variable rs. See, for example, Courant-Hilbert [**1**, pp. 85–86]. This proof was given by Legendre [**6**, p. 250].

Jacobi gave a generating function for a more general set of orthogonal polynomials $P_n^{(\alpha,\beta)}(x)$. These polynomials can be defined by

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\;\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \quad (2)$$

It is easy to use (2) and integration by parts to prove

$$\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx=0, \qquad m\neq n,\ \alpha,\beta>-1. \quad (3)$$

Jacobi's generating function is

$$\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)r^n=2^{\alpha+\beta}R^{-1}(1-r+R)^{-\alpha}(1+r+R)^{-\beta} \quad (4)$$

where $R=(1-2xr+r^2)^{1/2}$. His original proof used Lagrange's extension of Taylor's theorem. A second proof of this generating function was given a few years later by Tchebychef [**9**]. His proof was modeled on Legendre's proof mentioned above. He seems to have found this generating function independently from Jacobi, since he does not mention Jacobi's paper. Tchebychef's proof is a very complicated one which involves a number of changes of variables to reduce the integral

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Abstract

An idea of Hermite is used to give a simple proof of Jacobi's generating function for Jacobi polynomials.