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An involutory Pascal matrix

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Abstract

An involutory upper triangular Pascal matrix U_n is investigated. Eigenvectors of U_n and of U_n^{T} are considered. A characterization of U_n is obtained, and it is shown how the results can be extended to matrices over a commutative ring with unity.

Keywords: Pascal matrices; Involutory matrices; Eigenvectors; Matrices over a ring

1. Introduction

Let $U_n = (u_{ij})$ be the real upper triangular matrix of order n with

$$u_{ij} = (-1)^{i-1}\binom{j-1}{i-1} \quad \text{for } 1 \leqslant i \leqslant j \leqslant n.$$

For example,

$$U_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Such Pascal matrices are found in the book by Klein [2]. Moreover, the MATLAB command pascal$(n, 1)$ yields the lower triangular matrix U_n^{T}.

Klein mentioned that $U_n^{-1} = U_n$ (that is, U_n is involutory). In fact a somewhat more general result holds. Let p and q be integers with $1 \leqslant p \leqslant q \leqslant n$. Using the identity

$$\delta_{nk} = \sum_{j=k}^{n} (-1)^{j+k} \binom{n}{j} \binom{j}{k},$$

which can be found on page 44 of [3], it is not difficult to see that the principal submatrix of U_n that lies in rows and column $p, p+1, \ldots, q$ is involutory.

The matrix U_n is closely related to two other "Pascal matrices". Let $P_n = (p_{ij})$ be the real lower triangular matrix of order n with

$$p_{ij} = \binom{j-1}{i-1} \quad \text{for } 1 \leqslant i \leqslant j \leqslant n,$$

and let $S_n = (s_{ij})$ be the real symmetric matrix of order n with

$$s_{ij} = \binom{i+j-2}{j-1} \quad \text{for } i, j = 1, 2, \ldots, n.$$

Clearly $P_n = U_n^{\mathrm{T}} D_n$ for the $n \times n$ diagonal matrix $D_n = ((-1)^{i-1}\delta_{ij})$. Hence, it follows from the Cholesky factorization $S_n = P_n P_n^{\mathrm{T}}$ obtained by Brawer and Pirovino [1] that $S_n = (U_n^{\mathrm{T}} D_n)(U_n^{\mathrm{T}} D_n)^{\mathrm{T}} = U_n^{\mathrm{T}} U_n$. Thus, the involutory matrices U_n^{T} and U_n can be used to obtain an LU factorization for S_n.

Other properties of U_n are investigated in this paper. Eigenvectors of U_n and of U_n^{T} are considered in Section 2. A characterization of U_n is presented next, and then it is shown how the results can be extended to matrices over a commutative ring with unity.

2. Eigenvectors

It is easy to see that U_n is similar to the diagonal matrix $D_n = ((-1)^{i-1}\delta_{ij})$. We now consider eigenvectors of U_n. For each positive integer k, let

$$x_k = \begin{bmatrix} \binom{k}{0} \\ -\binom{k}{1} \\ \vdots \\ (-1)^{k-1}\binom{k}{k-1} \end{bmatrix}.$$

Lemma 2.1. *For each positive integer k, x_k is an eigenvector of U_k corresponding to the eigenvalue $(-1)^{k-1}$.*

Proof. Since

$$U_{k+1} = \begin{bmatrix} U_k & x_k \\ 0 & (-1)^k \end{bmatrix},$$

we have

$$I = U_{k+1}^2 = \begin{bmatrix} I & U_k x_k + (-1)^k x_k \\ 0 & 1 \end{bmatrix}$$

and thus $U_k x_k = (-1)^{k-1} x_k$. $\square$

For integers $1 \leqslant k \leqslant n$ we define the vector $y_{nk} \in \mathbb{R}^n$ by letting

$$y_{nk} = \begin{bmatrix} x_k \\ 0 \end{bmatrix}.$$

Let $Y_{n1} = \{y_{nk} : k \text{ is odd}\}$ and $Y_{n2} = \{y_{nk} : k \text{ is even}\}$.

Theorem 2.2. *The set Y_{n1} is a basis for the eigenspace of U_n corresponding to the eigenvalue* 1, *and Y_{n2} is a basis for the eigenspace of U_n corresponding to the eigenvalue* -1 (*when* $n \geqslant 2$).

Proof. Lemma 2.1 implies that $y_{nn} = x_n$ is an eigenvector of U_n corresponding to the eigenvalue $(-1)^{n-1}$. Let $1 \leqslant k < n$. Partition U_n as

$$U_n = \begin{bmatrix} U_k & A \\ 0 & B \end{bmatrix}.$$

Using Lemma 2.1, we see that

$$U_n y_{nk} = \begin{bmatrix} U_k & A \\ 0 & B \end{bmatrix} \begin{bmatrix} x_k \\ 0 \end{bmatrix} = \begin{bmatrix} (-1)^{k-1} x_k \\ 0 \end{bmatrix} = (-1)^{k-1} y_{nk}.$$

Hence, for each $1 \leqslant k \leqslant n$, y_{nk} is an eigenvector of U_n corresponding to the eigenvalue $(-1)^{k-1}$. Moreover, it is easy to see that Y_{n1} and Y_{n2} are linearly independent sets. $\square$

Let $H_n = (h_{ij})$ be the upper triangular matrix with

$$h_{ij} = (-1)^{i+j} \binom{j-1}{i-1} 2^{i-1} \quad \text{for } 1 \leqslant i \leqslant j \leqslant n,$$

and let $M_n = (m_{ij})$ be the lower triangular matrix with

$$m_{ij} = \binom{i-1}{j-1} 2^{n-j} \quad \text{for } 1 \leqslant j \leqslant i \leqslant n.$$

For example,

$$H_6 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 2 & -4 & 6 & -8 & 10 \\ 0 & 0 & 4 & -12 & 24 & -40 \\ 0 & 0 & 0 & 8 & -32 & 80 \\ 0 & 0 & 0 & 0 & 16 & -80 \\ 0 & 0 & 0 & 0 & 0 & 32 \end{bmatrix},$$

$$M_6 = \begin{bmatrix} 32 & 0 & 0 & 0 & 0 & 0 \\ 16 & 16 & 0 & 0 & 0 & 0 \\ 8 & 16 & 8 & 0 & 0 & 0 \\ 4 & 12 & 12 & 4 & 0 & 0 \\ 2 & 8 & 12 & 8 & 2 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}.$$

It will be shown that the columns of H_n are eigenvectors of U_n, and that the columns of M_n are eigenvectors of U_n^{T}.

Lemma 2.3. *For each positive integer n, $U_n H_n = H_n D_n$.*

Proof. Clearly $U_n H_n = (a_{ij})$ and $H_n D_n = (b_{ij})$ are upper triangular. For $1 \leqslant i \leqslant j \leqslant n$, we see that

$$\begin{aligned} a_{ij} &= \sum_{k=i}^{j} (-1)^{i-1} \binom{k-1}{i-1} (-1)^{k+j} \binom{j-1}{k-1} 2^{k-1} \\ &= (-1)^{i+1} \sum_{k=i-1}^{j-1} (-1)^{k+j-1} \binom{k}{i-1} \binom{j-1}{k} 2^{k} \\ &= (-1)^{i+2j-1} \binom{j-1}{i-1} 2^{i-1} \\ &= b_{ij}, \end{aligned}$$

where we used the identity

$$\sum_{k=m}^{n} (-1)^{n+k} \binom{n}{k} \binom{k}{m} 2^{k-m} = \binom{n}{m},$$

which can be found on page 32 of [3]. □

The columns of H_n yield different bases for the eigenspaces of U_n than those given in Theorem 2.2. Let $V_{n1} = \{v_{nk} : k \text{ is odd}\}$ and $V_{n2} = \{v_{nk} : k \text{ is even}\}$, where v_{nk} is the kth column of H_n.

Theorem 2.4. *The set V_{n1} is a basis for the eigenspace of U_n corresponding to the eigenvalue* 1, *and V_{n2} is a basis for the eigenspace of U_n corresponding to the eigenvalue* -1.

Proof. Lemma 2.3 implies that v_{nk} is an eigenvector of U_n corresponding to the eigenvalue $(-1)^{k-1}$. Moreover, V_{n1} and V_{n2} are linearly independent sets. $\square$

We now consider eigenvectors of U_n^{T}. Let $W_{n1} = \{w_{nk} : k \text{ is odd}\}$ and $W_{n2} = \{w_{nk} : k \text{ is even}\}$, where w_{nk} is the kth column of M_n. Define the diagonal matrices Q_n and R_n of order n by letting $Q_n = (2^{i-1}\delta_{ij})$ and $R_n = (2^{n-i}\delta_{ij})$.

Lemma 2.5. *For each positive integer n, $M_n = 2^{n-1}(H_n^{\mathrm{T}})^{-1}$.*

Proof. We see that $H_n = Q_n U_n D_n$ and $M_n = R_n U_n^{\mathrm{T}} D_n$. Hence, using $D_n^2 = I = U_n^2$, it follows that

$$M_n H_n^{\mathrm{T}} = (R_n U_n^{\mathrm{T}} D_n)(D_n U_n^{\mathrm{T}} Q_n) = R_n Q_n = 2^{n-1} I,$$

and thus $M_n = 2^{n-1}(H_n^{\mathrm{T}})^{-1}$. $\square$

Lemma 2.6. *For each positive integer n, $U_n^{\mathrm{T}} M_n = M_n D_n$.*

Proof. Using Lemma 2.3, we see that

$$U_n^{\mathrm{T}}(H_n^{\mathrm{T}})^{-1} = ((U_n H_n)^{\mathrm{T}})^{-1} = ((H_n D_n)^{\mathrm{T}})^{-1} = (H_n^{\mathrm{T}})^{-1} D_n,$$

and it follows from Lemma 2.5 that $U_n^{\mathrm{T}} M_n = M_n D_n$. $\square$

Theorem 2.7. *The set W_{n1} is a basis for the eigenspace of U_n^{T} corresponding to the eigenvalue* 1, *and W_{n2} is a basis for the eigenspace of U_n^{T} corresponding to the eigenvalue* -1.

Proof. Lemma 2.6 implies that w_{nk} is an eigenvector of U_n^{T} corresponding to the eigenvalue $(-1)^{k-1}$. Moreover, W_{n1} and W_{n2} are linearly independent. $\square$

3. A characterization of U_n

Let $K_n = (k_{ij})$ be the (0,1)-matrix of order n with $k_{ij} = 1$ if and only if $j = i + 1$, and let $G_n = (g_{ij}) = U_n + (K_n^{\mathrm{T}} U_n - U_n)K_n$. An easy computation shows that G_n is a (0,1)-matrix with $g_{ij} = 1$ if and only if $i = j = 1$. Thus G_n is a symmetric matrix. We will show that such symmetry and the property that each leading principal submatrix is involutory characterizes $\pm U_n$ for $n \geqslant 4$. The following lemmas will be used.

Lemma 3.1. *Let $X = (x_{ij})$ be an involutory matrix of order 2 such that $x_{11} = 1$, $X + (K_2^{\mathrm{T}}X - X)K_2$ is symmetric and $X \neq U_2$. Then*

$$X = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Proof. We see that

$$X + (K_2^{\mathrm{T}}X - X)K_2 = \begin{bmatrix} 1 & x_{12} - 1 \\ x_{21} & 1 - x_{21} + x_{22} \end{bmatrix}.$$

Since this matrix is symmetric and $X^2 = I$, it follows that

$$X = \begin{bmatrix} 1 & x_{12} \\ x_{12} - 1 & -1 \end{bmatrix},$$

where $x_{12} = 1$ or $x_{12} = 0$. □

Lemma 3.2. *Let X be a matrix of order $n \geqslant 3$ and let Y be the leading principal submatrix of X of order $n - 1$. If $X + (K_n^{\mathrm{T}}X - X)K_n$ is symmetric then $Y + (K_{n-1}^{\mathrm{T}}Y - Y)K_{n-1}$ is symmetric.*

Proof. Partition K_n and X as

$$K_n = \begin{bmatrix} K_{n-1} & L \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} Y & C \\ R & d \end{bmatrix}.$$

We see that

$$X + (K_n^{\mathrm{T}}X - X)K_n = \begin{bmatrix} Y + (K_{n-1}^{\mathrm{T}}Y - Y)K_{n-1} & C + (K_{n-1}^{\mathrm{T}}Y - Y)L \\ R + (L^{\mathrm{T}}Y - R)K_{n-1} & d + (L^{\mathrm{T}}Y - R)L \end{bmatrix}.$$

□

Lemma 3.3. *Let $X = (x_{ij})$ be a matrix of order 3 such that each leading principal submatrix of X is involutory, $x_{11} = 1$, $X + (K_3^{\mathrm{T}}X - X)K_3$ is symmetric and $X \neq U_3$. Then*

$$X = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. It follows from Lemmas 3.1 and 3.2 that $X = X_1$ or $X = X_2$ where

$$X_1 = \begin{bmatrix} 1 & 1 & x_{13} \\ 0 & -1 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 0 & x_{13} \\ -1 & -1 & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}.$$

In both cases, since $X^2 = I$, we see that either $x_{13} = x_{23} = 0$ or $x_{31} = x_{32} = 0$. First suppose that $X = X_1$. It then follows that

$$G = X + (K_3^{\mathrm{T}}X - X)K_3 = \begin{bmatrix} 1 & 0 & x_{13} - 1 \\ 0 & 0 & x_{23} + 2 \\ x_{31} & x_{32} - x_{31} & x_{33} - 1 - x_{32} \end{bmatrix}.$$

Since G is symmetric, if $x_{13} = x_{23} = 0$, then $x_{31} = -1$ and $x_{32} = 1$. However, this would imply that $X^2 \neq I$. Hence, we must have $x_{31} = x_{32} = 0$. Therefore, since G is symmetric, we see that $x_{13} = 1$ and $x_{23} = -2$. It now follows that $X = U_3$. Thus we assume that $X = X_2$, and it follows that

$$G = X + (K_3^{\mathrm{T}}X - X)K_3 = \begin{bmatrix} 1 & -1 & x_{13} \\ -1 & 1 & x_{23} + 1 \\ x_{31} & x_{32} - 1 - x_{31} & x_{33} - 1 - x_{32} \end{bmatrix}.$$

Since G is symmetric, if $x_{13} = x_{23} = 0$, then $x_{31} = 0$ and $x_{32} = 2$. However, this would imply that $X^2 \neq I$. Hence, we must have $x_{31} = x_{32} = 0$. Therefore, since G is symmetric, we see that $x_{13} = 0$ and $x_{23} = -2$. It now follows that

$$X = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Lemma 3.4. *Let $X = (x_{ij})$ be a matrix of order 4 such that each leading principal submatrix of X is involutory, $x_{11} = 1$, and $X + (K_4^{\mathrm{T}}X - X)K_4$ is symmetric. Then $X = U_4$.*

Proof. It follows from Lemmas 3.2 and 3.3 that $X = X_1$ or $X = X_2$ where

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & x_{14} \\ -1 & -1 & -2 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 1 & 1 & x_{14} \\ 0 & -1 & -2 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}.$$

In both cases, since $X^2 = I$, we see that either $x_{14} = x_{24} = x_{34} = 0$ or $x_{41} = x_{42} = x_{43} = 0$. First suppose that $X = X_1$. It then follows that,

$$G = X + (K_4^{\mathrm{T}}X - X)K_4 = \begin{bmatrix} 1 & -1 & 0 & x_{14} \\ -1 & 1 & -1 & x_{24} + 2 \\ 0 & -1 & 0 & x_{34} - 3 \\ x_{41} & x_{42} - x_{41} & x_{43} - x_{42} & x_{44} + 1 - x_{43} \end{bmatrix}.$$

Since G is symmetric, if $x_{14} = x_{24} = x_{34} = 0$, then $x_{41} = 0$, $x_{42} = 2$ and $x_{43} = -1$. However, this would imply that $X^2 \neq I$. Moreover, since G is symmetric, if $x_{41} = x_{42} = x_{43} = 0$, then $x_{14} = 0$, $x_{24} = -2$ and $x_{34} = 3$. However, this would imply that $X^2 \neq I$. Thus we must have $X = X_2$, and it follows that

$$G = X + (K_4^{\mathrm{T}}X - X)K_4 = \begin{bmatrix} 1 & 0 & 0 & x_{14} - 1 \\ 0 & 0 & 0 & x_{24} + 3 \\ 0 & 0 & 0 & x_{34} - 3 \\ x_{41} & x_{42} - x_{41} & x_{43} - x_{42} & x_{44} + 1 - x_{43} \end{bmatrix}.$$

Since G is symmetric, if $x_{14} = x_{24} = x_{34} = 0$, then $x_{41} = -1$ and $x_{42} = 2$. However, this would imply that $X^2 \neq I$. Hence, we must have $x_{41} = x_{42} = x_{43} = 0$. Therefore, since G is symmetric, we see that $x_{14} = 1$, $x_{24} = -3$ and $x_{34} = 3$. It now follows that $X = U_4$. $\square$

Lemma 3.5. *Let $X = (x_{ij})$ be a matrix of order $n \geqslant 4$ such that each leading principal submatrix of X is involutory, $x_{11} = 1$, and $X + (K_n^{\mathrm{T}}X - X)K_n$ is symmetric. Then $X = U_n$.*

Proof. We use induction on n. Lemma 3.4 implies that Lemma 3.5 holds for $n = 4$. Let X be a matrix of order $n \geqslant 5$ that satisfies the hypotheses of Lemma 3.5 and suppose that this lemma holds for matrices of order $n-1$. Using Lemma 3.2, we see that

$$X = \begin{bmatrix} U_{n-1} & C \\ R & x_{nn} \end{bmatrix}$$

for some $1 \times (n-1)$ matrix R and $(n-1) \times 1$ matrix C. Since $X^2 = I$, it follows that $x_{in}x_{nj} = 0$ for $i, j = 1, 2, \ldots, n-1$. This implies that either $C = 0$ or $R = 0$. Let $G = (g_{ij}) = X + (K_n^{\mathrm{T}}X - X)K_n$. It is not difficult to see that

$$\begin{aligned} g_{n1} &= x_{n1}, \\ g_{nj} &= x_{nj} - x_{n,j-1} \quad \text{for } j = 2, 3, \ldots, n-1, \\ g_{in} &= x_{in} - (-1)^{i-1}\binom{n-1}{i-1} \quad \text{for } i = 1, 2, \ldots, n-1. \end{aligned}$$

Since G is symmetric, if $C = 0$, then it follows that $x_{n1} = -1$ and $x_{n2} = n-2$. However, we see that now there is no value of x_{nn} that will ensure that both the $(n, 1)$ and the $(n, 2)$ entries of X^2 are zero. Thus $X^2 \neq I$. Hence, we must have $R = 0$. Therefore, since G is symmetric, we see that $g_{in} = 0$ for $i = 1, 2, \ldots, n-1$. Thus

$$x_{in} = (-1)^{i-1}\binom{n-1}{i-1} \quad \text{for } i = 1, 2, \ldots, n-1,$$

and it follows that $X = U_n$. $\square$

Theorem 3.6. *Let X be a matrix of order $n \geqslant 4$. Then $X = \pm U_n$ if and only if $X + (K_n^{\mathrm{T}}X - X)K_n$ is symmetric and each leading principal submatrix of X is involutory.*

Proof. Suppose that $X + (K_n^{\mathrm{T}}X - X)K_n$ is symmetric and that each leading principal submatrix of $X = (x_{ij})$ is involutory. Then $x_{11} = \pm 1$. If $x_{11} = 1$, then Lemma 3.5 implies that $X = U_n$. If $x_{11} = -1$, let $Y = (y_{ij}) = -X$. Then each leading principal submatrix of Y is involutory, $y_{11} = 1$ and $Y + (K_n^{\mathrm{T}}Y - Y)K_n$ is symmetric. Hence, Lemma 3.5 implies that $X = -Y = -U_n$. Therefore, if $X + (K_n^{\mathrm{T}}X - X)K_n$

is symmetric and each leading principal submatrix of X is involutory, then $X = \pm U_n$. As discussed earlier, it is easy to see that the converse also holds. □

4. Extensions to matrices over a ring

Let A be a matrix of order n over a commutative ring R with unity e, let $\lambda \in R$, and let x be a nonzero column vector over R. We say that λ is an eigenvalue of A with corresponding eigenvector x if $Ax = \lambda x$. Since U_n and the vectors in Y_{ni}, V_{ni}, and W_{ni} have integer entries, we can obtain the corresponding matrices and vectors over R by replacing each entry k by ke. Thus we have the following.

Theorem 4.1. *Let R be a commutative ring with unity e.*

(a) *Each vector in $Y_{n1}(V_{n1})$ is an eigenvector of U_n corresponding to the eigenvalue e.*
(b) *Each vector in $Y_{n2}(V_{n2})$ is an eigenvector of U_n corresponding to the eigenvalue $-e$.*
(c) *Each vector in W_{n1} is an eigenvector of U_n^{T} corresponding to the eigenvalue e.*
(d) *Each vector in W_{n2} is an eigenvector of U_n^{T} corresponding to the eigenvalue $-e$.*

There are difficulties in attempting such an extension of our characterization of U_n. For example, Lemma 3.1 cannot be extended to general commutative rings with unity. To see this, let $k \geqslant 2$ be an integer, and let $m = k(k+1)$. Over the ring $\mathbb{Z}_m$ of integers modulo m, the matrix

$$X = \begin{bmatrix} 1 & k+1 \\ k & m-1 \end{bmatrix}$$

is involutory with $X + (K_2^{\mathrm{T}} X - X)K_2$ symmetric. However, it is not difficult to obtain the following extension of Theorem 3.6.

Theorem 4.2. *Let D be an integral domain, and let X be a matrix over D of order $n \geqslant 4$. Then $X = \pm U_n$ if and only if $X + (K_n^{\mathrm{T}} X - X)K_n$ is symmetric and each leading principal submatrix of X is involutory.*

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