Weighted de Bruijn Graphs for the Menage Problem and Its Generalizations

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Abstract

We address the problem of enumeration of seating arrangements of married couples around a circular table such that no spouses seat next to each other and no k consecutive persons are of the same gender. While the case of $k = 2$ corresponds to the classical *problème des ménages* with a well-studied solution, no closed-form expression for number of arrangements is known when $k \geq 3$.

We propose a novel approach to this type of problems based on enumeration of circuits in certain algebraically weighted de Bruijn graphs. Our approach leads to a new expression for the menage numbers and their exponential generating function, and allows one to efficiently compute the number of seating arrangements in general cases. We work out all the details for $k = 3$.

1 Introduction

The famous *ménage problem* asks for the number M_n of seating arrangements of n married couples of opposite sex around a circular table such that

1. no spouses seat next to each other;
2. females and males alternate.

The problem was formulated by Edouard Lucas in 1891 [3]. A complete solution was first obtained by Touchard in 1934 [5].

Let us call a couple seating next to each other *close*. The restriction of the menage problem can equivalently stated as

1. there are no close couples;
2. no $k = 2$ of consecutive people are of the same sex.

This reformulation allows us to generalize the menage problem to other values of k, such as $k = 3$ which we refer to as the *ternary menage problem*. The ternary menage problem was posed by Hugo Pfoertner in 2006 as the sequence `A114939` in the OEIS [4], for which he then managed to compute only first 3 terms.

In this work, we propose a novel approach for the generalized menage problem. We work out its details for the classical case $k = 2$, where we obtain new formulae for the menage numbers M_n and their exponential generating function, and for the ternary case $k = 3$, which apparently has not been addressed in the literature before.

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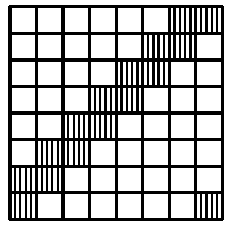


Figure 1: A board corresponding to the menage problem with $n = 8$ couples. The 8 non-attacking rooks need to be placed on non-shaded cells of this board.

2 Classical Approaches for Menage Problem

To the best of our knowledge, there exist three major approaches for solving the menage problem, which we briefly discuss below.

Ladies First. A straightforward approach to the menage problem is first to seat all ladies (in $2 \cdot n!$ ways) and then to seat all gentlemen, obeying the close couple restriction. This way the problem reduces to enumerating placements of non-attacking rooks on a chess board like the one shown in Fig. 1. Using the rook theory, this leads to the *Touchard formula*:

$$M_n = 2 \cdot n! \cdot \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)! \,. \tag{1}$$

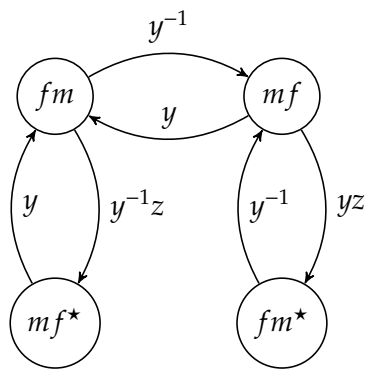
Hamiltonian Cycles in Crown Graphs. The seating arrangements satisfying the menage problem correspond to directed Hamiltonian cycles in the *crown graph* on $2n$ vertices obtained from the complete bipartite graph $K_{n,n}$ with removal of a perfect matching. Here males/females represent the partite sets of $K_{n,n}$ with every male vertex connected to every female vertex, except for the spouses (corresponding to the removed perfect 08). For odd n, crown graphs on $2n$ vertices represent circulant graphs, for which there exists a general formula for the number of Hamiltonian cycles [2].

Non-Sexist Inclusion-Exclusion. Bogart and Doyle [1] suggested to compute M_n by inclusion-exclusion with the following formula:

$$M_n = 2 \cdot \sum_{j=0}^{n} (-1)^j \cdot \binom{n}{j} \cdot (n-j)!^2 \cdot \frac{2n}{2n-j} \binom{2n-j}{j} \cdot j! \,, \tag{2}$$

where:

- the factor 2 accounts for two ways to reserve alternating seats for males/females;
- j stands for the number of close couples;
- $\binom{n}{j}$ is the number of ways to select j close couples out of total n couples;
- $\frac{2n}{2n-j}\binom{2n-j}{j}$ is the number of ways to select $2j$ seats for j close couples;



$$A = \begin{bmatrix} 0 & y^{-1} & y^{-1}z & 0 \\ y & 0 & 0 & yz \\ y & 0 & 0 & 0 \\ 0 & y^{-1} & 0 & 0 \end{bmatrix}$$

Figure 2: De Bruijn graph for the menage problem and its adjacency matrix A.

- $j!$ is the number of seating arrangements of the j close couples at $2j$ selected seats;
- $(n-j)!^2 = (n-j)! \cdot (n-j)!$ is the number of ways to seat females and males from the $n-j$ non-selected couples.

The formula (2) trivially simplifies to (1).

The aforementioned approaches for the menage problem do not seem to easily extend to the ternary case, since there is no nice male-female alternating structure anymore. In particular, the ladies-first approach does not reduce the problem to an uniform board and there is no obvious reduction to a Hamiltonian cycle problem. The (non-sexist) inclusion-exclusion approach is most prominent, but it is unclear what should be in place of $\frac{2n}{2n-j}\binom{2n-j}{j}$. In order to generalize solution to the menage problem to the ternary case, we suggest to look at this problem at a different angle as described below.

3 De Bruijn Graph Approach for Menage Problem

So far, a seating arrangement in the menage problem was viewed as a cyclic (clockwise) sequence of females (f_i) and males (m_j):

$$f_{i_1} \to m_{j_1} \to f_{i_2} \to m_{j_2} \to \cdots \to f_{i_n} \to m_{j_n} \to f_{i_1}\,.$$

However, it can also be viewed as a cyclic sequence of *pairs* of people seating next to each other:

$$(f_{i_1}, m_{j_1}) \to (m_{j_1}, f_{i_2}) \to (f_{i_2}, m_{j_2}) \to \cdots \to (f_{i_n}, m_{j_n}) \to (m_{j_n}, f_{i_1}) \to (f_{i_1}, m_{j_1})\,.$$

The same idea was used by Nicolaas de Bruijn to construct a sequence, which contains every subsequence of a fixed length m (called *m-mer*) over a fixed alphabet. He introduced directed graphs, now named after him, whose nodes represent $(m-1)$-mers and arcs represent m-mers (the arc corresponding to an m-mer s connects the prefix of s with the suffix of s).

We will employ de Bruijn graphs for $m = 3$ for solving the menage problem. However, in contrast to conventional *unweighted* de Bruijn graphs, we will use algebraic weights to account for (i) the balance between females and males; and (ii) the number of close couples.

The (weighted) de Bruijn graph the menage problem and its adjacency matrix A are shown in Fig. 2. This graph has 4 nodes labeled fm (for clockwise adjacent female–male pair), mf (clockwise

adjacent male–female pair), and their starred variants standing for close couples. There is an arc connecting every pair of nodes uv and vw (at most one of which may be starred) for $u, v, w \in \{f, m\}$. Each such arc has an algebraic weight $y^p z^q$ with $p = \pm 1$ and $q \in \{0, 1\}$ such that the degree of indeterminate y accounts for the males-females balance, while the degree of indeterminate z accounts for the number of close couples. Namely, $p = 1$ whenever $w = m$ and $p = -1$ whenever $w = f$, while $q = 1$ iff vw is starred.

Any seating arrangement corresponds to a cyclic sequence of nodes fm and mf, some of which may be starred to indicate close couples. Such sequence with j close couples corresponds to a circuit (with a labeled starting/ending node) of length $2n$ and algebraic weight $y^0 z^j$. It follows that the number of such circuits equals $[y^0 z^j]\ \mathrm{tr}(A^{2n})$, i.e., the coefficient of $y^0 z^j$ in the trace of matrix A^{2n}.

So, by inclusion-exclusion we get the following matrix formula for M_n:

$$M_n = \sum_{j=0}^{n} (-1)^j \cdot \binom{n}{j} \cdot (n-j)!^2 \cdot j! \cdot [y^0 z^j]\ \mathrm{tr}(A^{2n})\,,$$

where the terms bear the same meaning as in (2). Our formula trivially simplifies to

$$M_n = n! \cdot \sum_{j=0}^{n} (-1)^j \cdot (n-j)! \cdot [y^0 z^j]\ \mathrm{tr}(A^{2n})\,. \tag{3}$$

Comparison of this formula to (1) suggests the following identity, which we will prove explicitly:

Lemma 1. *For the matrix A defined in Fig. 2 and any integers $n > 1$, $j \geq 0$,*

$$[y^0 z^j]\ \mathrm{tr}(A^{2n}) = 2 \cdot \frac{2n}{2n-j}\binom{2n-j}{j}.$$

Proof. The eigenvalues of A are $\frac{1 \pm \sqrt{1+4z}}{2}$, each of multiplicity 2.[1] It follows that

$$[y^0 z^j]\ \mathrm{tr}(A^{2n}) = 2 \cdot [z^j] \left(\left(\frac{1 + \sqrt{1+4z}}{2} \right)^{2n} + \left(\frac{1 - \sqrt{1+4z}}{2} \right)^{2n} \right).$$

We further remark that $\frac{1-\sqrt{1+4z}}{2} = -zC(-z)$ and $\frac{1+\sqrt{1+4z}}{2} = 1 + zC(-z)$, where $C(x) = \frac{1-\sqrt{1-4x}}{2}$ is the ordinary generating function for Catalan numbers.

Since $j \leq n$ and $n > 1$, we have $[z^j]\ (-zC(-z))^{2n} = 0$. So it remains to compute $[z^j]\ (1 + zC(-z))^{2n}$. For $j = 0$, we trivially have $[z^j]\ (1 + zC(-z))^{2n} = 1 = \frac{2n}{2n-j}\binom{2n-j}{j}$. For $j > 0$, we have

$$\begin{aligned}
[z^j]\ (1 + zC(-z))^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} [z^j]\ z^k C(-z)^k = \sum_{k=0}^{j} \binom{2n}{k} [z^{j-k}]\ C(-z)^k \\
&= \sum_{k=1}^{j} \binom{2n}{k} (-1)^{j-k} \frac{k}{j} \binom{2j-k-1}{j-1} = \frac{2n}{j} \sum_{k=1}^{j} (-1)^{j-k} \binom{2n-1}{k-1} \binom{2j-k-1}{j-1} \\
&= \frac{2n}{j} \binom{2n-1-j}{j-1} = \frac{2n}{2n-j} \binom{2n-j}{j}.
\end{aligned}$$

Here we used the fact that $[x^n]\ C(x)^m = \frac{m}{n+m}\binom{2n+m-1}{n+m-1}$. □ □

[1] We remark that A^2 does not depend on y, so it is not surprising that the eigenvalues of A do not depend on y either.

Lemma 1 proves that our formula (3) implies Touchard formula (1). In the next section, we show that it also implies another (apparently new) formula for M_n. But most importantly, the matrix formula (3) can be generalized for the ternary menage problem as we show in Section 5.

4 New Formulae for Menage Numbers and Their EGF

Lemma 2. *Let U, V be same-size square matrices that do not depend on indeterminate z. Then for any integer $n \geq 0$,*

$$\sum_{j=0}^{n} (-1)^j \cdot (n-j)! \cdot [z^j] \operatorname{tr}((U + V \cdot z)^n) = \int_{t=0}^{\infty} \operatorname{tr}((U \cdot t - V)^n) \cdot e^{-t} \cdot dt\,.$$

Proof. We have

$$[z^j] \operatorname{tr}((U + V \cdot z)^n) = [z^{n-j}] \operatorname{tr}((U \cdot z + V)^n) = [z^{n-j}] \operatorname{tr}((U \cdot z - V)^n) \cdot (-1)^j\,.$$

Hence

$$\sum_{j=0}^{n} (-1)^j \cdot (n-j)! \cdot [z^j] \operatorname{tr}((U + V \cdot z)^n) = \sum_{j=0}^{n} (n-j)! \cdot [z^{n-j}] \operatorname{tr}((U \cdot z - V)^n) = \sum_{j=0}^{n} j! \cdot [z^j] \operatorname{tr}((U \cdot z - V)^n)\,.$$

It remains to notice that the last sum represents the series Laplace transform $\mathcal{L}\left[\operatorname{tr}((U \cdot t - V)^n)\right](s)$ evaluated at $s = 1$, that is

$$\sum_{j=0}^{n} j! \cdot [z^j] \operatorname{tr}((U \cdot z - V)^n) = \int_{t=0}^{\infty} \operatorname{tr}((U \cdot t - V)^n) \cdot e^{-t} \cdot dt\,.$$

□ □

We are now ready to derive a closed-form expression for numbers M_n and their exponential generating function.

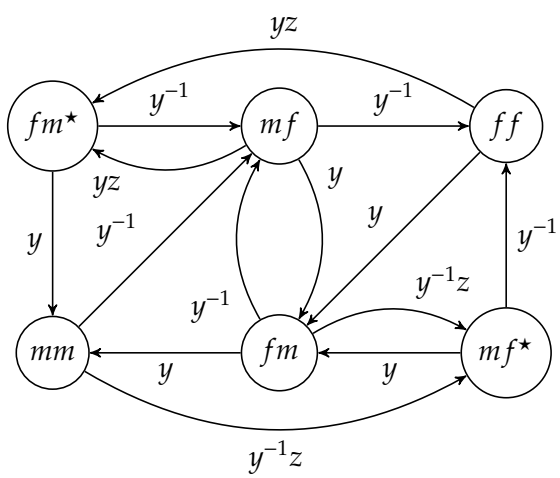
Theorem 3. *For all integer $n > 1$,*

$$M_n = 2 \cdot n! \cdot \int_{t=0}^{\infty} \left(\left(\frac{t - 2 + \sqrt{t^2 - 4t}}{2} \right)^n + \left(\frac{t - 2 - \sqrt{t^2 - 4t}}{2} \right)^n \right) \cdot e^{-t} \cdot dt\,. \tag{4}$$

Furthermore,

$$\begin{aligned} \sum_{n=0}^{\infty} M_n \frac{x^n}{n!} &= 2 \int_{t=0}^{\infty} \frac{xt - 2(x+1)}{xt - (x+1)^2} e^{-t}\, dt \\ &= -1 + 2x + \frac{2(1-x^2)}{x} e^{-(x+1)^2/x} \operatorname{Ei}\left(\frac{(x+1)^2}{x} \right), \end{aligned}$$

where $\operatorname{Ei}(t)$ *is the exponential integral.*



$$B = \begin{bmatrix} 0 & y^{-1} & 0 & y & 0 & 0 \\ yz & 0 & y^{-1} & 0 & y & 0 \\ yz & 0 & 0 & 0 & y & 0 \\ 0 & y^{-1} & 0 & 0 & 0 & y^{-1}z \\ 0 & y^{-1} & 0 & y & 0 & y^{-1}z \\ 0 & 0 & y^{-1} & 0 & y & 0 \end{bmatrix}$$

Figure 3: De Bruijn graph for the ternary menage problem and its adjacency matrix B.

Proof. For the matrix A defined in Fig. 2, we have $A^2 = U + V \cdot z$, where

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then Lemma 2 and formula (3) imply

$$M_n = \int_{t=0}^{\infty} \operatorname{tr}((U \cdot t - V)^n) \cdot e^{-t} \cdot dt\,.$$

Since the eigenvalues of the matrix $U \cdot t - V$ are $\frac{t-2\pm\sqrt{t^2-4t}}{2}$, each of multiplicity 2, we obtain formula (4).

To derive the exponential generating function for M_n, it remains to notice that

$$\left(\frac{t-2+\sqrt{t^2-4t}}{2}\right)^n + \left(\frac{t-2-\sqrt{t^2-4t}}{2}\right)^n = [x^n]\,\frac{xt-2(x+1)}{xt-(x+1)^2}$$

and take special care of the initial values $M_0 = 1$ and $M_1 = 0$. □ □

5 De Bruijn Graph Approach for Ternary Menage Problem

In contrast to the menage problem, in the ternary case two females or two males can seat to each other. Hence, the de Bruijn graph in this case can be obtained from the de Bruijn graph for the menage problem by adding two more nodes labeled ff and mm, connected to the other nodes following the same rules (Fig. 3).

Theorem 4. *For $n > 1$, the number T_n of seating arrangements for the ternary menage problem can be computed in the following ways:*

$$T_n = n! \cdot \sum_{j=0}^{n} (-1)^j \cdot (n-j)! \cdot [y^0 z^j]\,\operatorname{tr}(B^{2n})\,, \tag{5}$$

where B is defined in Fig. 3; or

$$T_n = n! \cdot \int_{t=0}^{\infty} [y^n] \operatorname{tr}(B_2^n) \cdot e^{-t} \cdot dt\,, \text{ where } B_2 = \begin{pmatrix} -y & yt & t & 0 & yt & -y \\ -y & y(t-1) & 0 & y^2(t-1) & yt & -y \\ 0 & y(t-1) & 0 & y^2(t-1) & 0 & -y \\ -y & 0 & t-1 & 0 & y(t-1) & 0 \\ -y & yt & t-1 & 0 & y(t-1) & -y \\ -y & yt & 0 & y^2 t & yt & -y \end{pmatrix}; \quad (6)$$

or

$$T_n = n! \cdot \int_{t=0}^{\infty} [x^n y^n] \frac{a(xy,t) + b(xy,t) \cdot (x + xy^2)}{c(xy,t) + d(xy,t) \cdot (x + xy^2)} \cdot e^{-t} \cdot dt\,, \quad (7)$$

where

$$\begin{aligned} a(p,t) &= -2p^5t^3 + 2p^4t^4 + 4p^5t^2 - 8p^4t^3 - 2p^5t + 12p^4t^2 - 8p^4t + 6p^3t - 4p^2t^2 \\ &\quad + 16p^2 - 10pt + 20p + 6\,, \\ b(p,t) &= -p^2t(2+p-t)(p-3t+6)\,, \\ c(p,t) &= p^6t^2 - 2p^5t^3 + p^4t^4 + 4p^5t^2 - 4p^4t^3 - 2p^5t + 6p^4t^2 - 4p^4t + 2p^3t - p^2t^2 \\ &\quad + 4p^2 - 2pt + 4p + 1\,, \\ d(p,t) &= -p^2t(2+p-t)^2\,. \end{aligned}$$

Proof. Formula (5) is similar to (3) and follows directly from the definition of de Bruijn graph in Fig. 3.

To avoid dealing with negative powers, we notice that $[y^0z^j] \operatorname{tr}(B^{2n}) = [y^{2n}z^j] \operatorname{tr}((yB)^{2n})$. Furthermore, the matrix $(yB)^2$ has entries that are polynomial in y^2 and z with the degree with respect to z being at most 1, that is $(yB)^2 = U + V \cdot z$, where matrices U, V do not depend on z. Since the specified matrix B_2 equals $U \cdot t - V$ where y^2 is replaced with y, formula (6) easily follows from (5) and Lemma 2.

The sequence of traces $\operatorname{tr}(B_2^n)$ $(n = 0, 1, \dots)$ is linear recurrent with the characteristic polynomial being the same as for the matrix B_2. The ordinary generating function for this sequence can be found with the standard technique and happens to have the form:

$$\sum_{n=0}^{\infty} \operatorname{tr}(B_2^n) \cdot x^n = \frac{a(xy,t) + b(xy,t)(x + xy^2)}{c(xy,t) + d(xy,t)(x + xy^2)}\,.$$

Substituting this expression into (6) yields (7). □ □

While formulae (5) and (6) provide an efficient way for computing T_n for a given integer $n > 1$, the special form of the rational function in x, y appearing in (7) allows one to obtain a closed-form expression for the exponential generating function for numbers T_n as we demonstrate below.

Lemma 5. *Let $a(z)$, $b(z)$, $c(z)$, $d(z)$ be polynomials such that $c(z)d(z) \neq 0$ and $c(z) \neq \pm 2zd(z)$. Then for any integer $n \geq 0$,*

$$[x^n y^n] \frac{a(xy) + b(xy) \cdot (x + xy^2)}{c(xy) + d(xy) \cdot (x + xy^2)} = [p^n] \left(\frac{a(p)d(p) - b(p)c(p)}{d(p) \cdot \sqrt{c(p)^2 - 4p^2 d(p)^2}} + \frac{b(p)}{d(p)} \right).$$

Proof. Let $p = xy$. Then we need to extract the terms from $\frac{a(p)+b(p)\cdot(x+py)}{c(p)+d(p)\cdot(x+py)}$ that have the same degree in x and y. In such terms we then replace every product xy with p. We start with the following expansion:

$$\begin{aligned}\frac{a(p)+b(p)\cdot(x+py)}{c(p)+d(p)\cdot(x+py)} &= \frac{a(p)+b(p)\cdot(x+py)}{c(p)}\sum_{k=0}^{\infty}\left(\frac{-d(p)}{c(p)}\right)^k(x+py)^k \\ &= \frac{a(p)}{c(p)}\sum_{k=0}^{\infty}\left(\frac{-d(p)}{c(p)}\right)^k(x+py)^k + \frac{b(p)}{c(p)}\sum_{k=0}^{\infty}\left(\frac{-d(p)}{c(p)}\right)^k(x+py)^{k+1}.\end{aligned}$$

Here from each power of $x+py$ we extract the term with the same degree of x and y and replace them with the corresponding power of p. This yields

$$\begin{aligned}&\frac{a(p)}{c(p)}\sum_{k=0}^{\infty}\left(\frac{-d(p)}{c(p)}\right)^{2k}\binom{2k}{k}p^{2k} + \frac{b(p)}{c(p)}\sum_{k=0}^{\infty}\left(\frac{-d(p)}{c(p)}\right)^{2k+1}\binom{2k+2}{k+1}p^{2k+2} \\ &= \frac{a(p)}{c(p)}\cdot f\left(\left(\frac{d(p)}{c(p)}\right)^2p^2\right) - \frac{b(p)}{d(p)}\cdot\left(f\left(\left(\frac{d(p)}{c(p)}\right)^2p^2\right)-1\right) \\ &= \frac{a(p)d(p)-b(p)c(p)}{c(p)d(p)}\cdot f\left(\left(\frac{d(p)}{c(p)}\right)^2p^2\right) + \frac{b(p)}{d(p)} \\ &= \frac{a(p)d(p)-b(p)c(p)}{d(p)\cdot\sqrt{c(p)^2-d(p)^2p^2}} + \frac{b(p)}{d(p)},\end{aligned}$$

where $f(z) = (1-4z)^{-1/2} = \sum_{k=0}^{\infty}\binom{2k}{k}z^k$. The coefficient of x^ny^n in the original expression equals the coefficient of p^n in the last expression. □ □

Formula (7) and Lemma 5 imply the following expression for the exponential generating function for T_n.

Theorem 6.

$$\begin{aligned}\sum_{n=0}^{\infty}T_n\frac{x^n}{n!} &= -2+2x-2xe^{-x-2}\mathrm{Ei}(x+2) \\ &+ \int_{t=0}^{\infty}\frac{\left(t^3x^2+\left(-2x^3-4x^2-x\right)t^2+\left(x^4+4x^3+7x^2+4x+3\right)t-6x^2-9x-6\right)\cdot e^{-t}}{(t-(x+2))\cdot\sqrt{(t^2x^2-tx^3-2tx^2-3xt+4x^2+4x+1)\cdot(t^2x^2-tx^3-2tx^2+xt+1)}}\,dt.\end{aligned}$$

6 Sequences in the OEIS

The Online Encyclopedia of Integer Sequences [4] contains a number of sequences related to menage problem:

Sequence	Terms for $n = 1, 2, 3, \ldots$	OEIS index
M_n	$0, 0, 12, 96, 3120, 115200, 5836320, \ldots$	A059375
$M_n/2n!$	$0, 0, 1, 2, 13, 80, 579, 4738, 43387, \ldots$	A000179
$M_n/2n$	$0, 0, 2, 12, 312, 9600, 416880, 23879520, \ldots$	A094047
$T_n/4n$	$0, 1, 7, 216, 10956, 803400, 83003040, \ldots$	A114939
T_n	$0, 8, 84, 3456, 219120, 19281600, 2324085120, \ldots$	A258338

Acknowledgements

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