Characterized subgroups of the circle group

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31st Summer Conference on Topology and its Applications
Leicester - August 2nd, 2016
(Topological Methods in Algebra and Analysis)

Introduction

Let G be an abelian group and p a prime.
The *p-torsion subgroup* of G is

$$t_p(G) = \{x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N}\}.$$

The *torsion subgroup* of G is

$$t(G) = \{x \in G : nx = 0 \text{ for some } n \in \mathbb{N}_+\}.$$

The circle group is $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ written additively $(\mathbb{T}, +)$;
for $r \in \mathbb{R}$, we denote $\bar{r} = r + \mathbb{Z} \in \mathbb{T}$.
We consider on $\mathbb{T}$ the quotient topology of the topology of $\mathbb{R}$ and μ is the (unique) Haar measure on $\mathbb{T}$.

- $t_p(\mathbb{T}) = \mathbb{Z}(p^\infty)$.
- $t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$.

Let now G be a *topological* abelian group.

[Bracconier 1948, Vilenkin 1945, Robertson 1967, Armacost 1981]:

The *topologically p-torsion subgroup* of G is

$$\boxed{t_{\underline{p}}(G) = \{x \in G : p^n x \to 0\}.}$$

The *topologically torsion subgroup* of G is

$$\boxed{G! = \{x \in G : n!x \to 0\}.}$$

Clearly, $t_p(G) \subseteq t_{\underline{p}}(G)$ and $t(G) \subseteq G!$.

[Armacost 1981]:

- $t_{\underline{p}}(\mathbb{T}) = t_p(\mathbb{T}) = \mathbb{Z}(p^\infty)$;
- $\bar{e} \in \mathbb{T}!$, but $\bar{e} \notin t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$. Problem: describe $\mathbb{T}!$.

[Borel 1991; Dikranjan-Prodanov-Stoyanov 1990; D-Di Santo 2004]:

For every $x \in [0,1)$ there exists a unique $(c_n)_n \in \mathbb{N}^{\mathbb{N}_+}$ such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{(n+1)!},$$

$c_n < n+1$ for every $n \in \mathbb{N}_+$ and $c_n < n$ for infinitely many $n \in \mathbb{N}_+$.

Theorem (Dikranjan-Prodanov-Stoyanov 1990; D-Di Santo 2004)

Let $x \in [0,1)$. Then $\bar{x} \in \mathbb{T}!$ $\Leftrightarrow$ $\frac{c_n}{n+1} \to 0$ in $\mathbb{T}$.

[Dikranjan-Di Santo 2004]:

$\mathbb{T}!$ has Haar measure 0, $\mathbb{T}!$ has size $\mathfrak{c}$ and $\mathbb{T}!$ is not divisible.

Topologically **u**-torsion subgroup

Definition (Dikranjan-Prodanov-Stoyanov 1990; Dikranjan 2001)

Let G be a topological abelian group and $\mathbf{u} = (u_n)_n \in \mathbb{Z}^{\mathbb{N}}$.
The *topologically* ***u****-torsion subgroup* of G is

$$\boxed{t_{\mathbf{u}}(G) = \{x \in G : u_n x \to 0\}.}$$

- $t_{\underline{p}}(G) = \{x \in G : p^n x \to 0\} = t_{\mathbf{p}}(G)$, where $\mathbf{p} = (p^n)_n$.
- $G! = \{x \in G : n!x \to 0\} = t_{\mathbf{u}}(G)$, where $\mathbf{u} = ((n+1)!)_n$.

Example (Dikranjan-Kunen 2007)

$t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z} = t_{\mathbf{u}}(\mathbb{T})$, where $\mathbf{u}$ is the sequence
$(1!, 2!, 2{\cdot}2!, 3!, 2{\cdot}3!, 3{\cdot}3!, 4!, \ldots, n!, 2{\cdot}n!, 3{\cdot}n!, \ldots, n{\cdot}n!, (n+1)!, \ldots)$.

Problem: given $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, describe $t_{\mathbf{u}}(\mathbb{T})$.

Which property is shared by $(p^n)_n$ and $((n+1)!)_n$?

Let $\mathbf{u} = (u_n)_n$ be an arithmetic sequence
(i.e., $u_0 = 1$, $u_n \in \mathbb{N}_+$ and $u_n \mid u_{n+1}$ for every $n \in \mathbb{N}$)
and for every $n \in \mathbb{N}$ let

$$\boxed{d^{\mathbf{u}}_{n+1} = \frac{u_{n+1}}{u_n} \in \mathbb{N}_+.}$$

Theorem

For every $x \in [0,1)$, there exists a unique $(c^{\mathbf{u}}_n(x))_n \in \mathbb{N}^{\mathbb{N}_+}$ with

$$\boxed{x = \sum_{n=1}^{\infty} \frac{c^{\mathbf{u}}_n(x)}{u_n},}$$

$c^{\mathbf{u}}_n(x) < d^{\mathbf{u}}_n$ *for every* $n \in \mathbb{N}_+$,
and $c^{\mathbf{u}}_n(x) < d^{\mathbf{u}}_n - 1$ *for infinitely many* $n \in \mathbb{N}_+$.

Theorem (Dikranjan-Prodanov-Stoyanov 1990; D-Di Santo 2004)

Let $x = \sum_{n=1}^{\infty} \frac{c_n^{\mathbf{u}}(x)}{u_n} \in [0,1)$.

- *If* $(d_n^{\mathbf{u}})_n$ *is bounded, then* $\bar{x} \in t_{\mathbf{u}}(\mathbb{T}) \Leftrightarrow (c_n^{\mathbf{u}}(x))_n$ *is ev.* 0.
- *If* $d_n^{\mathbf{u}} \to +\infty$, *then* $\bar{x} \in t_{\mathbf{u}}(\mathbb{T}) \Leftrightarrow \frac{c_n^{\mathbf{u}}(x)}{d_n^{\mathbf{u}}} \to 0$ *in* $\mathbb{T}$.

[Dikranjan-Di Santo 2004, Dikranjan-Impieri 2014]:
Complete description of $t_{\mathbf{u}}(\mathbb{T})$ for $\mathbf{u}$ arithmetic sequence.

Corollary

The following conditions are equivalent:

- $(d_n^{\mathbf{u}})_n$ *is bounded;*
- $t_{\mathbf{u}}(\mathbb{T})$ *is countable;*
- $t_{\mathbf{u}}(\mathbb{T})$ *is torsion.*

Characterized subgroups of $\mathbb{T}$

Definition (Bíró-Deshouillers-Sós 2001)

A subgroup H of $\mathbb{T}$ is *characterized* if $H = t_{\mathbf{u}}(\mathbb{T})$ for some $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$.
(H is *characterized* by $\mathbf{u}$; $\mathbf{u}$ *characterizes* H.)

Problem

Describe the characterized subgroups of $\mathbb{T}$.
In other words, given $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, *describe* $t_{\mathbf{u}}(\mathbb{T})$.

The "inverse problem": given $H \leq \mathbb{T}$, is H characterized?

Problem (Di Santo 2002; Maharam-Stone 2001)

Given $H \leq \mathbb{T}$ *characterized, describe*
$S_H = \{\mathbf{u} \in \mathbb{Z}^{\mathbb{N}} : H = t_{\mathbf{u}}(\mathbb{T})\} \leq \mathbb{Z}^{\mathbb{N}}$.

- If H is a finite subgroup of $\mathbb{T}$, then H is characterized.
- $\mathbf{u}$ characterizes $\mathbb{T}$ if and only if $\mathbf{u}$ is eventually zero.
- If $H \lneq \mathbb{T}$ is characterized, then:
  - $H = t_{\mathbf{u}}(\mathbb{T})$ for $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ strictly increasing;
  - $\mu(H) = 0$.
- Since $t_{\mathbf{u}}(\mathbb{T}) = \bigcap_{N\geq 2} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \left\{x \in \mathbb{T} : \|u_n x\| \leq \frac{1}{N}\right\}$ is a Borel set, so

$$\text{either } t_{\mathbf{u}}(\mathbb{T}) \text{ is countable or } |t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}.$$

Theorem (Eggleston 1952)

- $u_{n+1}/u_n \to +\infty \Rightarrow |t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$.
- $(u_{n+1}/u_n)_n$ *bounded* $\Rightarrow t_{\mathbf{u}}(\mathbb{T})$ *countable.*

Theorem (Borel 1983)

All countable subgroups of $\mathbb{T}$ are characterized by some $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$.

[Beiglböck-Steineder-Winkler 2006]: $\mathbf{u}$ can be chosen with $(u_{n+1}/u_n)_n$ bounded, but also arbitrarily fast increasing.

Borel's motivation:

$(x_n)_n \in \mathbb{R}^{\mathbb{N}}$ is *uniformly distributed mod* 1 if for all $[a, b] \subseteq [0, 1)$,

$$\frac{|\{j \in \{0, \dots, n\} : \{x_j\} \in [a, b]\}|}{n} \longrightarrow a - b.$$

Theorem (Weyl, 1916; Kuipers-Niederreiter 1974)

Let $\mathbf{u} \in \mathbb{N}^{\mathbb{N}}$ be a strictly increasing sequence. Then $\lambda(\{\beta \in \mathbb{R} : (\{u_n\beta\})_n$ is uniformly distributed mod $1\}) = 1$.

This does not hold for all $\beta \in \mathbb{R}$, so study the "opposite case" $t_{\mathbf{u}}(\mathbb{T}) = \{\bar{\beta} \in \mathbb{T} : u_n\bar{\beta} \to 0\}$.

The infinite cyclic subgroups of $\mathbb{T}$

Let α be an irrational number and consider $\langle\bar{\alpha}\rangle \leq \mathbb{T}$.

The irrational α has a unique continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

denoted by $\alpha = [a_0; a_1, a_2, \dots]$.

For every $n \in \mathbb{N}$, let $\mathbf{q} = (q_n)_n \in \mathbb{N}_+^{\mathbb{N}}$, where

$$[a_0; a_1, \dots, a_n] = \frac{p_n}{q_n}, \quad \text{with } q_n > 0.$$

Since $q_n\bar{\alpha} \to 0$ in $\mathbb{T}$, so

$$\langle\bar{\alpha}\rangle \subseteq t_{\mathbf{q}}(\mathbb{T}).$$

Theorem (Larcher 1988)

$\langle\bar{\alpha}\rangle = t_{\mathbf{q}}(\mathbb{T})$ *when* $(a_n)_n$ *is bounded.*

The equality does not hold true in general, more precisely:

Theorem (Kraaikamp-Liardet 1992)

$\langle\bar{\alpha}\rangle = t_{\mathbf{q}}(\mathbb{T})$ *if and only if* $(a_n)_n$ *is bounded.*

Example

Consider the Golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. Then

$$\langle\bar{\varphi}\rangle = t_{\mathbf{f}}(\mathbb{T}),$$

where $\mathbf{f} = (f_n)_n$ is the Fibonacci sequence
$f_0 = 1,\ f_1 = 1,\ f_2 = 2,\ f_3 = 3,\ f_4 = 5,\ \ldots.$
Indeed, $\varphi = [1; 1, 1, \ldots]$ and $q_n = f_n$ for every $n \in \mathbb{N}_+$.

[Barbieri-Dikranjan-Milan-Weber 2008]:

Sequences $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ verifying a linear recurrence.

For every $n \geq 2$, $\boxed{u_n = a_n u_{n-1} + b_n u_{n-2},}$ $a_n, b_n \in \mathbb{N}_+$.

Theorem (Barbieri-Dikranjan-Milan-Weber 2008)

$|t_{\mathbf{u}}(\mathbb{T}))| = \mathfrak{c} \;\Leftrightarrow\; (u_{n+1}/u_n)_n$ *not bounded.*

If $\alpha = [a_0; a_1, \ldots]$ is an irrational and $\mathbf{q} = (q_n)_n$, then
for every $n \geq 2$, $\boxed{q_n = a_n q_{n-1} + q_{n-2},}$ $q_0 = 1, q_1 = a_1$.

Corollary

The following conditions are equivalent:

- $\langle \bar{\alpha} \rangle = t_{\mathbf{q}}(\mathbb{T})$;
- $(a_n)_n$ *is bounded;*
- $t_{\mathbf{q}}(\mathbb{T})$ *is countable.*

Let $\alpha = [a_0; a_1, \dots]$ irrational, $\mathbf{q} = (q_n)_n$ as above and let $\mathbf{v}_\alpha$:

$$q_0 \leq q_1 < 2q_1 < \dots < a_2 q_1 < q_2 < 2q_2 < \dots < a_3 q_2 < q_3 < 2q_3 < \dots$$

Theorem (Bíró-Deshouillers-Sós 2001)

$\langle \bar{\alpha} \rangle = t_{\mathbf{v}_\alpha}(\mathbb{T})$.

Problem (Bíró-Deshouillers-Sós 2001)

Find all sequences $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ characterizing $\langle \bar{\alpha} \rangle$.

In particular, if $\mathbf{q} \subseteq \mathbf{u} \subseteq \mathbf{v}_\alpha$, one has

$$\langle \bar{\alpha} \rangle = t_{\mathbf{v}_\alpha}(\mathbb{T}) \subseteq t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{q}}(\mathbb{T}),$$

and the question is: under which hypotheses one has $\langle \bar{\alpha} \rangle = t_{\mathbf{u}}(\mathbb{T})$?

Theorem (Marconato 2016)

If $(u_{n+1}/u_n)_n$ is bounded, then $\langle \bar{\alpha} \rangle = t_{\mathbf{v}_\alpha}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$.

Precompact topologies on $\mathbb{Z}$ with converging sequences

[Raczkowski 2002; Barbieri-Dikranjan-Milan-Weber 2003]:

Given $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, $\exists$ a precompact gr. top. τ on $\mathbb{Z}$ such that $u_n \xrightarrow{\tau} 0$?

A topological abelian group (G, τ) is:

- *totally bounded* if $\forall \emptyset \neq U \in \tau$, $\exists F \subseteq G$ finite, $G = U + F$;
- *precompact* if τ is Hausdorff and totally bounded.

For G abelian group and $H \leq \widehat{G}$, let T_H be the weakest group topology on G such that all $\chi \in H$ are continuous.

Theorem (Comfort-Ross 1964)

- *T_H is totally bounded and $w(G, T_H) = |H|$.*
- *If (G, τ) is totally bounded, then $\tau = T_H$ for some $H \leq \widehat{G}$.*
- *T_H is Hausdorff if and only if H is dense in $\widehat{G}$.*

For $G = \mathbb{Z}$, we have $\widehat{G} = \mathbb{T}$.

Let $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ and τ a totally bounded group topology on $\mathbb{Z}$. Then:

- $\tau = T_H$ for some $H \leq \mathbb{T}$;
- $u_n \xrightarrow{T_H} 0 \;\Leftrightarrow\; H \leq t_{\mathbf{u}}(\mathbb{T})$.

Then $T_{t_{\mathbf{u}}(\mathbb{T})}$ is the finest tot. bounded gr. top. τ on $\mathbb{Z}$ with $u_n \xrightarrow{\tau} 0$;

- $w(\mathbb{Z}, T_{t_{\mathbf{u}}(\mathbb{T})}) = |t_{\mathbf{u}}(\mathbb{T})|$;
- $T_{t_{\mathbf{u}}(\mathbb{T})}$ is precompact if and only if $t_{\mathbf{u}}(\mathbb{T})$ is infinite;
- $T_{t_{\mathbf{u}}(\mathbb{T})}$ is metrizable if and only if $|t_{\mathbf{u}}(\mathbb{T})| = \omega$.

Theorem (Barbieri-Dikranjan-Milan-Weber 2003)

- *If $u_{n+1}/u_n \to +\infty$, then there exists a precompact group topology τ on $\mathbb{Z}$ such that $w(\mathbb{Z}, \tau) = \mathfrak{c}$ and $u_n \xrightarrow{\tau} 0$.*
- *If $(u_{n+1}/u_n)_n$ is bounded, then every precompact group topology τ on $\mathbb{Z}$ such that $u_n \xrightarrow{\tau} 0$ is metrizable.*

Dirichlet sets and Arbault sets

Definition (Arbault 1952, Kahane 1969)

A set $A \subseteq [0,1]$ is:

- an *Arbault set* (briefly, *A-set*) if there exists $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ increasing such that $\sin(\pi u_n x) \to 0$ for all $x \in A$.
- a *Dirichlet set* (briefly, *D-set*) if there exists $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ increasing such that $\sin(\pi u_n x) \to 0$ uniformly on A.

Let $\varpi : \mathbb{R} \to \mathbb{T}$ and $\varphi := \varpi \restriction_{[0,1)} : [0,1) \to \mathbb{T}$ bijection.

Definition

$A \subseteq \mathbb{T}$ *A-set* (resp., *D-set*) if $\varphi^{-1}(A) \subseteq [0,1]$ A-set (resp., D-set); that is, there exists $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ increasing such that $u_n x \to 0$ $\forall x \in A$ (resp., $u_n x \to 0$ uniformly on A).

$A \subseteq \mathbb{T}$ A-set if and only if $A \subseteq t_{\mathbf{u}}(\mathbb{T})$ for some $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$.

[Eliaš 2003, 2005]:

Let $\mathcal{A}$ be the family of all Arbault sets of $\mathbb{T}$.
Let $\mathcal{S} = \{\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}} : \ u_{n+1}/u_n \to +\infty\}$.

- $X \in \mathcal{A} \Rightarrow X \subseteq t_{\mathbf{u}}(\mathbb{T})$ for some $\mathbf{u} \in \mathcal{S}$.

Theorem (Eliaš 2003; Barbieri-GB-Weber 2015 for $\mathbb{R}$)

Description of when $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{v}}(\mathbb{T})$ for $\mathbf{u}, \mathbf{v} \in \mathcal{S}$.

Eliaš' motivation:

An $X \subseteq \mathbb{T}$ is *A-permitted* if $X \cup Y \in \mathcal{A}$ for every $Y \in \mathcal{A}$.
[Kholshchevnikova 1994]: $X \subseteq \mathbb{T}$ countable $\Rightarrow$ A-permitted.

Theorem (Eliaš 2005)

A set $X \subseteq \mathbb{T}$ is A-permitted if and only if for every $\mathbf{u} \in S$ there exists $\mathbf{v} \in \mathcal{S}$ such that $X \cup t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{v}}(\mathbb{T})$.

Properties of A-permitted sets.

Let $\mathbf{u} = (u_n)_n$ be an arithmetic sequence (i.e., $u_n \mid u_{n+1}, \forall n \in \mathbb{N}$) and for every $n \in \mathbb{N}$ let $d^{\mathbf{u}}_{n+1} = \frac{u_{n+1}}{u_n} \in \mathbb{N}_+$.
For every $x \in [0,1)$, there exists a unique $(c^{\mathbf{u}}_n(x))_n \in \mathbb{N}^{\mathbb{N}_+}$ with

$$\boxed{x = \sum_{n=1}^{\infty} \frac{c^{\mathbf{u}}_n(x)}{u_n},}$$

$c^{\mathbf{u}}_n(x) < d^{\mathbf{u}}_n \ \forall n \in \mathbb{N}_+$, $c^{\mathbf{u}}_n(x) < d^{\mathbf{u}}_n - 1$ for infinitely many $n \in \mathbb{N}_+$.
For $x \in [0,1)$, let $\boxed{\mathrm{supp}_{\mathbf{u}}(x) := \{n \in \mathbb{N}_+ : c^{\mathbf{u}}_n(x) \neq 0\}.}$

Definition (Marcinkiewicz 1938; Barbieri-Dikranjan-GB-Weber 2016)

For $L \subseteq \mathbb{N}_+$, let $\boxed{K^{\mathbf{u}}_L = \{\bar{x} \in \mathbb{T} : \mathrm{supp}_{\mathbf{u}}(x) \subseteq L\}.}$

Clearly, $0 \in K^{\mathbf{u}}_L$ since $\mathrm{supp}_{\mathbf{u}}(0) = \emptyset$.

- $K^{\mathbf{u}}_L$ finite $\Rightarrow$ $K^{\mathbf{u}}_L$ D-set.
- $K^{\mathbf{u}}_L$ D-set $\Rightarrow$ L non-cofinite.

[Barbieri-Dikranjan-GB-Weber 2016]:

If $L \subseteq \mathbb{N}_+$ is infinite non-cofinite, then:

- $K_L^{\mathbf{u}}$ is closed;
- $K_L^{\mathbf{u}}$ is perfect (i.e., has no isolated points).

Theorem (Barbieri-Dikranjan-GB-Weber 2016)

If $L \subseteq \mathbb{N}_+$ is infinite non-cofinite, then $K_L^{\mathbf{u}}$ is a D-set if and only if either $\{d_n^{\mathbf{u}} : n \in \mathbb{N}_+ \setminus L\}$ is not bounded or L is not large.

$L \subseteq \mathbb{N}_+$ is *large* if there exists $F \subseteq \mathbb{Z}$ finite such that $\mathbb{N}_+ \subseteq L + F$.

Corollary

If $L \subseteq \mathbb{N}_+$ is infinite and not large, then $K_L^{\mathbf{u}}$ is a D-set.

If $L_1 \subseteq \mathbb{N}_+$ and $L_2 = \mathbb{N}_+ \setminus L$, then $\boxed{\mathbb{T} = K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2}.}$

If L_1 and L_2 are both non-large, then $K^{\mathbf{u}}_{L_1}$ and $K^{\mathbf{u}}_{L_2}$ are D-sets.

Theorem (Barbieri-Dikranjan-GB-Weber 2016)

$\mathbb{T}$ can be written as the sum of two closed perfect D-sets.

(Related to Erdös-Kunen-Mauldin Theorem 1981: for $\emptyset \neq P \subseteq \mathbb{T}$ perfect, there exists $D \subseteq \mathbb{T}$ perfect D-set with $\mathbb{T} = P + D$.)

In particular, there exist $\mathbf{v}$, $\mathbf{w}$ subsequences of $\mathbf{u}$ such that $\mathbf{v}$, $\mathbf{w}$ witness that $K^{\mathbf{u}}_{L_1}$, $K^{\mathbf{u}}_{L_2}$ are D-sets; then $\boxed{\mathbb{T} = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T}).}$

Corollary

$\mathbb{T}$ can be written as the sum of two proper subgroups characterized by arithmetic sequences.

This answers a question from [Barbieri-Dikranjan-Milan-Weber 2003].

Characterized subgroups of topological abelian groups

Definition (Dikranjan-Milan-Tonolo 2005)

Let G be a topological abelian group, $\mathbf{u} = (u_n)_n \in \widehat{G}^{\mathbb{N}}$ and

$$s_{\mathbf{u}}(G) = \{x \in G : u_n(x) \to 0\}.$$

$H \leq G$ is *characterized* if $H = s_{\mathbf{u}}(G)$ for some $\mathbf{u} \in \widehat{G}^{\mathbb{N}}$.

If $G = \mathbb{T}$, then $\widehat{\mathbb{T}} = \mathbb{Z}$; so $t_{\mathbf{u}}(\mathbb{T}) = s_{\mathbf{u}}(\mathbb{T})$ for $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$.
Since $s_{\mathbf{u}}(G) = \bigcap_{N\geq 2} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \left\{x \in G : \|u_n(x)\| \leq \frac{1}{N}\right\}$,
if $G = K$ is compact, then:

- $s_{\mathbf{u}}(K)$ is a Borel set, and so
  either $s_{\mathbf{u}}(K)$ is countable or $|s_{\mathbf{u}}(K)| = \mathfrak{c}$;
- $\mu(s_{\mathbf{u}}(K)) = 0$ if $\mathbf{u}$ is non-trivial.
  [Comfort-Trigos-Arrieta-Wu 1993; Raczkowski 2002 for LCA]

Let K be a compact abelian group.

Theorem (Dikranjan-Kunen; Beiglböck-Steineder-Winkler 2006)

If K is metrizable, then every countable $H \leq K$ is characterized.

For G an abelian group, $\mathbf{u} \in \widehat{G}^{\mathbb{N}}$ and $H \leq G$, let

$$\mathfrak{g}_G(H) = \bigcap \{ s_{\mathbf{u}}(G) : \mathbf{u} \in \widehat{G}^{\mathbb{N}},\ H \leq s_{\mathbf{u}}(G) \};$$

H is *$\mathfrak{g}$-closed* if $H = \mathfrak{g}_G(H)$.

Theorem (Lukács 2006)

All countable subgroups of K are $\mathfrak{g}$-closed.

Every characterized subgroup of K is $F_{\sigma\delta}$.
[Biró 2007]: There exist F_σ-subgroups of K not characterized.

Theorem (Dikranjan-Gabriyelyan 2013)

Every G_δ-subgroup of K is characterized.

Definition

A non-trivial sequence $\mathbf{u}$ in an abelian group G is:

- a *T-sequence* if there is a Hausdorff group topology τ on G such that $u_n \xrightarrow{\tau} 0$;
- a *TB-sequence* if there exists a precompact topology τ on G such that $u_n \xrightarrow{\tau} 0$.

[Protasov-Zelenyuk 1999]: complete criterion for T-sequences.

[Dikranjan-Milan-Tonolo 2005]:

- If τ is a totally bounded group topology on G, then $\tau = T_H$ for some $H \leq \widehat{G}$ and $u_n \xrightarrow{\tau} 0 \iff H \leq s_{\mathbf{u}}(\widehat{G})$.
- $T_{s_{\mathbf{u}}(\widehat{G})}$ is the finest tot. bounded gr. top. τ on G with $u_n \xrightarrow{\tau} 0$.
- $\mathbf{u}$ is a TB-sequence if and only if $s_{\mathbf{u}}(\widehat{G})$ is dense in $\widehat{G}$.

For other results on characterized subgroups and related topics see recent papers by Dikranjan, Gabriyelyan, Impieri, etc.

- THE END -

Thank you for the attention