

Topological entropy on totally disconnected locally compact groups

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Topological entropy (h_{top})

- [Adler, Konheim, McAndrew 1965]:
for continuous selfmaps of compact spaces.
- [Bowen 1971]:
for uniformly continuous selfmaps of metric spaces.
- [Hood 1974]:
for uniformly continuous selfmaps of uniform spaces.
- We consider it:
for continuous endomorphisms of locally compact groups.
- These entropies coincide on compact groups.
- [Stojanov 1978]:
characterization of topological entropy for continuous endomorphisms of compact groups.

Let G be a locally compact group, μ a Haar measure on G , $\mathcal{C}(G)$ the family of all compact neighborhoods of 1 in G , $\phi : G \rightarrow G$ a continuous endomorphism.

- For $n > 0$, the n -th ϕ -cotrajectory of $U \in \mathcal{C}(G)$ is

$$C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \dots \cap \phi^{-n+1}(U) \in \mathcal{C}(G).$$

- The *topological entropy* of ϕ with respect to $U \in \mathcal{C}(G)$ is

$$H_{top}(\phi, U) = \limsup_{n \rightarrow \infty} \frac{-\log \mu(C_n(\phi, U))}{n}.$$

(It does not depend on the choice of the Haar measure μ .)

- The *topological entropy* of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{C}(G)\}.$$

Problem (Additivity of topological entropy)

Let G be a locally compact group, $\phi : G \rightarrow G$ a continuous endomorphism and N a ϕ -invariant closed normal subgroup of G . Is it true that

$$h_{\text{top}}(\phi) = h_{\text{top}}(\phi \upharpoonright_N) + h_{\text{top}}(\bar{\phi}),$$

where $\bar{\phi} : G/N \rightarrow G/N$ is the endomorphism induced by ϕ ?

$$\begin{array}{ccccc}
 N & \longrightarrow & G & \longrightarrow & G/N \\
 \phi \upharpoonright_N \downarrow & & \phi \downarrow & & \bar{\phi} \downarrow \\
 N & \longrightarrow & G & \longrightarrow & G/N
 \end{array}$$

[Yuzvinski 1965]: for separable compact groups.

[Bowen 1971]: for compact metric spaces.

[Alcaraz-Dikranjan-Sanchis 2014]: for compact groups.

We consider the case when

G is a totally disconnected locally compact group
and $\phi : G \rightarrow G$ is a continuous endomorphism.

Let $\mathcal{B}(G) = \{U \leq G : U \text{ compact, open}\}$.

[van Dantzig 1931]: $\mathcal{B}(G)$ is a base of the neighborhoods of 1 in G .

[Dikranjan-Sanchis-Virili 2012]:

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}(G)\};$$

moreover, for $U \in \mathcal{B}(G)$,

$$H_{top}(\phi, U) = \lim_{n \rightarrow \infty} \frac{\log[U : C_n(\phi, U)]}{n}.$$

(Recall that $C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \dots \cap \phi^{-n+1}(U) \in \mathcal{B}(G)$.)

Let G be a totally disconnected locally compact group and $\phi : G \rightarrow G$ a continuous endomorphism.

For $U \in \mathcal{B}(G)$, let:

- $U_0 = U$;
- $U_{n+1} = U \cap \phi(U_n)$ for every $n > 0$;
- $U_+ = \bigcap_{n=0}^{\infty} U_n$.

Then:

- $U_{n+1} \subseteq U_n$ for every $n > 0$;
- U_+ is a compact subgroup of G such that $U_+ \subseteq \phi(U_+)$.

Theorem (Limit-free formula; GB-Virili 2016)

$$H_{\text{top}}(\phi, U) = \log[\phi(U_+) : U_+],$$

[GB 2015]: for topological automorphisms.

Let G be a totally disconnected locally compact group,

$\phi : G \rightarrow G$ a continuous endomorphism,

N a ϕ -invariant closed normal subgroup of G ,

$\bar{\phi} : G/N \rightarrow G/N$ the endomorphism induced by ϕ .

Theorem (Addition Theorem; GB-Virili 2016)

If $\ker \phi \subseteq N$ and $\phi(N) = N$, then

$$h_{\text{top}}(\phi) = h_{\text{top}}(\phi \upharpoonright_N) + h_{\text{top}}(\bar{\phi}).$$

$\ker \phi \subseteq N$ and $\phi(N) = N$ if and only if

$\phi \upharpoonright_N$ is injective and $\bar{\phi}$ is surjective.

Corollary

If $\phi : G \rightarrow G$ is a topological automorphism, then

$$h_{\text{top}}(\phi) = h_{\text{top}}(\phi \upharpoonright_N) + h_{\text{top}}(\bar{\phi}).$$

Let G be a totally disconnected locally compact group,
 $\phi : G \rightarrow G$ a continuous endomorphism.

If $N \leq G$ compact (not necessarily normal) with $\phi(N) = N$,
 then $G/N = \{xN : x \in G\}$ is a locally compact uniform space
 and $\bar{\phi} : G/N \rightarrow G/N$ is a uniformly continuous map.

Then

$$h_{top}(\bar{\phi}) = \sup\{H_{top}(\phi, U) : N \subseteq U \in \mathcal{B}(G)\}.$$

Theorem (GB-Virili 2016)

If $\ker \phi \subseteq N$ and $\phi(N) = N$, then

$$h_{top}(\phi) = h_{top}(\phi \upharpoonright_N) + h_{top}(\bar{\phi}).$$

[Willis 2015] (in 2001 for topological automorphisms):

- The *scale* of a continuous endomorphism $\phi : G \rightarrow G$ of a totally disconnected locally compact group G is

$$s(\phi) = \min\{[\phi(U) : U \cap \phi(U)] : U \in \mathcal{B}(G)\}.$$

- $U \in \mathcal{B}(G)$ is *minimizing* if $s(\phi) = [\phi(U) : U \cap \phi(U)]$.
- $\text{nub } \phi \leq G$ is compact and $\phi(\text{nub } \phi) = \text{nub } \phi$;

$$\text{nub } \phi := \bigcap \{U \in \mathcal{B}(G) : U \text{ minimizing}\}.$$

Theorem (GB-Virili 2016)

For $\bar{\phi} : G/\text{nub } \phi \rightarrow G/\text{nub } \phi$ the map induced by ϕ ,

$$h_{\text{top}}(\bar{\phi}) = \log s(\phi).$$

Corollary (Berlai-Dikranjan-GB and Spiga 2013)

$h_{\text{top}}(\phi) = \log s(\phi)$ if and only if $\text{nub } \phi = \{1\}$.

Algebraic entropy (h_{alg})

- [Adler, Konheim, McAndrew 1965; Weiss 1974; Dikranjan-Goldsmith-Salce-Zanardo 2009]:
for endomorphisms of discrete (torsion) abelian groups.
- [Peters 1979]:
for automorphisms of discrete abelian groups.
- [Dikranjan-GB 2012, 2016]:
for endomorphisms of discrete (abelian) groups.
- [Peters 1981]:
for top. automorphisms of locally compact abelian groups.
- [Virili 2010; Dikranjan-GB 2012]:
for cont. endomorphisms of locally compact (abelian) groups.

Let G be a locally compact group, μ a Haar measure on G , $\mathcal{C}(G)$ the family of all compact neighborhoods of 1 in G , $\phi : G \rightarrow G$ a continuous endomorphism.

- For $n > 0$, the n -th ϕ -trajectory of $U \in \mathcal{C}(G)$ is

$$T_n(\phi, U) = U \cdot \phi(U) \cdot \dots \cdot \phi^{n-1}(U) \in \mathcal{C}(G).$$

- The *algebraic entropy* of ϕ with respect to $U \in \mathcal{C}(G)$ is

$$H_{alg}(\phi, U) = \limsup_{n \rightarrow \infty} \frac{\log \mu(T_n(\phi, U))}{n}.$$

(It does not depend on the choice of the Haar measure μ .)

- The *algebraic entropy* of ϕ is

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, U) : U \in \mathcal{C}(G)\}.$$

Let G be a totally disconnected locally compact **abelian** group and $\phi : G \rightarrow G$ a continuous endomorphism.

Let \widehat{G} be the Pontryagin dual of G and $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$ the dual of ϕ .

Then $\mathcal{B}(\widehat{G})$ is cofinal in $\mathcal{C}(\widehat{G})$. Hence,

$$h_{alg}(\phi) = \sup\{H_{alg}(\phi, U) : U \in \mathcal{B}(\widehat{G})\};$$

moreover, for $U \in \mathcal{B}(\widehat{G})$,

$$H_{alg}(\widehat{\phi}, U) = \lim_{n \rightarrow \infty} \frac{\log[T_n(\widehat{\phi}, U) : U]}{n}.$$

Theorem (Bridge Theorem; Dikranjan-GB 2014)

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}).$$

[Weiss 1974]: for totally disconnected compact abelian groups.

[Dikranjan-GB 2011]: for compact abelian groups.

[Virili 2015]: for actions of amenable groups on LCA groups.

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