Entropy in locally linearly compact vector spaces

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Recent Advances in Commutative Ring and Module Theory
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- Measure entropy [Kolmogorov, Sinai 1957]
- Topological entropy h_{top} [Adler, Konheim, McAndrew 1965; Bowen 1971; Hood 1974; . . .]
- Algebraic entropies h_{alg} [A, K, M 1965; Weiss 1975; Peters 1979; Dikranjan, Goldsmith, Salce, Zanardo 2009; D, GB 2009–16; Virili 2010; . . .]
- i-Entropies for modules [S, Z 2009; Z 2009; GB, S 2010; S, Vàmos, V 2010; V, V 2011; S, V 2016; . . .]
- Adjoint entropy $h^{\star}_{alg}$ [D, GB, S 2010; Goldsmith, Gong 2011; GB 2011; . . .]
- . . .

[AGB & Luigi Salce 2010]

Let $\mathbb{K}$ be a field and V a vector space over $\mathbb{K}$.
Let $\phi : V \to V$ be a linear transformation.

For a subspace F of V and $n > 0$, let

$$T_n(\phi, F) = F + \phi(F) + \phi^2(F) + \ldots + \phi^{n-1}(F).$$

If F is a finite-dimensional subspace of V,
the algebraic entropy of ϕ with respect to F is

$$H(\phi, F) = \lim_{n\to\infty} \frac{1}{n} \dim T_n(\phi, F).$$

The algebraic entropy of ϕ is

$$\mathrm{ent}(\phi) = \sup\{H(\phi, F) : F \leq V,\ \dim F \text{ finite}\}.$$

1. For any vector space V, we have $\mathrm{ent}(\mathrm{id}_V) = 0$.
2. The right Bernoulli shift

$$\beta_{\mathbb{K}} : \mathbb{K}^{(\mathbb{N})} \to \mathbb{K}^{(\mathbb{N})}, \quad (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots)$$

   has $\mathrm{ent}(\beta_{\mathbb{K}}) = 1$.
   The left Bernoulli shift

$$_{\mathbb{K}}\beta : \mathbb{K}^{(\mathbb{N})} \to \mathbb{K}^{(\mathbb{N})}, \quad (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$$

   has $\mathrm{ent}(_{\mathbb{K}}\beta) = 0$.
3. If $\mathbb{K} = \mathbb{Z}(p)$ with p a prime, a vector space V over $\mathbb{K}$ is a torsion abelian group and

$$\mathrm{ent}(\phi) = \frac{h_{alg}(\phi)}{\log |\mathbb{K}|}.$$

Invariance under conjugation: If $\phi = \xi^{-1}\psi\xi$, $\psi : W \to W$ linear transformation, $\xi : V \to W$ isomorphism

$$\begin{array}{ccc} V & \xrightarrow{\phi} & V \\ \xi\downarrow & & \downarrow\xi \\ W & \xrightarrow{\psi} & W \end{array},$$

then $\mathrm{ent}(\phi) = \mathrm{ent}(\psi)$.

Monotonicity: W ϕ-invariant subspace of V, $\overline{\phi} : V/W \to V/W$ induced by ϕ; then $\mathrm{ent}(\phi) \geq \max\{\mathrm{ent}(\phi \restriction_W), \mathrm{ent}(\overline{\phi})\}$.

Logarithmic Law: $\mathrm{ent}(\phi^k) = k \cdot \mathrm{ent}(\phi)$ for every $k \geq 0$.

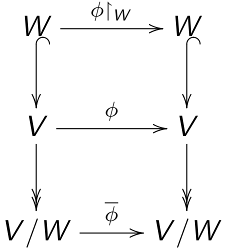
Continuity: If V is direct limit of ϕ-invariant subspaces $\{V_i : i \in I\}$, then $\mathrm{ent}(\phi) = \sup_{i \in I} \mathrm{ent}(\phi \restriction_{V_i})$.

weak Addition Theorem: If $V = V_1 \times V_2$ and $\phi_i : V_i \to V_i$ linear transformation, $i = 1, 2$, then $\mathrm{ent}(\phi_1 \times \phi_2) = \mathrm{ent}(\phi_1) + \mathrm{ent}(\phi_2)$.

Theorem (Addition Theorem)

Let V be a vector space, $\phi : V \to V$ a linear transformation, W a ϕ-invariant subspace of V, $\overline{\phi} : V/W \to V/W$ induced by ϕ. Then

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_W) + \mathrm{ent}(\overline{\phi}).$$



Let $\mathbb{K}$ be a field and V a vector space over $\mathbb{K}$.
Let $\phi : V \to V$ be a linear transformation.

For a subspace N of V and $n > 0$, let

$$C_n(\phi, N) = N \cap \phi^{-1}(N) \cap \phi^{-2}(N) \cap \ldots \cap \phi^{-n+1}(N).$$

If N is a subspace of V of finite codimension,
the adjoint algebraic entropy of ϕ with respect to N is

$$H^\star(\phi, N) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{V}{C_n(\phi, N)}.$$

The adjoint algebraic entropy of ϕ is

$$\mathrm{ent}^\star(\phi) = \sup\{H^\star(\phi, N) : N \leq V,\ \dim V/N \text{ finite}\}.$$

Invariance under conjugation: If $\phi = \xi^{-1}\psi\xi$, $\psi : W \to W$ linear transformation, $\xi : V \to W$ isomorphism, then $\mathrm{ent}^\star(\phi) = \mathrm{ent}^\star(\psi)$.

Monotonicity: W ϕ-invariant subspace of V, $\overline{\phi} : V/W \to V/W$ induced by ϕ; then $\mathrm{ent}^\star(\phi) \geq \max\{\mathrm{ent}(\phi \restriction_W), \mathrm{ent}^\star(\overline{\phi})\}$.

Logarithmic Law: $\mathrm{ent}^\star(\phi^k) = k \cdot \mathrm{ent}^\star(\phi)$ for every $k \geq 0$.

weak Addition Theorem: If $V = V_1 \times V_2$ and $\phi_i : V_i \to V_i$ linear transformation, $i = 1, 2$, then
$\mathrm{ent}^\star(\phi_1 \times \phi_2) = \mathrm{ent}^\star(\phi_1) + \mathrm{ent}^\star(\phi_2)$.

Theorem (Addition Theorem)

Let V be a vector space, $\phi : V \to V$ a linear transformation, W a ϕ-invariant subspace of V, $\overline{\phi} : V/W \to V/W$ induced by ϕ. Then

$$\mathrm{ent}^\star(\phi) = \mathrm{ent}^\star(\phi \restriction_W) + \mathrm{ent}^\star(\overline{\phi}).$$

1. For any vector space V, we have $\mathrm{ent}^\star(\mathrm{id}_V) = 0$.
2. The right Bernoulli shift

$$\beta_{\mathbb{K}} : \mathbb{K}^{(\mathbb{N})} \to \mathbb{K}^{(\mathbb{N})}, \quad (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots)$$

   has $\mathrm{ent}^\star(\beta_{\mathbb{K}}) = \infty$.
   The left Bernoulli shift

$$_{\mathbb{K}}\beta : \mathbb{K}^{(\mathbb{N})} \to \mathbb{K}^{(\mathbb{N})}, \quad (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$$

   has $\mathrm{ent}^\star({}_{\mathbb{K}}\beta) = \infty$.
3. If $\mathbb{K} = \mathbb{Z}(p)$ with p a prime, a vector space V over $\mathbb{K}$ is a torsion abelian group and

$$\mathrm{ent}^\star(\phi) = \frac{h^\star_{alg}(\phi)}{\log |\mathbb{K}|}.$$

Theorem (Bridge Theorem)

Let V be a vector space, $\phi : V \to V$ a linear transformation, $V^\wedge$ the dual space of V and $\phi^\wedge : V^\wedge \to V^\wedge$ the dual linear transformation. Then

$$\mathrm{ent}^\star(\phi) = \mathrm{ent}(\phi^\wedge).$$

Corollary (Dichotomy)

Let V be a vector space, $\phi : V \to V$ a linear transformation. Then

$$\mathrm{ent}^\star(\phi) \in \{0, \infty\}.$$

[Lefschetz 1942]

Let $\mathbb{K}$ be a discrete field and let V be a linearly topologized vector space over $\mathbb{K}$

- V is linearly compact if, for every family $\mathcal{M} = \{m_i + M_i : i \in I\}$ of closed linear varieties of V with the finite intersection property, $\mathcal{M}$ has non-empty intersection.
- V is locally linearly compact if there exists an open linear subspace of V that is linearly compact.

Discrete vector spaces and linearly compact vector spaces are l.l.c.; V is linearly compact and discrete if and only if $\dim V$ is finite; V l.l.c. $\cong_{top} V_{lc} \times V_d$.

V is locally linearly compact if and only if it admits a neighborhood basis of 0 consisting of linearly compact subspaces; so let $\mathcal{B}(V) = \{U \leq V : U \text{ open, linearly compact}\}$.

If $\mathbb{K} = \mathbb{Z}(p)$, V l.c. implies that V is a t.d. compact abelian group; and V l.l.c. implies that V is a t.d. locally compact abelian group.

Let V be a locally linearly compact vector space.

Let $\mathrm{CHom}(V,\mathbb{K}) = \{\chi : V \to \mathbb{K} \text{ continuous character}\} \leq V^\wedge$.
For $A \leq V$, let $A^\perp = \{\chi \in \mathrm{CHom(V,\mathbb{K})} : \chi(A) = 0\}$.

The topological dual V^* of V is $\mathrm{CHom}(V,\mathbb{K})$ with the topology generated by $\{A^\perp : A \leq V,\ A \text{ l.c.}\}$ as a basis of nbhs of 0.

- V^* is a locally linearly compact vector space.
- V is discrete if and only if V^* is linearly compact.
- V is linearly compact if and only if V^* is discrete.

The dual of $\phi : V \to W$ is $\phi^* : W^* \to V^*$ such that $\chi \mapsto \chi \circ \phi$.

$^* : \mathrm{LLC}_\mathbb{K} \to \mathrm{LLC}_\mathbb{K}$ duality functor.
$(^* : \mathrm{Vect}_\mathbb{K} \to \mathrm{LC}_\mathbb{K};\ ^* : \mathrm{LC}_\mathbb{K} \to \mathrm{Vect}_\mathbb{K})$

Duality Theorem: V is topological isomorphic to V^{**}.
($^{**} : \mathrm{LLC}_\mathbb{K} \to \mathrm{LLC}_\mathbb{K}$ and $\mathrm{id} : \mathrm{LLC}_\mathbb{K} \to \mathrm{LLC}_\mathbb{K}$ are naturally iso.)
($^* : \mathrm{Vect}_\mathbb{K} \to \mathrm{LC}_\mathbb{K}$ and $^* : \mathrm{LC}_\mathbb{K} \to \mathrm{Vect}_\mathbb{K}$ form a duality.)

[Ilaria Castellano & AGB 2016]

Let V be a linearly compact vector space.
Then $\mathcal{B}(V) = \{U \leq V : U \text{ open}\}$.

If $U \in \mathcal{B}(V)$, then dim V/U is finite.

Let $\phi : V \to V$ a continuous linear transformation.

The (topological) adjoint entropy of ϕ is

$$\mathrm{ent}^\star_t(\phi) = \sup\{H^\star(\phi, U) : U \in \mathcal{B}(V)\} \leq \mathrm{ent}^\star(\phi).$$

Invariance under conjugation: If $\phi = \xi^{-1}\psi\xi$, $\psi : W \to W$ continuous linear transformation, $\xi : V \to W$ topological isomorphism, then $\mathrm{ent}^\star_t(\phi) = \mathrm{ent}^\star_t(\psi)$.

Monotonicity: $W \leq V$ closed ϕ-invariant, $\overline{\phi} : V/W \to V/W$ induced by ϕ; then $\mathrm{ent}^\star_t(\phi) \geq \max\{\mathrm{ent}^\star_t(\phi \restriction_W), \mathrm{ent}^\star_t(\overline{\phi})\}$.

Logarithmic Law: $\mathrm{ent}^\star_t(\phi^k) = k \cdot \mathrm{ent}^\star_t(\phi)$ for every $k \geq 0$.

Continuity: If V is inverse limit of $\{V/V_i : i \in I\}$ where each $V_i \leq V$ is closed ϕ-invariant, then $\mathrm{ent}^\star_t(\phi) = \sup_{i \in I} \mathrm{ent}^\star_t(\overline{\phi}_{V/V_i})$.

weak Addition Theorem: If $V = V_1 \times V_2$ and $\phi_i : V_i \to V_i$ continuous, $i = 1, 2$, then $\mathrm{ent}^\star_t(\phi_1 \times \phi_2) = \mathrm{ent}^\star_t(\phi_1) + \mathrm{ent}^\star_t(\phi_2)$.

Theorem (Addition Theorem)

Let V be a linearly compact vector space, $\phi : V \to V$ a continuous linear transformation, W a closed ϕ-invariant subspace of V, $\overline{\phi} : V/W \to V/W$ induced by ϕ. Then

$$\mathrm{ent}^\star_t(\phi) = \mathrm{ent}^\star_t(\phi \restriction_W) + \mathrm{ent}^\star_t(\overline{\phi}).$$

1. For any linearly compact vector space V, $\mathrm{ent}_t^\star(\mathrm{id}_V) = 0$.
2. The right Bernoulli shift
$$\tilde{\beta}_{\mathbb{K}} : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}, \quad (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots)$$
has $\mathrm{ent}_t^\star(\tilde{\beta}_{\mathbb{K}}) = 0$.
The left Bernoulli shift
$$_{\mathbb{K}}\tilde{\beta} : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}, \quad (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$$
has $\mathrm{ent}_t^\star({}_{\mathbb{K}}\tilde{\beta}) = 1$.
3. If $\mathbb{K} = \mathbb{Z}(p)$ with p a prime, a linearly compact vector space V over $\mathbb{K}$ is a totally disconnected compact abelian group and
$$\mathrm{ent}_t^\star(\phi) = \frac{h_{top}(\phi)}{\log |\mathbb{K}|}.$$

Let V be a discrete vector space. Then V^* is a linearly compact vector space.

- $(\beta_{\mathbb{K}})^* = {}_{\mathbb{K}}\tilde{\beta}$, so $\mathrm{ent}(\beta_{\mathbb{K}}) = 1 = \mathrm{ent}^\star_t({}_{\mathbb{K}}\tilde{\beta}) = \mathrm{ent}^\star_t((\beta_{\mathbb{K}})^*)$.
- $({}_{\mathbb{K}}\beta)^* = \tilde{\beta}_{\mathbb{K}}$, so $\mathrm{ent}({}_{\mathbb{K}}\beta) = 0 = \mathrm{ent}^\star_t(\tilde{\beta}_{\mathbb{K}}) = \mathrm{ent}^\star_t(({}_{\mathbb{K}}\beta)^*)$.

Theorem (Bridge Theorem)

Let V be a discrete vector space and $\phi : V \to V$ a linear transformation. Then

$$\mathrm{ent}(\phi) = \mathrm{ent}^\star_t(\phi^*).$$

If $\mathbb{K} = \mathbb{Z}(p)$ with p a prime, V is a torsion abelian group and V^* is a totally disconnected compact abelian group. For $\phi : V \to V$ a continuous linear transformation,

$$\mathrm{ent}(\phi) = \frac{h_{alg}(\phi)}{\log |\mathbb{K}|} \quad \text{and} \quad \widetilde{ent}^\star_t(\phi^*) = \frac{h_{top}(\phi^*)}{\log |\mathbb{K}|}.$$

Moreover, $h_{alg}(\phi) = h_{top}(\phi^*)$ [Dikranjan & GB 2012].

Let V be a locally linearly compact vector space and $\mathcal{B}(V) = \{U \leq V : U \text{ open, linearly compact}\}$.

Let $\phi : V \to V$ be a continuous linear transformation.

For $U \in \mathcal{B}(V)$, the algebraic entropy of ϕ with respect to U is

$$\widetilde{H}(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{T_n(\phi, U)}{U};$$

the algebraic entropy of ϕ is

$$\widetilde{ent}(\phi) = \sup\left\{\widetilde{H}(\phi, U) : U \in \mathcal{B}(V)\right\}.$$

For $U \in \mathcal{B}(V)$, the adjoint entropy of ϕ with respect to U is

$$\widetilde{H}^\star(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{U}{C_n(\phi, U)};$$

the adjoint entropy of ϕ is

$$\widetilde{ent}^\star_t(\phi) = \sup\left\{\widetilde{H}^\star(\phi, U) : U \in \mathcal{B}(V)\right\}.$$

Let V be a locally linearly compact vector space and $\phi : V \to V$ a continuous linear transformation.

$$\widetilde{ent}(\phi) = \begin{cases} \mathrm{ent}(\phi) & \text{if } V \text{ is discrete,} \\ 0 & \text{if } V \text{ is linearly compact.} \end{cases}$$

$$\widetilde{ent}_t^\star(\phi) = \begin{cases} 0 & \text{if } V \text{ is discrete,} \\ \mathrm{ent}_t^\star(\phi) & \text{if } V \text{ is linearly compact.} \end{cases}$$

All the basic properties of ent and $\mathrm{ent}_t^\star$ extend to the general case of $\widetilde{ent}$ and $\widetilde{ent}_t^\star$, also the Addition Theorems.

If $\mathbb{K} = \mathbb{Z}(p)$ for p a prime, then a locally linearly compact vector space V over $\mathbb{K}$ is a totally disconnected locally compact abelian group such that V^* is a totally disconnected locally compact abelian group too; and

$$\widetilde{ent}(\phi) = \frac{h_{alg}(\phi)}{\log |\mathbb{K}|} \quad \text{and} \quad \widetilde{ent}_t^\star(\phi) = \frac{h_{top}(\phi)}{\log |\mathbb{K}|}.$$

Theorem (Bridge Theorem)

Let V be a locally linearly compact vector space and $\phi : V \to V$ a continuous linear transformation. Then

$$\widetilde{ent}(\phi) = \widetilde{ent}^{\star}_{t}(\phi^*).$$

Assume that $\mathbb{K} = \mathbb{Z}(p)$ with p a prime, V is a locally linearly compact vector space and $\phi : V \to V$ is a continuous linear transformation. Then V and V^* are totally disconnected locally compact abelian groups; and

$$\widetilde{ent}(\phi) = \frac{h_{alg}(\phi)}{\log |\mathbb{K}|} \quad \text{and} \quad \widetilde{ent}^{\star}_{t}(\phi^*) = \frac{h_{top}(\phi^*)}{\log |\mathbb{K}|}.$$

Moreover, $h_{alg}(\phi) = h_{top}(\phi^*)$ [Dikranjan & GB 2014].

END

Thank you!