Topological entropy for automorphisms of totally disconnected locally compact groups

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Topological entropy

- 1965 Adler, Konheim, McAndrew: for continuous selfmaps of compact spaces;
- 1971 Bowen: for uniformly continuous selfmaps of metric spaces;
- 1974 Hood: for uniformly continuous selfmaps of uniform spaces;
  in particular, for continuous endomorphisms of locally compact groups.
- These entropies coincide on continuous endomorphisms of compact groups.
  1987 Stojanov: characterization of topological entropy for continuous endomorphisms of compact groups.

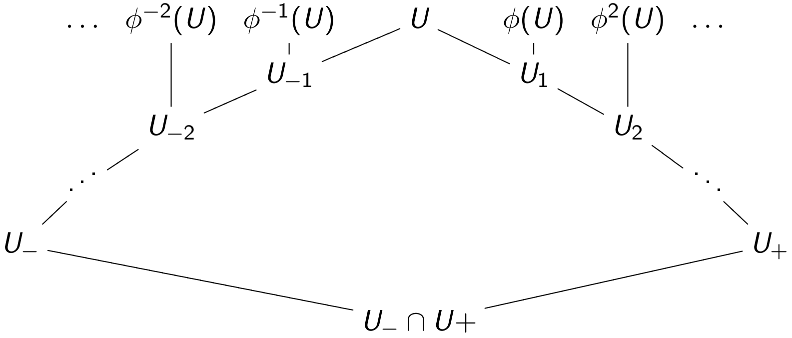
G locally compact group, μ right Haar measure on G.
$\mathcal{C}(G)$ a local base at 1 of compact neighborhoods; $U \in \mathcal{C}(G)$.
$\phi : G \to G$ topological automorphism;
n non-negative integer.

Let

- $U_n = U \cap \phi(U) \cap \ldots \cap \phi^n(U)$;
- $U_+ = \bigcap_{n=0}^{\infty} \phi^n(U)$.

Analogously,

- $U_{-n} = U \cap \phi^{-1}(U) \cap \ldots \cap \phi^{-n}(U)$;
- $U_- = \bigcap_{n=0}^{\infty} \phi^{-n}(U)$.


. . . $\phi^{-2}(U)$ $\phi^{-1}(U)$ U $\phi(U)$ $\phi^{2}(U)$ . . .
U_{-1}
U_1
U_{-2}
U_2
U_-
U_+
$U_- \cap U+$

The topological entropy of ϕ with respect to U is

$$\boxed{H_{top}(\phi, U) = \limsup_{n\to\infty} -\frac{\log \mu(U_{-n})}{n}.}$$

The topological entropy of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{C}(G)\}.$$

Assume that G is also totally disconnected;

$$\mathcal{B}(G) = \{U \leq G : \text{open compact}\} \subseteq \mathcal{C}(G).$$

van Dantzig 1931: $\mathcal{B}(G)$ is a local base at 1.

For $U \in \mathcal{B}(G)$,

$$\boxed{H_{top}(\phi, U) = \limsup_{n\to\infty} \frac{\log[U : U_{-n}]}{n}.}$$

Moreover,

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}(G)\}.$$

Theorem (Limit-free Formula)

$$H_{top}(\phi, U) = \log[\phi(U_+) : U_+]$$

Sketch of the proof.

1. $c_n := [U : U_{-n}]$, $c_n | c_{n+1}$ for every $n \geq 0$;
2. $\alpha_n := \frac{c_{n+1}}{c_n} = [U_{-n} : U_{-(n+1)}]$, $\alpha_{n+1} \leq \alpha_n$ for every $n \geq 0$;
3. $\alpha_n = \alpha$ for every $n \gg 0$; so
$$H_{top}(\phi, U) = \log \alpha;$$
4. Then
$$\begin{aligned}[\phi(U_+) : U_+] &= [\phi(U_+) : U \cap \phi(U_+)] = [\phi(U_+)U : U] \\ &= [\phi(U_n)U : U] \text{ for every } n \gg 0 \\ &= [\phi(U_n) : U \cap \phi(U_n)] = [\phi(U_n) : U_{n+1}] \\ &= [\phi^{-(n+1)}(\phi(U_n)) : \phi^{-(n+1)}(U_{n+1})] \\ &= [U_{-n} : U_{-(n+1)}] = \alpha.\end{aligned}$$

The modulus $\Delta : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is defined by

$$\Delta(\phi) = \frac{\mu(\phi(U))}{\mu(U)}.$$

We have

$$\begin{aligned} H_{top}(\phi, U) &= \log[\phi(U_+) : U_+] \\ &= \log[\phi^{-1}(U_-) : U_-] + \log \Delta(\phi). \end{aligned}$$

Hence,

$$H_{top}(\phi^{-1}, U) = H_{top}(\phi, U) - \log \Delta(\phi).$$

Monotonicity: N closed normal subgroup of G, $\phi(N) = N$, $\overline{\phi} : G/N \to G/N$ induced by ϕ, then $h_{top}(\phi) \geq \max\{h_{top}(\phi \restriction_N), h_{top}(\overline{\phi})\}$.

Invariance under conjugation: $\xi : G \to H$ topological isomorphism, then $h_{top}(\xi\phi\xi^{-1}) = h_{top}(\phi)$.

Logarithmic law: $h_{top}(\phi^k) = k \cdot h_{top}(\phi)$ for every natural k.

Continuity: $G = \varprojlim G/G_i$ with G_i closed normal ϕ-invariant subgroup, then $h_{top}(\phi) = \sup_{i\in I} h_{top}(\phi \restriction_{G_i})$.

Additivity for direct products: $G = G_1 \times G_2$, $\phi_i : G_i \to G_i$ topological automorphism, $i = 1, 2$, then $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.

Willis 2002: the scale of ϕ is

$$s(\phi) = \min\{[\phi(U) : U \cap \phi(U)] : U \in \mathcal{B}(G)\}.$$

By the Limit-free Formula

$$h_{top}(\phi) = \sup\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\},$$

and by Willis' Tidying Procedure

$$\log s(\phi) = \min\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}.$$

This gives

Theorem

$$h_{top}(\phi) \geq \log s(\phi)$$

Equality holds precisely when the minimizing subgroups form a local base of neighborhoods of 1.

Problem (1)

Extend the Limit-free Formula to continuous endomorphisms.

Available in the compact case.

Problem (2)

Prove an analogous Limit-free Formula for the algebraic entropy.

Available for endomorphisms of discrete torsion groups.

Problem (3)

Compare the scale function with the algebraic entropy.

Problem (4)

Does the Addition Theorem hold for topological automorphisms of totally disconnected locally compact groups?

In other words, we ask whether

$$h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\overline{\phi}),$$

where N is closed normal subgroup of G with $\phi(N) = N$.

$$\begin{array}{ccc} N & \xrightarrow{\phi\restriction_N} & N \\ \downarrow & & \downarrow \\ G & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow \\ G/N & \xrightarrow{\overline{\phi}} & G/N \end{array}$$

Stojanov 1987: the Addition Theorem holds for compact groups.

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